Gauged duality, conformal symmetry, and spacetime with two times

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We construct a duality between several simple physical systems by showing that they are different aspects of the same quantum theory. Examples include the free relativistic massless particle and the hydrogen atom in any number of dimensions. The key is the gauging of the Sp(2) duality symmetry that treats position and momentum (x,p) as a doublet in phase space. As a consequence of the gauging, the Minkowski spacetime vectors x^μ, p^μ get enlarged by one additional spacelike and one additional timelike dimension to (x^M, p^M). A manifest global symmetry SO$(d,2)$ rotates (x^M, p^M)-like $(d+2)$-dimensional vectors. The SO$(d,2)$ symmetry of the parent theory may be interpreted as the familiar conformal symmetry of quantum field theory in Minkowski spacetime in one gauge or as the dynamical symmetry of a totally different physical system in another gauge. Thanks to the gauge symmetry, the theory permits various choices of ''time'' which correspond to different looking Hamiltonians, while avoiding ghosts. Thus we demonstrate that there is a physical role for a spacetime with two times when taken together with a gauged duality symmetry that produces appropriate constraints. [S0556-2821(98)04016-8]

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I. INTRODUCTION

The purpose of this paper is to introduce some new points of view on duality as a gauge symmetry and to connect duality to the concept of a spacetime with two timelike dimensions. This is an attempt at finding a physical role for the idea that there may be more than one timelike dimension to describe our universe at the fundamental level. We will show that certain familiar physical systems, such as the free massless relativistic particle, hydrogen atom, harmonic oscillator, and others, do fit such a concept, as reported in this paper and in a companion paper [1]. We will show that these and other apparently different physical systems correspond to the same quantum Hilbert space characterized by a *unique* unitary representation of the conformal group SO$(d,2)$. We will argue that the presence of conformal symmetry or dynamical symmetry in these special cases *is* evidence for the presence of two timelike coordinates. The physics looks different because the choice of ''time'' is not unique and hence the Hamiltonians look different, although they describe the same parent system for which we present an action. These special physical systems are related to each other by a duality that is a gauge symmetry. Thanks to the gauge symmetry ghosts are eliminated from the two-time Hilbert space.

Clues for two or more timelike dimensions have been emerging from various points of view, including the brane scan [2], the structure of extended supersymmetry of p-branes [3], extensions of M theory [4] to F theory [5] and S theory [6,7], (1,2) strings [8], 12D super Yang-Mills and supergravity theories in backgrounds of constant lightlike vectors [9], and finally the discovery of models of multisuperparticles that are fully covariant in (10,2) and (11,3) dimensions [10–13].

Two or more timelike dimensions are possible only with appropriate gauge symmetry and constraints that reduce the theory to an effective theory with a single timelike dimension and no ghosts. The gauged Sp(2) duality symmetry suggested here is an evolution of the local bosonic symmetry introduced in [10–12] for the same purpose. The difference is that we apply the concept to the phase space doublet (X^M, P^M) for a single particle rather than to a multiplet of the positions of several particles $(X_1^M, X_2^M, \ldots)$. We suggest an action principle in phase space, including invariant interactions with background fields, with and without supersymmetry.

We have suggestively named our local symplectic symmetry Sp(2) ''duality'' because we see signs that our duality is related to the generalized concept of electric-magnetic duality in super Yang-Mills theories and M theory. However, this connection remains to be established by further detailed study.

II. GAUGING DUALITY

The quantization rules of quantum mechanics are symmetric under the interchange of coordinates and momenta. This is known as the symplectic symmetry Sp(2) that transforms (x,p) as a doublet. Maxwell's equations for electricity and magnetism are symmetric under the interchange of electricity and magnetism in the absence of sources. The electric and magnetic fields are generalized coordinates and momenta. In the presence of particles with quantized electric and magnetic charges the symmetry is a discrete version of Sp(2). This symmetry, known as ''electric-magnetic duality,'' is apparently broken in our part of the universe by the absence of magnetic monopoles and dyons. The idea of electric-magnetic duality symmetry has been generalized in recent nonperturbative studies of supersymmetric field theory [14] and string theory [15], which are now believed to be only some aspect of a larger, duality-invariant, mysterious theory (M theory, F theory, S theory, U theory, etc.). In the context of the mysterious theory, ''duality,'' which is a much

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larger symmetry than Sp(2), but containing it, is believed to be a gauge symmetry.

