# Conformal symmetry and duality between free particle, H atom, and harmonic oscillator

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We establish a duality between the free massless relativistic particle in $d$ dimensions, the non-relativistic hydrogen atom ($1/r$ potential) in $(d-1)$ space dimensions, and the harmonic oscillator in $(d\text{-}2)$ space dimensions with its mass given as the light cone momentum of an additional dimension. The duality is in the sense that the classical action of these systems are gauge fixed forms of the same worldline gauge theory action at the classical level, and they are all described by the same unitary representation of the conformal group SO $(d,2)$ at the quantum level. The world line action has a gauge symmetry Sp(2) which treats canonical variables $(x,p)$ as doublets and exists only with a target spacetime that has $d$ spacelike dimensions and two timelike dimensions. This spacetime is constrained due to the gauge symmetry, and the various dual solutions correspond to solutions of the constraints with different topologies. For example, for the H atom the two timelike dimensions $X^{0'},X^0$ live on a circle. The model provides an example of how realistic physics can be viewed as existing in a larger covariant space that includes two timelike coordinates, and how the covariance in the larger space unifies different looking physics into a single system.
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## I. GAUGE SECTORS AND DUALITY

In a recent paper [1] a duality was constructed between several simple physical systems by showing that they are different aspects of the same quantum theory. The theory is based on gauging the Sp(2) duality symmetry that treats position and momentum $(x,p)$ as a doublet in phase space. The worldline action has a manifest SO$(d,2)$ symmetry acting linearly on a target spacetime $X^M(\tau)$ with two times. Thanks to the gauge symmetry the theory is equivalent to a theory with a single time, but the choice of ''time'' is not unique. For different gauge choices of ''time'' the Hamiltonian looks different and appears to describe different physical systems. However these systems are gauge equivalent, i.e. duality equivalent. It was shown that the Sp(2) duality *gauge invariant sector* is fully characterized in the quantum theory by a *unique* unitary representation of the conformal group SO$(d,2)$. The quadratic Casimir coefficient of SO$(d,2)$ takes the value $C_2=1-d^2/4$ and all higher Casimir coefficients $C_n$ are also fixed. In [1] it was shown that the free relativistic particle is described by this representation at the quantum level. In this paper we will show that the hydrogen atom and the harmonic oscillator are also described by the same unitary representation and hence they are dual to the free particle at the classical as well as quantum levels.

First we give the action for the model. To remove the distinction between position and momentum we rename them $X_1^M\equiv X^M$ and $X_2^M\equiv P^M$ and define the doublet $X_i^M=(X_1^M,X_2^M)$. The local Sp(2) acts as follows:

$$\delta_\omega X_i^M(\tau)=\varepsilon_{ik}\omega^{kl}(\tau)X_l^M(\tau). \tag{1}$$

Here $\omega^{ij}(\tau)=\omega^{ji}(\tau)$ is a symmetric matrix containing three local parameters, and $\varepsilon_{ij}$ is the Levi-Civita symbol that is invariant under Sp$(2,R)$ and serves to raise or lower indices. The Sp$(2,R)$ gauge field $A^{ij}(\tau)$ is symmetric in $(ij)$ and transforms in the standard way $\delta_\omega A^{ij}=\partial_\tau\omega^{ij}+\omega^{ik}\varepsilon_{kl}A^{lj}+\omega^{jk}\varepsilon_{kl}A^{il}$. The covariant derivative is $D_\tau X_i^M=\partial_\tau X_i^M-\varepsilon_{ik}A^{kl}X_l^M$. An action that is invariant under this gauge symmetry is

$$S_0=\frac{1}{2}\int_0^T d\tau(D_\tau X_i^M)\varepsilon^{ij}X_j^N\eta_{MN}$$
$$=\int_0^T d\tau\left(\partial_\tau X_1^M X_2^N-\frac{1}{2}A^{ij}X_i^M X_j^N\right)\eta_{MN}. \tag{2}$$

As argued in [1] this system exists non-trivially only if $\eta_{MN}$ has signature $(d,2)$ including two timelike dimensions. Thus there is a manifest global SO$(d,2)$ symmetry. The canonical conjugates are $X_1^M=X^M$ and $\partial S/\partial\dot X_1^M=X_2^M=P^M$. They are consistent with the idea that $(X_1^M,X_2^M)$ is the doublet $(X^M,P^M)$. There has been some discussion in the past of an action related to this one [2–5], but not including our non-trivial classical and quantum solutions or our point view on duality. This action can be generalized in several ways consistently with the Sp(2) gauge invariance, including supersymmetry, and interactions with background gravitational fields $G_{(MN)}$ and $B_{[MN]}$ and/or background gauge fields $A_i^M$ that are doublets of Sp(2) [1]. In the presence of background fields the global symmetry SO$(d,2)$ is replaced by the Killing symmetries of the background fields.

The equations of motion for $X_i^M, A^{ij}$ that follows from the Lagrangian (2) are

$$\begin{pmatrix}\partial_\tau X^M\\ \partial_\tau P^M\end{pmatrix}=\begin{pmatrix}A^{12} & A^{22}\\ -A^{11} & -A^{12}\end{pmatrix}\begin{pmatrix}X^M\\ P^M\end{pmatrix} \tag{3}$$

$$X\cdot X=X\cdot P=P\cdot P=0. \tag{4}$$

The global symmetry generators for SO$(d,2)$ are

 

$$L^{MN}=\varepsilon^{ij}X_i^M X_j^N=X^M P^N-X^N P^M. \tag{5}$$

They are manifestly Sp(2) *gauge invariant*. At the classical level all Casimir coefficients of SO($d$,2) vanish due to the constraints (4)

$$C_n(\mathrm{SO}(d,2))=\frac{1}{n!}\mathrm{Tr}(iL)^n=0, \quad \text{classical.} \tag{6}$$

It was shown that this is sufficient to characterize completely all the classical solutions without making any gauge choice for ''time'' [1].

When the theory is quantized and orders of operators $X,P$ are taken into account, there is a similar statement. Before taking the constraints (4) into account, the quadratic Casimir operator of the gauge group is

$$C_2(\mathrm{Sp}(2))=\frac{1}{4}\left[X^M P^2 X_M-(X\cdot P)(P\cdot X)+\frac{d^2-4}{4}\right], \tag{7}$$

where the last term results from operator reordering. For $L^{MN}$ of the form (5) all the Casimir operator $C_n(SO(d,2))$ of Eq. (6) can all be written in terms of the quadratic Casimir of the gauge group $C_2(\mathrm{Sp}(2))$ plus operator reordering constants that depend on $d$. In particular,

$$C_2(\mathrm{SO}(d,2))=\frac{1}{2}L_{MN}L^{MN}=\left[4C_2(\mathrm{Sp}(2))+1-\frac{d^2}{4}\right]. \tag{8}$$

In the gauge invariant sector the physical states are singlets of Sp(2) and therefore, for physical states,

$$C_2(\mathrm{Sp}(2))=0, \quad C_2(\mathrm{SO}(d,2))=1-\frac{d^2}{4}. \tag{9}$$

Similarly all $C_n$ are fixed at the quantum level by demanding $C_2(\mathrm{Sp}(2))=0$. Thus, the quantum solution of the theory corresponds to a unique unitary representation of SO($d$,2), with specific eigenvalues of the Casimir coefficients $C_n(d)$. This important information obtained in covariant quantization completely determines the unitary physical Hilbert space. There are no ghosts in the physical space because this SO ($d$,2) representation is unitary.

The physical content of the system is better understood in non-covariant quantization by choosing a ''time'' and constructing a Hamiltonian. The choice of time is not unique because the spacetime of our model has more than one timelike dimension $X^{0'},X^0$. Since there is more than one ''time'' there are different looking Hamiltonians that are canonically conjugate to the given choice of time. In such physical gauges the system is automatically unitary but one must verify that quantization is consistent with the global SO($d$,2) symmetry. This requires some non-trivial ordering of canonical operators in the construction of the *gauge invariant* quantum generators $L^{MN}$ expressed in fixed gauges. After doing so, we show that the unitary representation is identical to the one that emerged from covariant quantization, with the same Casimir eigenvalues $C_n(d)$, but now expressed in the physical basis of some Hamiltonian. Examples of this procedure include the relativistic massless particle, the H atom, the harmonic oscillator, and more.

## II. FREE PARTICLE AND H ATOM AS GAUGE CHOICES

Consider the basis $X^M=(X^{+'},X^{-'},X^+,X^-,X^i)$ with the metric $\eta^{MN}$ taking the values $\eta^{+'-'}=\eta^{+-}=-1$ in the light cone type dimensions, while $\eta^{ij}=\delta^{ij}$ for the remaining $d-2$ space dimensions. Thus one time $X^{0'}$ is a linear combination of $X^{\pm'}$, and the other $X^0$ is a linear combination of $X^{\pm}$. The gauge group Sp(2) has three gauge parameters, hence we can make three gauge choices. The free particle lightcone gauge is $X^{+'}=1$, $P^{+'}=0$, $X^+=\tau$. This is a legitimate gauge choice as shown in [1]. Inserting this gauge into the constraints (4), and solving them, one finds the following components expressed in terms of the remaining independent degrees of freedom $(x^-,p^+,\vec{x}^i,\ \vec{p}^i)$:

$$\begin{aligned} &M=[+',\ -',+,\ -,\ i]\\ &X^M=[1,(\vec{x}^2/2-\tau x^-),\ \tau,\ x^-,\ \vec{x}^i]\\ &P^M=\left[0,\left(\vec{x}\cdot\vec{p}-x^-p^+-\frac{\tau\vec{p}^2}{2p^+}\right),\ p^+,\ \frac{\vec{p}^2}{2p^+},\ \vec{p}^i\right].\end{aligned} \tag{10}$$

One can verify that this gauge corresponds to the free relativistic massless particle, by inserting the gauge fixed form (10) into the action (2). Since all constraints have been solved, the $A^{ij}$ terms are absent, and we get

$$\begin{aligned} S_0&=\int_0^T d\tau\,\partial_\tau X_1^M X_2^N\,\eta_{MN}\\ &=\int_0^T d\tau\left(\partial_\tau\vec{x}\cdot\vec{p}-\partial_\tau x^-p^+-\frac{\vec{p}^2}{2p^+}\right).\end{aligned} \tag{11}$$

This is the action of the free massless relativistic particle in the lightcone gauge, in the first order formalism, with the correct Hamiltonian $P^-=\vec{p}^2/2p^+$. Note that both time coordinates have been gauge fixed, $X^{+'}=1$ and $X^+=\tau$, to describe the free particle. This is the free particle ''time.'' The SO($d$,2) symmetry generators in this gauge were given in [1]. They will be used in Sec. IV to discuss the duality between the free massless particle and the harmonic oscillator.

We now show that the hydrogen atom corresponds to another gauge choice in this system, with a rather different choice of ''time'' as a function of the two timelike dimensions $X^{0'},X^0$. Consider the basis $X^M=(X^{0'},X^0,X^I)$ and $P^M=(P^{0'},P^0,P^I)$ with metric $\eta^{0'0'}=\eta^{00}=-1$ and $\eta^{IJ}=\delta^{IJ}$. Choose one gauge such that the four functions $X^{0'},X^0,P^{0'},P^0$ are expressed in terms of three functions $F,G,u$:

$$X^{0'}=F\cos u, \quad X^0=F\sin u \tag{12}$$

$$P^{0'} = -G\ \sin u, \quad P^0 = G\ \cos u. \tag{13}$$

Inserting this form in the constraints (4) gives

$$X^M = F[\cos u,\ \ \sin u,\ n^I] \tag{14}$$

$$P^M = G[-\sin u,\ \cos u,\ m^I], \tag{15}$$

where $n^I, m^I$ are *Euclidean* unit vectors that are orthogonal. We choose the following parametrization for these unit vectors in the basis $I=[1',i]$ where $I=1'$ denotes the extra space dimension and $i=1,2,\cdots,(d-1)$ labels ordinary space:

$$n^I = \left[ -\frac{1}{\alpha}\sqrt{-2H}\ \mathbf{r}\cdot\mathbf{p},\ \left( \frac{1}{r}\mathbf{r}^i - \frac{\mathbf{r}\cdot\mathbf{p}}{\alpha}\mathbf{p}^i \right) \right], \tag{16}$$

$$m^I = \left[ \left( 1 - \frac{r\mathbf{p}^2}{\alpha} \right),\ \sqrt{-2H}\ \frac{r}{\alpha}\mathbf{p}^i \right],$$

where

$$H = \frac{\mathbf{p}^2}{2} - \frac{\alpha}{r}, \tag{17}$$

is the hydrogen atom Hamiltonian. We emphasize that this is a general solution of the constraints (4) that have taken the form $n^I n^I = m^I m^I = 1$ and $m^I n^I = 0$. Even though the solution is expressed with a particular choice of coordinates, this does not involve a gauge choice. We still have the freedom of choosing two gauge functions. One gauge choice is

$$GF = \frac{\alpha}{\sqrt{-2H}}, \tag{18}$$

and the last gauge choice is a gauge for ''time''

$$u(\tau) = \frac{1}{\alpha}\int^{\tau} (\mathbf{r}\cdot\mathbf{p}\partial_\tau\sqrt{-2H} + H\sqrt{-2H})\,d\tau'. \tag{19}$$

Note that time $\tau$ is embedded in $X^{0'}, X^0$ in a rather complicated way given through Eqs. (12), (18), (19). While $\tau$ takes values on the infinite real line, $X^{0'}, X^0$ live on a circle. Thus, the topology of the $(d+2)$-dimensional space is different as compared to the free massless particle, although both topologies are permitted as solutions of the same action (2).