In this paper we study an elementary system with local continuous Sp(2) duality symmetry. We start by reformulating the world line description of the standard free massless relativistic point particle by gauging the Sp(2) duality symmetry. What we find in doing so is a more general theory capable of describing not only the free particle but other physical systems dual to it, such as the hydrogen atom, harmonic oscillator, and others.

To remove the distinction between x and p we will rename them $X_1^M \equiv X^M$ and $X_2^M \equiv P^M$ and define the doublet $X_i^M = (X_1^M, X_2^M)$. The local Sp(2) acts as follows:

$$\delta_\omega X_i^M(\tau) = \varepsilon_{ik}\omega^{kl}(\tau) X_l^M(\tau). \tag{1}$$

Here $\omega^{ij}(\tau) = \omega^{ji}(\tau)$ is a symmetric matrix containing three local parameters, and ε_{ij} is the Levi-Civitá symbol that is invariant under Sp(2,R) and serves to raise or lower indices. We also introduce an Sp(2,R) gauge field $A^{ij}(\tau)$ which is symmetric in (ij) which transforms in the standard way:

$$\delta_\omega A^{ij} = \partial_\tau \omega^{ij} + \omega^{ik}\varepsilon_{kl} A^{lj} + \omega^{jk}\varepsilon_{kl} A^{il}. \tag{2}$$

The covariant derivative is

$$D_\tau X_i^M = \partial_\tau X_i^M - \varepsilon_{ik} A^{kl} X_l^M. \tag{3}$$

An action that is invariant under this gauge symmetry is

$$\begin{aligned} S_0 &= \frac{1}{2}\int_0^T d\tau (D_\tau X_i^M)\varepsilon^{ij} X_j^N \eta_{MN} \\ &= \int_0^T d\tau \left(\partial_\tau X_1^M X_2^N - \frac{1}{2} A^{ij} X_i^M X_j^N \right) \eta_{MN}. \end{aligned} \tag{4}$$

Here η_{MN} is a flat metric in $d+2$ dimensions and a total derivative has been dropped in rewriting the first term. The signature of the metric η_{MN} is not specified at this stage, but we will see that it will be *imposed* on us that it must have signature for two timelike dimensions. From the second form of the action one may identify the canonical conjugates as $X_1^M = X^M$ and $\partial S / \partial \dot{X}_1^M = X_2^M = P^M$, so that the action is consistent with the idea that (X_1^M, X_2^M) is the doublet (X^M, P^M) rather than describing two particles.

If instead of the full Sp(2) group we had gauged a triangular Abelian subgroup containing only $\omega^{22}(\tau)$, and kept only the gauge potential $A^{22}(\tau)$, then the resulting action would have been the free massless particle action in the first order formalism, with $\eta_{\mu\nu}$ the standard Minkowski metric. Thus ω^{22} is closely related to τ reparametrization invariance, but ω^{12}, ω^{11} are new local symmetry parameters that permit the removal of redundant gauge degrees of freedom. In the presence of the gauge degrees of freedom we are able to see the structure of duality and the role it plays in exhibiting higher symmetries in higher dimensions.