To verify that this gauge choice really corresponds to the H atom, we insert it in the action (2) and verify that it reduces to the action for the H atom. Since all the constraints are explicitly solved, the $A^{ij}$ terms drop out and we get

$$S_0 = \int_0^T d\tau\, \partial_\tau X_1^M X_2^N \eta_{MN}$$

$$= \int_0^T d\tau\, GF(-\partial_\tau u + m^I \partial_\tau n^I) \tag{20}$$

$$= \int_0^T d\tau (\mathbf{p}^i \partial_\tau \mathbf{r}^i - H).$$

A total derivative $\partial_\tau(-\mathbf{r}\cdot\mathbf{p})$ has been dropped in the last line. To derive the third line we have used the gauge choices (18),(19) and

$$m^I \partial_\tau n^I = \frac{\sqrt{-2H}}{\alpha}[\mathbf{r}\cdot\mathbf{p}\partial_\tau \ln\sqrt{-2H} - \partial_\tau(\mathbf{r}\cdot\mathbf{p}) + \mathbf{p}\cdot\partial_\tau \mathbf{r}] \tag{21}$$

which follows from the $m^I, n^I$ given above.

The last form of the action (20) is the first order formalism, with the H atom Hamiltonian given in Eq. (17). This form shows that the unconstrained variables $(\mathbf{r}^i, \mathbf{p}^i)$ are the standard canonical variables. The middle line of Eq. (20) shows that the H atom in $(d-1)$ space dimensions has SO$(d)$ symmetry. The first line shows that the H atom has a dynamical symmetry SO$(d,2)$ which mixes the two timelike coordinates with the $d$ space coordinates.

Through the two explicit examples discussed in this section, we have illustrated that the Sp$(2)$ gauge covariant action is capable of describing not only the free particle but also complicated systems like the H atom, and others. The underlying reason for this is the ability to choose time as a gauge in non-unique ways because we have more than one timelike coordinate in the $(d+2)$-dimensional spacetime. For each choice of time embedded in $X^0, X^{0'}$ the corresponding canonical Hamiltonian looks different. Nevertheless these special systems are Sp$(2)$ gauge equivalent, or dual to each other.

## III. SO$(d,2)$ AND THE H ATOM

The SO$(d,2)$ symmetry generators in the H atom gauge are obtained by inserting the gauge (12)–(19) in the gauge invariant $L^{MN}$. In a Hamiltonian formalism, at $\tau=u=0$, before ordering operators, we have

$$L^{0'I} = \frac{\alpha}{\sqrt{-2H}} m^I, \quad L^{0I} = -\frac{\alpha}{\sqrt{-2H}} n^I \tag{22}$$

$$L^{0'0} = \frac{\alpha}{\sqrt{-2H}}, \quad L^{IJ} = \frac{\alpha}{\sqrt{-2H}}(n^I m^J - n^J m^I). \tag{23}$$

By inserting the forms of $n^I, m^I$ given in the previous section one can verify that the SO$(d)$ subgroup has generators $L^{IJ} = (L^{ij}, L^{1'i})$ that are interpreted as angular momentum and the Runge-Lenz vector generalized to any dimension $d$

$$L^{ij}=\mathbf{r}^i\mathbf{p}^j-\mathbf{r}^j\mathbf{p}^i,\quad L^{1'i}=\frac{\alpha}{\sqrt{-2H}}\left(\frac{1}{2}L^{ij}\mathbf{p}_j+\frac{1}{2}\mathbf{p}_jL^{ij}-\alpha\frac{\mathbf{r}^i}{r}\right). \tag{24}$$

These SO($d$) generators are already written in their quantum ordered and Hermitian form (the factor $\sqrt{-2H}$ commutes with the Runge-Lenz vector and can be written on either side). The quantum ordered forms of $L^{0'i},L^{01'}$ are simple:

$$L^{0'i}=\frac{1}{2}(r\mathbf{p}^i+\mathbf{p}^ir),\quad L^{01'}=\frac{1}{2}(\mathbf{r}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{r}). \tag{25}$$

The quantum ordered forms of the remaining generators $L^{0'1'}$ and $L^{0i}$ are more complicated since the nonlinear function $\sqrt{-2H}$ is involved. The ordering of operators must be consistent with the commutators involving $L^{0'0}$ and the quantum ordered generator $L^{1'i}$ given above. In fact the quantum ordered operators can be defined through the commutators

$$[L^{01'},L^{1'i}]=-iL^{0i},\quad [L^{0'i},L^{1'j}]=i\delta^{ij}L^{0'1'} \tag{26}$$

$$[L^{0'0},L^{0'i}]=-iL^{0i},\quad [L^{0'0},L^{01'}]=iL^{0'1'}. \tag{27}$$

Another consistency requirement on the quantum ordering is that $(L^{0'0},L^{0'1'},L^{01'})$ must form an SO(1,2) algebra. The representation of the conformal group SO($d$,2) appropriate for the H atom can then be discussed. We will not do this here, but rather we will choose another gauge below where the ordering is much simpler but yet non-trivial. However, in the present gauge, an important observation at the quantum level is that the quadratic Casimir operators of the SO($d$) and SO(2) subgroups are both related to the H atom Hamiltonian as follows (watching orders of operators):

$$C_2(\mathrm{SO}(d))=(L^{1'i})^2+\frac{1}{2}(L^{ij})^2 \tag{28}$$

$$=\frac{\alpha^2}{-2H}-\frac{1}{4}(d-2)^2, \tag{29}$$

$$(L^{0'0})^2=\frac{\alpha^2}{-2H}. \tag{30}$$

Therefore, the SO($d$,2) basis labelled by the subgroups $|\mathrm{SO}(d),\mathrm{SO}(2)\rangle$ is of special interest since the representation consists of all the quantum states of the H atom taken together in a single irreducible representation. At a fixed energy level the SO($d$) subgroup explains the degeneracies. This is analogous to the well known SO(4) symmetry in three dimensions. The generators $L^{0'1'},L^{0'i},L^{01'},L^{0i}$ mix different energy levels.

There is another construction for SO($d$,2) that is simpler for discussing the H atom at the quantum level. We will discuss the quantum theory in more detail in the other basis, since we want to also show that there is another approach to find the H atom, without going through the arguments for the choice of time at the classical level given above. In the second approach we simply choose some generator of SO($d$,2) and call it ''Hamiltonian.'' This approach is also simpler for finding the harmonic oscillator as one of the dual sectors. For the H atom to emerge it is evident from the discussion above that diagonalizing the generator $L^{00'}$ would be of interest. This will be done below.

To discuss the second approach to the H atom we choose another gauge. In the basis $M=(+',-',0,i)$, where $i=1,2,\ldots,(d-1)$, we make the three gauge choices (at $\tau=0$), $X^{+'}=0$, $P^{+'}=1$ and $P^0=0$, and then solve the three constraints. The result is[1]

$$X^M=(0,\mathbf{r}\cdot\mathbf{p},r,\mathbf{r}^i),\quad P^M=\left(1,\frac{\mathbf{p}^2}{2},0,\mathbf{p}^i\right). \tag{33}$$

The canonical operators $(\mathbf{r},\mathbf{p})$ in this gauge should not be identified with the canonical operators in the other H atom gauge (12)–(19), they are not simply related. The generators of the conformal group SO($d$,2) are $L^{MN}=X^MP^N-X^NP^M$. They are invariant under the Sp(2) gauge transformations, therefore they can be evaluated in any gauge. Inserting our gauge fixed form we obtain (recall $\eta^{+'-'}=\eta^{00}=-1$)

$$L^{-'+'}=L_{+'-'}=\frac{1}{2}(\mathbf{r}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{r}) \tag{34}$$

$$L^{0+'}=L_{0-'}=r \tag{35}$$

$$L^{0-'}=L_{0+'}=\frac{1}{2}\mathbf{p}^ir\mathbf{p}^i+\frac{a}{r} \tag{36}$$

$$L^{i+'}=L_{-'i}=\mathbf{r}^i \tag{37}$$

$$L^{i-'}=L_{+'i}=-\frac{1}{2}\mathbf{p}\cdot\mathbf{r}\,\mathbf{p}^i-\frac{1}{2}\mathbf{p}^i\mathbf{r}\cdot\mathbf{p}+\frac{1}{2}\mathbf{p}^j\mathbf{r}^i\mathbf{p}^j+b\frac{\mathbf{r}^i}{\mathbf{r}^2} \tag{38}$$

$$L^{i0}=L_{0i}=-\frac{1}{2}(r\mathbf{p}^i+\mathbf{p}^ir) \tag{39}$$

$$L^{ij}=L_{ij}=\mathbf{r}^i\mathbf{p}^j-\mathbf{r}^j\mathbf{p}^i. \tag{40}$$

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[1]The relativistic massless particle of Eq. (10) can also be described in the timelike gauge in the basis $M=(+',-',0,i)$; at $\tau=0$ we have

$$X^M=\left(1,\frac{\mathbf{r}^2}{2},0,\mathbf{r}^i\right) \tag{31}$$

$$P^M=(0,\mathbf{r}\cdot\mathbf{p},|\mathbf{p}|,\mathbf{p}^i). \tag{32}$$

The second H atom gauge (33) is related to this particle gauge by a discrete Sp(2) duality transformation that interchanges $X^M$ and $P^M$. After the Sp(2) transformation we rename $\mathbf{r}\leftrightarrow\mathbf{p}$.

The system is quantized according to the standard commutation relations

$$[\mathbf{r}^i,\mathbf{p}^j]=i\delta^{ij}, \tag{41}$$

and all operators are ordered to insure that all components of $L^{MN}$ are Hermitian. In general there are ordering ambiguities. These are denoted by the constants $a$ and $b$ that appear in $L_{+'0}$ and $L_{+'i}$. For example, consider the classical expression for $L_{+'0}=\frac{1}{2}\mathbf{p}^2 r$. There are several possible quantum orderings, all of which are Hermitian, and all of which are consistent with rotation symmetry. For example, for any parameter $\lambda$ we have a Hermitian operator ordered as $\frac{1}{2}\mathbf{p}^2 r \to \frac{1}{2} r^\lambda \mathbf{p}^i r^{1-2\lambda}\mathbf{p}^i r^\lambda$. This may be reordered to the form

$$\frac{1}{2}r^\lambda \mathbf{p}^i r^{1-2\lambda}\mathbf{p}^i r^\lambda=\frac{1}{2}\mathbf{p}^i r\mathbf{p}^i+\frac{1}{2r}\lambda(\lambda-d+2) \tag{42}$$

showing that there is an ordering ambiguity parametrized by $a$ in $L_{+'0}$. Similarly, there is an ambiguity in $L_{+'i}$ parametrized by $b$ as indicated.

The operators $L^{MN}$ should form the algebra of SO($d$,2) in the quantum theory

$$[L_{MN},L_{RS}]=i\eta_{MR}L_{NS}+i\eta_{NS}L_{MR}-i\eta_{NR}L_{MS}-i\eta_{MS}L_{NR}. \tag{43}$$

By using the basic commutation relations among $(\mathbf{r},\mathbf{p})$ one can check that the SO($d$,2) commutation relations are indeed satisfied for any $a$, and that $b$ is fixed by demanding correct closure for the commutator

$$[L_{0+'},L_{0i}]=-iL_{+'i}, \quad \to b=-a-\frac{d-2}{4}. \tag{44}$$

The remaining parameter $a$ will be fixed by the Sp(2) gauge invariance, not by the SO($d$,2) algebra, as will be discussed below.