In addition to the local Sp(2,R) symmetry there is a manifest global symmetry SO(d,2) [assuming signature (d,2)] acting on the space time X_i^M with d spacelike and two timelike dimensions labelled by the index M. This symmetry contains the d-dimensional Poincaré symmetry ISO($d-1$,1) as a subgroup, but there is no translation symmetry in $d+2$ dimensions. Using Noëther's theorem one finds the generators of the symmetry SO(d,2):

$$L^{MN} = \varepsilon^{ij} X_i^M X_j^N = X^M P^N - X^N P^M. \tag{5}$$

They are manifestly *gauge invariant* under the local Sp(2,R) transformations.

To obtain spacetime supersymmetry in target space we use the Neveu-Schwarz approach but only for zero modes. To do so, phase space is enlarged by the addition of fermionic degrees of freedom $\psi^M(\tau)$ which are their own canonical conjugates (i.e., they form a Clifford algebra when quantized). The Sp(2) doublet is enlarged to an OSp(1/2) triplet (ψ^M, X_1^M, X_2^M) and the supergroup OSp(1/2) is gauged by adding two fermionic gauge potentials F_i in addition to the three bosonic gauge potentials A^{ij}. The action is the direct generalization of Eq. (4) to a gauge theory based on OSp(1/2). In $d+2=4,5,8,12$ dimensions, in a particular gauge, the degrees of freedom reduce correctly to the free $N=1$ *spacetime* supersymmetric particle in Minkowski space in dimensions $d=2,3,6,10$. This scheme can be enlarged to N supersymmetries by gauging OSp(N/2). Like the bosonic case, the supersymmetric case also has multiple physical sectors as seen from the point of view of various gauge choices for ''time.'' The supersymmetric case will be discussed in more detail in another paper [16].

Interactions with gravitational fields $G_{MN}(X_1, X_2)$ and gauge fields $A_j^N(X_1, X_2)$ in a way that respects the Sp(2) duality symmetry are possible (of course, also in the supersymmetric case):

$$S_{G,A} = \frac{1}{2}\int_0^T d\tau \left[\begin{array}{l} (D_\tau X_i^M)\varepsilon^{ij} X_j^N G_{MN}(X_1, X_2) \\ \quad + (D_\tau X_i^M)\varepsilon^{ij} A_{jN}(X_1, X_2) \end{array} \right]. \tag{6}$$

G_{MN} is a scalar under Sp(2) and a symmetric traceless tensor in $d+2$ dimensions. Similarly A_j^M is a doublet under Sp(2) and a vector in $d+2$ dimensions. It is tempting to suggest that the Sp(2) doublet of electromagnetic fields A_j^M is related to the electric-magnetic dual potentials of Maxwell's theory and its Yang-Mills generalizations. For the local invariance to hold, there must be restrictions on the functional forms of both $G_{MN}(X_1, X_2)$ and $A_j^N(X_1, X_2)$ since the arguments (X_1, X_2) also transform under Sp(2). These amount to a set of differential equations that restrict the functional forms of $G_{MN}(X_1, X_2)$ and $A_j^N(X_1, X_2)$. One automatic solution is to take any functions $G_{MN}(L)$, $A_j^N(L)$ where L^{MN} is the gauge invariant combination of (X_1, X_2) given in Eq. (5). In the presence of the background fields the global symmetry SO(d,2) is replaced by the Killing symmetries of the background fields. We see that, for consistency with the local symmetry, gravity and gauge interactions are more conveniently expressed in terms of bilocal fields $G_{MN}(X_1, X_2)$ and $A_j^N(X_1, X_2)$ in $d+2$ dimensions. Bilocal fields were advocated in [6] as a means of extending supergravity and super Yang-Mills theory to (10,2) dimensions based on clues from the Bogomol'nyi-Prasad-Sommerfield (BPS) solutions of extended supersymmetry.

We refer to the forms of the actions $S_0, S_{G,A}$ above as the first order formalism. Although not necessary, a second order formalism is obtained if X_2^M is integrated out in the path integral (or eliminated semiclassically through one of the equations of motion). Eliminating X_2^M is not easy for the interacting case, but for the free action S_0 the result is

$$S_0 = \int d\tau \left[\frac{1}{2A^{22}} (\partial_\tau X^M - A^{12} X^M)^2 - \frac{A^{11}}{2} X \cdot X \right]. \tag{7}$$

This form of the action may be thought of as "conformal gravity" on the world line, with the conformal group $SO(1,2) = Sp(2)$.

In this paper we will mainly analyze the simplest case S_0. The configuration space version of S_0, Eq. (7) was previously obtained with different reasoning and motivation [17],[1] and without the concept of duality. Our solutions to both the classical and quantum problems go well beyond previous discussion of this system [18–20]. More importantly, our interpretation of the system and its scope as a theory for duality and two times, and the applications to physical situations are new.

III. CLASSICAL SOLUTIONS AND DUAL SECTORS

The equation of motion for (X_1, X_2) in the case of S_0 is

$$\begin{pmatrix} \partial_\tau X^M \\ \partial_\tau P^M \end{pmatrix} = \begin{pmatrix} A_{12} & A_{22} \\ -A_{11} & -A_{12} \end{pmatrix} \begin{pmatrix} X^M \\ P^M \end{pmatrix}. \tag{8}$$

In addition, the equation of motion for the A_{ij} produces the constraints

$$X \cdot X = 0, \quad X \cdot P = 0, \quad P \cdot P = 0. \tag{9}$$

At least two timelike dimensions are required to obtain nontrivial solutions to the constraints [10], and our gauge symmetry does not allow more than two timelike dimensions without running into problems with ghosts. Thus our system exists physically only with the signature $(d,2)$.

To show that the massless Minkowski particle is one of the classical solutions of our system, we may choose the gauge $A^{12} = A^{11} = 0$ and $A^{22} = 1$, solve the equation $X^M = Q^M + P^M \tau$, and obtain the constraints $Q^2 = P^2 = Q \cdot P = 0$. There is a remaining gauge symmetry

$$\omega^{11}(\tau) = \omega_0^{11}, \quad \omega^{12}(\tau) = -\omega_0^{11}\tau + \omega_0^{12},$$
$$\omega^{22}(\tau) = \omega_0^{11}\tau^2 - 2\omega_0^{12}\tau + \omega_0^{22}, \tag{10}$$

where ω_0^{ij} are τ-independent constants. Next define the basis $Q^M = (Q^{+'}, Q^{-'}, q^\mu)$, $P^M = (P^{+'}, P^{-'}, p^\mu)$, where $\pm'$ indicate a light-cone-type basis for the extra (1,1) dimensions with metric $\eta^{+'-'} = -1$. Using two parameters of the remaining gauge freedom choose $Q^{+'} = 1$, $P^{+'} = 0$, and solve the two constraints $Q^2 = Q \cdot P = 0$, so that the solution takes the form

$$X^{+'}(\tau) = 1, \quad X^{-'}(\tau) = \frac{q^2}{2} + q \cdot p \tau,$$
$$X^\mu(\tau) = q^\mu + p^\mu \tau, \quad p^2 = 0 \quad \text{massless}. \tag{11}$$

There remains one free gauge parameter ω_0^{22} and one constraint $P^2 = p^2 = 0$, which is also what follows from τ reparametrizations on the world line. The motion in d-dimensional Minkowski subspace $x^\mu(\tau)$ is the same as the standard massless particle. Furthermore, the motion in the remaining two coordinates $X^{+'}, X^{-'}$ is fully determined by the position and momentum (q^μ, p^μ) in Minkowski space.

The free massless particle is not the only classical solution. For example, in the gauge $A^{12} = 0$, $A^{11} = A^{22} = \omega$ the solution is

$$X_M = a_M e^{i\omega\tau} + a_M^\dagger e^{-i\omega\tau},$$
$$a \cdot a = a^\dagger \cdot a = a^\dagger \cdot a^\dagger = 0. \tag{12}$$

This is an oscillatory motion with a different physical interpretation than the free relativistic particle. As we will see, our system has dual sectors that include the H atom and harmonic oscillator, which evidently are periodic systems. Some previously known solutions include a massive particle in Minkowski space [17], a massless particle in de Sitter space [17], etc. Thus, there are classical solutions of the same system with various physical meanings.