It is evident that the operators $\mathbf{L}_{ij}$ form the algebra of the rotation subgroup SO($d-1$). Its quadratic Casimir operator is given by

$$\mathbf{L}^2\equiv\frac{1}{2}L_{ij}L^{ij}=\mathbf{r}^j\mathbf{p}^2\mathbf{r}^j-\mathbf{r}\cdot\mathbf{p}\;\mathbf{p}\cdot\mathbf{r}. \tag{45}$$

Similarly, the following three operators form a SO(1,2) subalgebra

$$L_{+'-'}\equiv J_2, \quad L_{+'0}\equiv\frac{1}{2}(J_0+J_1), \quad L_{-'0}\equiv J_0-J_1, \tag{46}$$

$$J_2=\frac{1}{2}(\mathbf{r}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{r}), \quad (J_0+J_1)=\mathbf{p}^i r\mathbf{p}^i+\frac{2a}{r},$$

$$J_0-J_1=r. \tag{47}$$

For any anomaly coefficient $a$ they close correctly

$$[J_0,J_1]=iJ_2, \quad [J_0,J_2]=-iJ_1, \quad [J_1,J_2]=-iJ_0. \tag{48}$$

The compact generator $J_0=L_{0'0}$ is given in terms of the canonical operators as

$$J_0=L_{+'0}+\frac{1}{2}L_{-'0}=\frac{1}{2}\mathbf{p}^i r\mathbf{p}^i+\frac{a}{r}+\frac{r}{2}. \tag{49}$$

The quadratic Casimir operator for this subalgebra is

$$j(j+1)=J_0^2-J_1^2-J_2^2 \tag{50}$$

$$=L_{+'0}L_{-'0}+L_{-'0}L_{+'0}-(L_{+'-'})^2 \tag{51}$$

$$=\mathbf{L}^2+\frac{1}{4}(d-1)^2-\frac{1}{2}(d-1)+2a \tag{52}$$

$$=\mathbf{L}^2+\frac{1}{4}(d-2)^2-\frac{1}{4}+2a. \tag{53}$$

We see that the quadratic Casimir operators of the SO(1,2) subalgebra and that of the rotation subgroup SO($d-1$) are related to each other in this representation of SO($d$,2). The overall quadratic Casimir operator for SO($d$,2) may now be evaluated

$$C_2=\frac{1}{2}L_{MN}L^{MN} \tag{54}$$

$$=-(L_{+'-'})^2+L_{+'0}L_{-'0}+L_{-'0}L_{+'0} \tag{55}$$

$$-L_{+'i}L_{-'i}-L_{-'i}L_{+'i}-L_{0i}L_{0i}+\frac{1}{2}L_{ij}L^{ij} \tag{56}$$

$$=\mathbf{L}^2+\frac{1}{4}(d-2)^2-\frac{1}{4}+2a \tag{57}$$

$$-2\mathbf{L}^2-\frac{1}{2}(d-2)^2+2a-\frac{5}{4}+\mathbf{L}^2 \tag{58}$$

$$=-\frac{d^2}{4}+d-\frac{3}{2}+4a \tag{59}$$

$$=1-\frac{d^2}{4} \tag{60}$$

In the last line we required a definite value for the SO($d$,2) Casimir operator, $C_2=1-d^2/4$, because this is equivalent to requiring Sp(2) gauge singlets, thus insuring that the states are physical, as in Eq. (9). The last step fixes the values of $a$ and $b$ uniquely in the gauge invariant sector

$$a=\frac{1}{8}(5-2d), \quad b=-\frac{1}{8}. \tag{61}$$

These values correspond to an interesting resolution of the quantum ordering ambiguity (42) of the operators in $L_{+'0}$

$$L_{+'0}=\frac{1}{2}\mathbf{p}^i r\mathbf{p}^i+\frac{1}{8r}(5-2d)=r^{1/2}\left[\frac{1}{2}\mathbf{p}^2\right]r^{1/2}.$$

A basis for the quantum theory is chosen to diagonalize the Hamiltonian. As explained earlier, since we have two timelike dimensions the choice of ''time'' corresponds to a choice of Hamiltonian as a linear combination of the generators of SO($d$,2). One such choice is dual to another via Sp(2) gauge transformations. We now make the following choice for ''Hamiltonian'' $h=J_0=L^{0'0}$ which is consistent with Eq. (30):

$$h=J_0=L_{+'0}+\frac{1}{2}L_{-'0}=r^{1/2}\left[\frac{1}{2}\mathbf{p}^2+\frac{1}{2}\right]r^{1/2}. \tag{62}$$

Since this is a generator of the SO(1,2) algebra it is diagonalized on the usual SO(1,2) basis $|jm\rangle$ where $m$ is the quantized eigenvalue of the compact generator $J_0$. Evidently the operator $h$ is positive, therefore $m$ can only be positive. This is possible only in the positive unitary discrete series representation of SO(1,2) and the spectrum of $m$ must be

$$m=j+1+n_r,\quad n_r=0,1,2,\ldots, \tag{63}$$

where, as we will see shortly, the integer $n_r$ will play the role of the radial quantum number. Let us now show the relation to the Hydrogen atom Hamiltonian. Applying $h$ on these states we have

$$r^{1/2}\left[\frac{1}{2}\mathbf{p}^2+\frac{1}{2}\right]r^{1/2}|jm\rangle=m|jm\rangle. \tag{64}$$

Multiplying it with the operator $r^{-1/2}$ from the left, this equation is rewritten as

$$\left[\frac{1}{2}\mathbf{p}^2+\frac{1}{2}-\frac{m}{r}\right](r^{1/2}|jm\rangle)^{1/2}=0. \tag{65}$$

We now recognize that the states $|\psi_m\rangle=(r^{1/2}|jm\rangle)$ are eigenstates of the hydrogen atom Hamiltonian. Actually this is a rescaled form of the standard Hamiltonian equation written in terms of dimensionful coordinates and momenta $\tilde{\mathbf{r}},\tilde{\mathbf{p}}$

$$\left[\frac{\tilde{\mathbf{p}}^2}{2M}-\frac{\alpha}{\tilde{r}}\right]|\psi_m\rangle=E_m|\psi_m\rangle. \tag{66}$$

The following rescaling relates the two equations and gives the energy of the atom in terms of the quantum number $m=j+1+n_r$

$$\tilde{\mathbf{p}}=\frac{M\alpha}{m}\mathbf{p},\quad \tilde{r}=\frac{m}{M\alpha}r,\quad E_m=-\frac{M\alpha^2}{2}(j+1+n_r)^{-2}. \tag{67}$$