What is going on is that choosing "time" is tricky in our system since there is more than one timelike dimension. The dynamics of the system is arranged to evolve according to some gauge choice of "time" which is not unique in the system. For each such choice there is a corresponding canonical conjugate Hamiltonian which looks like different physics. However, there really is one single overall theory that follows from our action. It has various physical interpretations that are dual to each other, where duality is the $Sp(2)$ gauge symmetry that we have introduced. Under $Sp(2)$ transformations every classical solution which has a different physical interpretation in some gauge can be mapped to the free massless particle by a gauge transformation and a different choice of "time."

There is a gauge-invariant way to characterize the overall system at the classical as well as quantum levels. The $SO(d,2)$ global symmetry generators L^{MN} are gauge invariant, as well as constants of motion with respect to the "time" τ. Using the constraints, it is straightforward to compute that all the Casimir operators of $SO(d,2)$ vanish at the classical level:

$$C_n(SO(d,2)) = \frac{-1}{n!} \mathrm{Tr}(L)^n = 0 \quad \text{classical}. \tag{13}$$

[1] We thank K. Pilch for discovering this reference at the time of publication.

For a noncompact group such a representation is nontrivial. For example the free particle is such a representation. This can be verified by inserting the free particle gauge of Eq. (11) into Eq. (5). As we will see, the Casimir operators C_n will not all be zero at the quantum level, when ordering of operators is taken into account. We will find very specific values in the quantum gauge-invariant sector, in particular $C_2(\mathrm{SO}(d,2))=1-d^2/4$. Both at the classical and quantum levels, the Casimir invariants specify a *unique* unitary representation of SO(d,2) which fully characterizes the gauge-invariant physical space of the system. This approach does not involve a choice of ''time'' or Hamiltonian or effective Lagrangian in a fixed gauge.

Having realized this important observation one may now understand more generally that in a special gauge we find a rather nontrivial classical and quantum solution of our system, namely, the hydrogen atom in any dimension (the nonrelativistic central force problem with the $1/r$ potential). The essential reason for its existence is that all the levels of the H atom taken together form a single irreducible representation of the conformal group SO(d,2), in accordance with the observation above. In fact, the representation is precisely the unique one that emerges from quantum ordering (next section), with specific values of the Casimir operators. It was known that the H atom in three dimensions ($d-1=3$) forms a single irreducible representation of SO(4,2) [21]. The well-known SO(4) symmetry is the subgroup of SO(4,2). This solution will be fully explained and generalized to any dimension at the classical and quantum levels in a separate paper [1] (with quantum ordering and other technical aspects that differ from the old literature [21]). It will also be shown there that the harmonic oscillator in ($d-2$) dimensions, with its mass equal to a light cone momentum in an additional dimension, is also a solution of the system. As for all solutions, the H atom or harmonic oscillator are Sp(2) dual to the free massless relativistic particle.

To close this section we provide a general parametrization of classical solutions in any gauge. We take advantage of the fact that the SO(d,2) generators are constants of motion $\partial_\tau L^{MN}=0$ with respect to the ''time'' τ. A general classical solution in any gauge may be given in various bases $M=(+',-',\mu)$, $M=(0',1',\mu)$, $M=(0',0,I)$. The first is a light-cone-type basis in the extra dimensions $X^{+'}=(X^{0'}+X^{1'})$, $X^{-'}=\frac{1}{2}(X^{0'}-X^{1'})$, and the last distinguishes the two timelike coordinates from the spacelike ones. The first two are covariant under SO(1,1)$\otimes S(d-1,1)$ and the last is covariant under SO(2)$\otimes$SO(d). The general solution is