We now give an argument to compute $j$. Since SO(1,2) commutes with the SO($d-1$) rotation generators $L_{ij}$, the SO(1,2) basis can be taken to be simultaneously diagonal with the SO($d-1$) basis

$$|jml\rangle\sim|\mathrm{SO}(1,2),\mathrm{SO}(d-1)\rangle, \tag{68}$$

where $l$ stands for a collection of SO($d-1$) quantum numbers that we are about to explain. $L_{ij}$ is orbital angular momentum, and its basis must be constructed by taking direct products of the fundamental unit vector $\mathbf{\Omega}=\mathbf{r}/r$. The only irreducible representations that can be built in this way are the completely symmetric traceless tensors of SO($d-1$). Consider a tensor of rank $l$, i.e. $T_{i_1 i_2\cdots i_l}(\mathbf{\Omega})$ which is symmetric and traceless with indices in ($d-1$) dimensions. These provide a complete set of labels for the states $|jml\rangle$ and are the analogs of the spherical harmonics in 3 dimensions. The number of independent components of the tensor in ($d-1$) space dimensions is

$$N_l(d-1)=\frac{(l+d-4)!}{(d-3)!\,l!}(2l+d-3) \tag{69}$$

This reduces to $(2l+1)$ for $d-1=3$, in agreement with spherical harmonics. The value of the quadratic Casimir operator of SO($d-1$) for this representation is

$$\mathbf{L}^2|jml\rangle=l(l+d-3)|jml\rangle,\quad l=0,1,2,\ldots. \tag{70}$$

This reduces to $l(l+1)$ for $(d-1)=3$ in agreement with angular momentum in three dimensions. Now, we recall that we have established a relation between the quadratic Casimir operators of SO(1,2) and SO($d-1$) in Eq. (50). Using this we find

$$j(j+1)=l(l+d-3)+\frac{1}{4}(d-2)(d-4) \tag{71}$$

$$j=l+\frac{1}{2}(d-4). \tag{72}$$

Therefore, we have computed the full spectrum. Applying $h$ on these states $(r^{1/2}|jml\rangle)$ we now have

$$m=\frac{1}{2}(d-2)+l+n_r. \tag{73}$$

We may combine the orbital and radial quantum numbers into the total quantum number as done for the conventional H atom

$$l+n_r+1=n \tag{74}$$

and then write $|jml\rangle=|nl\rangle$ since the complete spectrum depends only on the total quantum number $n$

$$E_n=-\frac{M\alpha^2}{2}\left(\frac{1}{2}(d-4)+n\right)^{-2} \tag{75}$$

$$n=1,2,3,\ldots \tag{76}$$

$$l=0,1,\ldots,(n-1) \tag{77}$$

in agreement with the conventional labeling of the hydrogen atom states. We have computed its spectrum in any number of space dimensions ($d-1$) and found that there is a dependence on $d$ in the spectrum: the principal quantum number $n$

that appears in the denominator is shifted by a half integer $\frac{1}{2}(d-4)$. This shift disappears when $(d-1)=3$, which agrees with the standard result for the hydrogen atom in three space dimensions.

We have verified this group theoretical solution by solving directly the Schrödinger equation for the $1/r$ potential in $D$ space dimensions. The full wave function is $\psi(\mathbf{r})=r^{(D-1)/2}f(r)T_{i_1 i_2 \cdots i_l}(\mathbf{\Omega})$, and the radial equation for any rotationally invariant potential takes the form

$$\left(-\partial_r^2+\frac{1}{r^2}l_D(l_D+1)+v(r)-\varepsilon\right)f(r)=0, \tag{78}$$

where $l_D=l+(D-3)/2$ (try for example the two or three dimensional cases $D=2,3$). The solution of the radial equation for the $1/r$ potential proceeds just like the standard three dimensional case, except for replacing $l_D$ instead of $l$. To compare to our group theoretical results above we replace $D=d-1$.

We have shown that all the states of the H atom in $(d-1)$ dimensions form *a single irreducible representation* of the group SO$(d,2)$ with a quadratic Casimir operator $C_2=1-d^2/4$. This group includes a compact group SO$(d)$ which commutes with the generator $J_0$. Therefore the maximal compact symmetry that commutes with the Hamiltonian is the rotation group in one more dimension, and the energy eigenstates remain degenerate under its transformations. This symmetry is the generalization of the SO(4) symmetry of the H atom in 3 dimensions. Thus, the two dimensional H atom has an SO(3) symmetry, the 3 dimensional H atom has an SO(4) symmetry, the four dimensional H atom has an SO(5) symmetry, and so on. As a consequence of this symmetry the energy depends only on the total quantum number $n$ and all the states with different $l=0,1,\ldots,n-1$ are degenerate. We can compute this degeneracy $D_n$ at a fixed value of $n$ and find

$$D_n(d-1)=\sum_{l=0}^{n-1} N_l(d-1)=\frac{(n+d-3)!}{(d-2)!n!}(2n+d-2). \tag{79}$$

By comparing to Eq. (69) we see that it equals the number of components of a traceless symmetric tensor of rank $n$ in one higher dimension $D_n(d-1)=N_n(d)$. This is a result of the SO$(d)$ symmetry. The computed degeneracy confirms that indeed the states at a given energy level form a complete multiplet of SO$(d)$. The multiplet at a fixed energy level is identified as the completely symmetric traceless tensor in one more space dimension.

It has been known for a long time that the three dimensional hydrogen atom in 3 dimensions has a spectrum that can be described as a representation of the conformal group SO(4,2) [6]. We have generalized this result to any dimension. In comparing the details of our construction to previous work we find that the details of our construction are somewhat different. Note especially the issues of ordering of operators at the quantum level. For us this was crucial from the point of view of Sp(2) gauge invariance and the physical state conditions.

## IV. QUANTUM PARTICLE AND HARMONIC OSCILLATOR

Consider the free particle gauge (10). The generators of the conformal group SO$(d,2)$ are obtained at the classical level by inserting this gauge into the gauge invariant form (5). However, in the quantum theory operator ordering must be taken into account to insure that all the generators are Hermitian and that the algebra of SO$(d,2)$ closes correctly. In [1] it was shown that for this gauge the quantum generators are (at $\tau=0$)

$$L^{ij}=\vec{x}^i\vec{p}^j-\vec{x}^j\vec{p}^i \tag{80}$$

$$L^{+i}=-\vec{x}^i p^+,\quad L^{-i}=x^-\vec{p}^i-\frac{\vec{p}^j\vec{x}^i\vec{p}^j}{2p^+} \tag{81}$$

$$L^{+-}=-\frac{1}{2}(x^-p^++p^+x^-),\quad L^{-'+}=\frac{1}{2}\vec{x}^2p^+ \tag{82}$$

$$L^{+'+}=p^+,\quad L^{+'-}=\frac{\vec{p}^2}{2p^+},\quad L^{+'i}=\vec{p}^i \tag{83}$$