$$M=[+',-',\mu],$$

$$X^M=\left[a,b,\frac{-aL^{-'\mu}+bL^{+'\mu}}{ad-bc}\right],$$

$$P^M=\left[c,d,\frac{-cL^{-'\mu}+dL^{+'\mu}}{ad-bc}\right], \tag{14}$$

with

$$\begin{pmatrix} A_{12} & A_{22} \\ -A_{11} & -A_{12} \end{pmatrix}=\begin{pmatrix} \partial_\tau a & \partial_\tau b \\ \partial_\tau c & \partial_\tau d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}, \tag{15}$$

where the matrix $(a(\tau),b(\tau),c(\tau),d(\tau))$ is a group element of GL($2,R$). It can be checked that by inserting this form into Eq. (5) the constants $L^{\pm'\mu}$ that appear in Eqs. (14) are consistent with their definitions. Another constant of motion is the determinant of the matrix

$$L^{+'-'}=ad-bc. \tag{16}$$

So, effectively, the local gauge group is Sp(2) as parametrized by (a,b,c,d). The remaining generators $L^{\mu\nu}$, which are also constants of motion, are now written in terms of the constants $L^{+'-'}$, $L^{\pm'\mu}$:

$$L^{\mu\nu}=X^\mu P^\nu-X^\nu P^\mu=\frac{1}{L^{+'-'}}(L^{+'\mu}L^{-'\nu}-L^{-'\mu}L^{+'\nu}). \tag{17}$$

We may forget completely about the gauge potentials A^{ij} and concentrate instead on the local group element (a,b,c,d) and the global symmetry generators L^{MN}. The constraints (9) become conditions to be satisfied by the $L^{\pm'\mu}, L^{+'-'}, L^{\mu\nu}$ without any condition on the group element:

$$L^{+'\mu}L^{+'\nu}\eta_{\mu\nu}=L^{-'\mu}L^{-'\nu}\eta_{\mu\nu}=0,$$

$$\frac{1}{2}(L^{\mu\nu})^2=L^{+'\mu}L^{-'\nu}\eta_{\mu\nu}=-(L^{+'-'})^2. \tag{18}$$

With these conditions the Casimir operator for SO(d,2) becomes

$$C_2=\frac{1}{2}(L^{MN})^2=0, \tag{19}$$

at the classical level, and similarly for all higher Casimir coefficients. But we will see below that in quantum theory, when we watch the orders of the operators, the quadratic Casimir operators will be $C_2=1-d^2/4$. Similarly all higher Casimir operators of SO(d,2) vanish at the classical level, but not at the quantum level.

The same arguments may be repeated in the other bases. For example, in the basis $M=(0',0,I)$ we have

$$M=[0',\quad 0,\qquad I],$$

$$X^M=\left[a,\quad b,\quad \frac{-aL^{0I}+bL^{0'I}}{ad-bc}\right],$$

$$P^M=\left[c,\quad d,\quad \frac{-cL^{0I}+dL^{0'I}}{ad-bc}\right], \tag{20}$$

with

$$L^{0'0}=ad-bc,$$

$$L^{IJ}=\frac{1}{L^{0'0}}(L^{0'I}L^{0J}-L^{0'J}L^{0I}), \tag{21}$$

and

$$(L^{0I})^2=(L^{0'I})^2=-\frac{1}{2}(L^{IJ})^2=-(L^{0'0})^2, \tag{22}$$

so that again Eq. (19) and the same conditions on the higher order Casimir operators hold.

IV. QUANTUM THEORY

In any gauge the naive quantum rules that follows from the action S_0 are $[X_i^M,X_j^N]=i\varepsilon_{ij}\eta^{MN}$. These are subject to the constraint $X_i\cdot X_j=0$. We will rewrite these in any gauge as

$$[X^M,P^N]=i\eta^{MN}, \quad X^2=P^2=X\cdot P=0. \tag{23}$$

As usual one may approach the problem of quantization in a covariant formalism or in a noncovariant formalism.

In a covariant formalism one may apply the constraints on states constructed in a Hilbert space that obeys the naive quantization rules above. This approach would be manifestly covariant under both the duality symmetry $\mathrm{Sp}(2,R)$ as well as the $\mathrm{SO}(d,2)$ symmetry. But it does not seem to give direct insight into the physical content of the theory since ''time'' or ''Hamiltonian'' is not specified. In this paper we will obtain one crucial result on the values of the Casimir operators $C_n(\mathrm{SO}(d,2))$ that follow from covariant quantization.

In a noncovariant formalism both the duality symmetry and the manifest $\mathrm{SO}(d,2)$ symmetry are broken by the choice of gauges and solution of the constraints. One must then verify that the quantization procedure respects the gauge-invariant algebra of the global $\mathrm{SO}(d,2)$ generators L^{MN} in Eq. (5). In the fixed gauge formalism these generators incorporate the naive global transformation on the $d+2$ spacetime coordinates as well as the duality transformations $\mathrm{Sp}(2,R)$. This is because after an $\mathrm{SO}(d,2)$ transformation one goes out of the gauge slice, and a gauge transformation must be applied to go back to the gauge slice. Thus, the details of the $\mathrm{SO}(d,2)$ conformal generators in the fixed gauge provide information on the duality transformations. In a fixed gauge some of the L^{MN} require normal ordering of the canonical degrees of freedom and therefore there are anomaly coefficients. The closure of the algebra can fix some of these coefficients, but it turns out that this is not so in every gauge. It turns out that imposing the eigenvalues of the Casimir operators C_n obtained in the covariant quantization must be used to fully determine the anomaly coefficients. In particular for the hydrogen atom this additional constraint is needed. In this paper we will treat only the free particle in two different gauges and verify that we have the correct representation.

A. SO(d,2) and Sp(2) covariant quantization

The Hermitian quantum generators of $\mathrm{Sp}(2,R)$ are

$$J_0=\frac{1}{4}(P^2+X^2), \quad J_1=\frac{1}{4}(P^2-X^2), \tag{24}$$

$$J_2=\frac{1}{4}(X\cdot P+P\cdot X). \tag{25}$$

The Lie algebra that follows from the quantum rules is

$$[J_0,J_1]=iJ_2, \quad [J_0,J_2]=-iJ_1, \quad [J_1,J_2]=-iJ_0. \tag{26}$$

The quadratic Casimir operator $C_2(\mathrm{Sp}(2))=J_0^2-J_1^2-J_2^2$ takes the Hermitian form (watching the orders of operators)

$$C_2(\mathrm{Sp}(2))=\frac{1}{4}\left[X^MP^2X_M-(X\cdot P)(P\cdot X)+\frac{d^2}{4}-1\right], \tag{27}$$

where the constant term arises from reordering the operators $(d+2)^2-4(d+2)=d^2-4$. The gauge-invariant $\mathrm{SO}(d,2)$ Lorentz generators given in Eq. (5) are used to compute the quadratic Casimir operator for $\mathrm{SO}(d,2)$. One finds that the quadratic Casimir operator of the two groups are related:

$$C_2(\mathrm{SO}(d,2))=\frac{1}{2}L_{MN}L^{MN}=\left[C_2(\mathrm{Sp}(2))+1-\frac{d^2}{4}\right], \tag{28}$$

where $C_2(\mathrm{Sp}(2))$ is given by Eq. (27). Since L_{MN} is gauge invariant, both $C_2(\mathrm{SO}(d,2))$ and $C_2(\mathrm{Sp}(2))$ must have the same spectrum in any quantization scheme in any gauge.