$$L^{+'-'}=\frac{1}{2}(\vec{x}\cdot\vec{p}+\vec{p}\cdot\vec{x}-x^-p^+-p^+x^-) \tag{84}$$

$$L^{-'-}=\begin{bmatrix}\frac{1}{8p^+}(\vec{x}^2\vec{p}^2+\vec{p}^2\vec{x}^2-2\alpha)\\ -\frac{x^-}{2}(\vec{x}\cdot\vec{p}+\vec{p}\cdot\vec{x})+x^-p^+x^-\end{bmatrix} \tag{85}$$

$$L^{-'i}=\begin{bmatrix}\frac{1}{2}\vec{x}^j\vec{p}^i\vec{x}^j-\frac{1}{2}\vec{x}\cdot\vec{p}\vec{x}^i\\ -\frac{1}{2}\vec{x}^i\vec{p}\cdot\vec{x}+\frac{1}{2}\mathbf{x}^i(x^-p^++p^+x^-)\end{bmatrix}. \tag{86}$$

Hermiticity fixes almost all orders of operators, but the remaining ordering ambiguity introduces the parameter $\alpha$ in $L^{-'-}$. This is fixed to $\alpha=-1$ by the commutator $[L^{-'i},L^{-j}]=i\delta^{ij}L^{-'-}$. With this value of $\alpha$ the quadratic Casimir operator can then be verified to be precisely $C_2=1-d^2/4$ in agreement with covariant quantization.

The free particle Hilbert space is defined by diagonalizing the operators $p^+,\vec{p}$, which is the same as diagonalizing the commuting generators $L^{+'+},L^{+'i}$. The momentum eigenstates $|p^+,\vec{p}\rangle$ form a complete Hilbert space. On this space the free particle Hamiltonian, which is another generator of the conformal group $L^{+'-}=\vec{p}^2/2p^+$, is diagonal. These positive norm states provide a basis for a unitary representation of the conformal group SO$(d,2)$ through the representation of the generators given above. The Casimir eigenvalues for the representation are fixed as we have already discussed.

We now show the relation of this representation to the harmonic oscillator. Instead of diagonalizing $L^{+'+}, L^{+'-}, L^{+'i}$ we will choose a basis for SO$(d,2)$ in which the following operators that commute with each other are simultaneously diagonal

$$L^{ij}, L^{+'+}, (L^{+'-}+L^{-'+}). \tag{87}$$

More accurately, only an appropriate commuting subset of orbital angular momentum operators $L^{ij}$ will be simultaneously diagonal. These operators correspond to the SO$(d-2)$ orbital angular momentum $L^{ij}=\vec{x}^i\vec{p}^j-\vec{x}^j\vec{p}^i$, the lightcone momentum $L^{+'+}=p^+$ and the Hamiltonian of the harmonic oscillator in $(d-2)$ dimensions

$$H \equiv L^{+'-}+L^{-'+}=\frac{\vec{p}^2}{2p^+}+\frac{p^+\vec{x}^2}{2}, \tag{88}$$

where the lightcone momentum $L^{+'+}=p^+$ in another dimension plays the role of mass. This choice of Hamiltonian corresponds to another choice for ''time,'' as compared to the choice for ''time'' for the free particle. The spectrum of this Hamiltonian is well known from the study of the harmonic oscillator in $(d-2)$ dimensions

$$E_n=n+\frac{1}{2}(d-2), \quad n=0,1,2,\ldots. \tag{89}$$

We want to show how this quantum number $n$ and the angular momentum quantum numbers $l$ etc. are related to the representation space of SO$(d,2)$. To do so, consider the subgroups SO$(d-2)\otimes$SO$(2,2)$ and label the representation space of SO$(d,2)$ by the representations of these subgroups. Recall that SO$(2,2)=$SL$(2,R)_L\otimes$SL$(2,R)_R$. We will show that the Hamiltonian of the harmonic oscillator is the compact generator of SL$(2,R)_R$ and that the energy spectrum of the harmonic oscillator is classified as towers of states corresponding to the positive discrete series representation $|j_R, m_R\rangle$ of this SL$(2,R)_R$. For every SO$(d-2)$ angular momentum quantum number $l$, we will find a relation between $j_R, j_L$ and $l$.

From the general commutation rules for SO$(d,2)$ one can see that the generators of SO$(2,2)=$SL$(2,R)_L\otimes$SL$(2,R)_R$ are given by

$$G_2^L=\frac{1}{2}(L_{+'-'}+L_{+-}), \quad G_0^L\pm G_1^L=L_{\pm'\pm}, \tag{90}$$

$$G_2^R=\frac{1}{2}(L_{+'-'}-L_{+-}), \quad G_0^R\pm G_1^R=L_{\pm'\mp}. \tag{91}$$

These satisfy the commutation rules $[G_a^L, G_b^R]=0$ and

$$[G_0^{L,R}, G_1^{L,R}]=iG_2^{L,R},\ [G_0^{L,R}, G_2^{L,R}]=-iG_1^{L,R}, \tag{92}$$

$$[G_1^{L,R}, G_2^{L,R}]=-iG_0^{L,R}, \tag{93}$$

In the present gauge we have the construction

$$G_2^L=\frac{1}{4}(\vec{x}\cdot\vec{p}+\vec{p}\cdot\vec{x})-\frac{1}{2}(x^-p^++p^+x^-) \tag{94}$$

$$G_0^L+G_1^L=p^+, \tag{95}$$

$$G_0^L-G_1^L=\left[\begin{array}{l}\frac{1}{8p^+}(\vec{x}^2\vec{p}^2+\vec{p}^2\vec{x}^2-2\alpha)\\ -\frac{x^-}{2}(\vec{x}\cdot\vec{p}+\vec{p}\cdot\vec{x})+x^-p^+x^-\end{array}\right] \tag{96}$$

and

$$G_2^R=\frac{1}{4}(\vec{x}\cdot\vec{p}+\vec{p}\cdot\vec{x}), \tag{97}$$

$$G_0^R+G_1^R=\frac{\vec{p}^2}{2p^+}, \tag{98}$$

$$G_0^R-G_1^R=\frac{1}{2}\vec{x}^2p^+. \tag{99}$$

Now we see that the Hamiltonian of the harmonic oscillator is the compact generator of SL$(2,R)$

$$H=2G_{0R}=\frac{\vec{p}^2}{2p^+}+\frac{p^+\vec{x}^2}{2}. \tag{100}$$

In our special representation the quadratic Casimir operators of these subgroups are related to each other as follows. Defining $j_L(j_L+1)=G_{0L}^2-G_{1L}^2-G_{2L}^2$ and $j_R(j_R+1)=G_{0R}^2-G_{1R}^2-G_{2R}^2$ we find (for $\alpha=-1$ which corresponds to $C_2=1-d^2/4$ as seen above)

$$j_R(j_R+1)=j_L(j_L+1)=\frac{1}{4}L^2+\frac{1}{16}(d-2)(d-6), \tag{101}$$

where $L^2$ is the quadratic Casimir of SO$(d-2)$ given by the quantum ordered form

$$L^2\equiv\tfrac{1}{2}L_{ij}L^{ij}=\vec{p}^i\vec{x}^2\vec{p}^i-\vec{p}\cdot\vec{x}\vec{x}\cdot\vec{p}. \tag{102}$$

The unitary representation of SL$(2,R)_R$ is labelled by $|j_Rm_R\rangle$ where $m_R$ is the eigenvalue of $G_{0R}$. Since $G_{0R}$ is a positive operator in our construction, $m_R$ can take only positive values. This is possible only for the positive discrete series representation, and according to SL$(2,R)$ representation theory it is given by

$$m_R=j_R+1+n_r, \quad n_r=0,1,2,3,\ldots. \tag{103}$$

We will see that the integer $n_r\geqslant 0$ will find an interpretation as the radial quantum number of the harmonic oscillator. Next we need to find the allowed values of $j_R$. We saw in Eq. (101) that $j_R$ is related to angular momentum, therefore we must find the allowed values of orbital angular momentum SO$(d-2)$ (102). As already explained in the previous section the allowed states for orbital angular momentum correspond to tensors constructed from the unit vector $\vec{\Omega} = \vec{x}/|\vec{x}|$. The eigenvalues of $L^2$ and the number of states $N_l(d-2)$ are obtained from Eqs. (69),(70) by replacing $(d-1)$ by $(d-2)$

$$N_l(d-2) = \frac{(l+d-5)!}{(d-4)!l!}(2l+d-4) \tag{104}$$

$$L^2|l\rangle = l(l+d-4)|l\rangle, \quad l=0,1,2,\ldots .$$

Combining Eq. (101) and Eq. (104) yields the allowed values of both $j_L$ and $j_R$

$$j_R = j_L = \frac{1}{2}l + \frac{1}{4}d - \frac{3}{2}.$$

Inserting this result in Eq. (103) one finds

$$m_R = \frac{1}{2}l + \frac{1}{4}(d-2) + n_r. \tag{105}$$

Now we can compare to the energy spectrum of the harmonic oscillator (89) by using the relation (100). We see that we must identify $l+2n_r=n$ where $n$ is the total quantum number and $n_r$ is the radial quantum number in the usual interpretation of the solutions of the Schrödinger equation. Using the total quantum number $n$ instead of the radial quantum number $n_r$ we summarize our results

$$E_n = n + \frac{1}{2}(d-2), \quad n=0,1,2,\ldots \tag{106}$$

$$l = n,(n-2),(n-4),\ldots,(0 \ \ or \ \ 1) \tag{107}$$

$$m_R = \frac{1}{2}n + \frac{1}{4}(d-2) \tag{108}$$

$$j_R = j_L = \frac{1}{2}l + \frac{1}{4}d - \frac{3}{2}. \tag{109}$$

At a fixed energy level $n$ it is well known that the states with different values of $l$ belong together in SU$(d-2)$ multiplet corresponding to the single row Young tableau with $n$ boxes. Instead, here these states are rearranged vertically as multiplets at the same value of $l$ with different values of the energy $n$. Thus, at each $l$ there is an SL$(2,\mathrm{R})_R$ positive discrete series multiplet $|j_R,m_R\rangle$ which is a vertical multiplet of different energy levels.

In summary we have found the following labelling of our special representation by using the harmonic oscillator basis

$$|\mathrm{SO}(d-2);\mathrm{SL}(2,R)_L;\mathrm{SL}(2,R)_R\rangle \tag{110}$$

$$|l;j_L(l)p^+;j_R(l)m_R(n)\rangle.$$

The SL$(2,R)_L$ representation is labeled by the eigenvalues of the generator $G_0^L + G_1^L = p^+$ which plays the role of mass for the harmonic oscillator. All the levels taken together make up a single unitary representation of SO$(d,2)$.

We have seen that this representation of SO$(d,2)$ is the same as the free massless particle representation since it has the same Casimir eigenvalues. Hence the free massless relativistic particle and the harmonic oscillator with its mass defined as the lightcone momentum of the particle are dual to each other in our model.

## V. SUMMARY

There are two aspects of the model worth emphasizing as potentially more general than the model itself. One is duality and the other is a larger covariant space with two timelike dimensions. The concepts of Sp(2) duality and two times are inextricably connected to each other in our model.

As examples of dualities, we have shown that the H atom, the free particle and harmonic oscillator are dual to each other. These are some of the physical systems that can be described by this simple model. A complete classification of all of its dual sectors has not been obtained at this stage. At the quantum level the dual sectors are all described by the same unitary representation of SO$(d,2)$ with fixed Casimir eigenvalues. This representation of SO$(d,2)$ is realized in terms of different sets of unconstrained canonical variables. In each case a subset of the SO$(d,2)$ generators is simultaneously diagonalized and a particular combination of the generators is interpreted as the Hamiltonian. Each choice of Hamiltonian corresponds to a fixed gauge of the duality symmetry in which ''time'' is identified as a particular combination of the spacetime coordinates which includes two times $X^{0'},X^0$. The topology of the $(d+2)$ dimensional spacetime is not the same for each fixed gauge, but each such topology is an allowed solution of the constraint equations and equations of motion that follow from a single action $S_0$. ''Large'' gauge transformations map the gauge fixed solutions to each other. This is similar to M-theory dualities that also map physical systems that live in spaces of different topologies.

Besides dualities, the model shows that familiar physical systems can be viewed as embedded in a spacetime with two timelike dimensions. This provides an example for how it is possible to have more than one timelike dimension and yet describe realistic physics. Furthermore the model shows that a larger spacetime unifies these physical systems under the same umbrella.

The duality symmetry in our model is morally similar to the dualities encountered in M theory [7]. However, in M theory the analog of the action principle that gives rise to dualities remains to be discovered. It is hoped that our model may provide some new insight into the duality symmetries in M theory, and into the signals of more than one time or higher dimensions already noticed from different directions [8–19].

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