We will describe the general properties of the covariant Hilbert space we should find. The ''physical'' states form a subset of the Hilbert space for which the matrix elements of $\mathrm{Sp}(2,R)$ generators vanish weakly:

$$\langle \mathrm{phys}|J_{0,1,2}|\mathrm{phys}'\rangle \sim 0. \tag{29}$$

For $\mathrm{SL}(2,R)=\mathrm{Sp}(2,R)$ all the unitary representations are labelled by $|jm\rangle$. Within this space the singlet state $C_2(\mathrm{Sp}(2))=j(j+1)=0$ and $m=0$ satisfy the physical requirements. This is the module with only one state from the point of view of $\mathrm{Sp}(2,R)$. However, there can be an infinite number of such gauge-invariant states which are classified by the global symmetry $\mathrm{SO}(d,2)$. This must be the case since we already know that there is a nontrivial solution of the constraints in the classical limit when the signature of η^{MN} is $(d,2)$.[2] Thus, we have argued that for nontrivial states we must have

$$C_2(\mathrm{SO}(d,2))=1-\frac{d^2}{4},\quad C_2(\mathrm{Sp}(2))=0. \tag{30}$$

This will be confirmed by the noncovariant quantization below. To compute the eigenvalues of all the Casimir operators C_n we use the same approach. We find that all C_n at the quantum level can first be written in terms of $C_2(\mathrm{Sp}(2))$ plus normal ordering constants that depend on d. Once the general expression is obtained we set $C_2(\mathrm{Sp}(2))=0$ and obtain the eigenvalues of $C_n(\mathrm{SO}(d,2))$ for the gauge-invariant states. This procedure uniquely determines the physical space content of our theory as a *unique unitary representation* of the conformal group $\mathrm{SO}(d,2)$. We only need the quadratic Casimir operator in the present paper.

Although we have identified the physical representation of $\mathrm{Sp}(2,R)$ and $\mathrm{SO}(d,2)$, building explicitly the $\mathrm{Sp}(2,R)$ and $\mathrm{SO}(d,2)$ fully covariant Hilbert space in terms of the covariant canonical variables X^M,P^M remains as an open problem. For a physical interpretation this is desirable. A natural approach to study the general problem covariantly is in terms of bilocal fields $\phi(X_1^M,X_2^M)$. Recall that bilocal fields are also relevant as background fields in the general action $S_{G,A}$.

B. Fully gauge-fixed quantization

In the noncovariant approach we choose a gauge and solve all the constraints at the classical level, and then quantize the remaining degrees of freedom. The advantage of this approach is that unitarity is manifest and we work directly with the physical states. The disadvantage is that by choosing a gauge we hide the duality properties. We will discuss here only the free massless particle interpretation of the Hilbert space. In another paper we will show that the same Hilbert space is dual to the H atom and also to the harmonic oscillator. We fix three gauges that make evident the free particle interpretation as in the classical solution (11), $X^{+'}=1,P^{+'}=0,X^+=p^+\tau$. Since we will express the commutation rules at $\tau=0$, we have, in a light cone basis $M=(+',-',+,-,i)$,

$$X^M=(1,x^{-'};0,x^-;\mathbf{x}^i),\quad P^M=(0,p^{-'};p^+,p^-,\mathbf{p}^i), \tag{31}$$

where the transverse vectors $\mathbf{x}^i,\mathbf{p}^i$ are in $(d-2)$ dimensions. Inserting this form in the constraints gives

$$x^{-'}=\frac{\mathbf{x}^2}{2},\quad p^{-'}=(\mathbf{x}\cdot\mathbf{p}-x^-p^+),\quad p^-=\frac{\mathbf{p}^2}{2p^+}, \tag{32}$$

where we have used $\eta^{+'-'}=\eta^{+-}=-1$. The canonical pairs are

$$[\mathbf{x},\mathbf{p}],\quad [x^-,p^+],\quad \left[x^+=0,p^-=\frac{\mathbf{p}^2}{2p^+}\right],$$

$$\left[x^{-'}=\frac{\mathbf{x}^2}{2},p^{+'}=0\right],\quad [x^{+'}=1,p^{-'}=(\mathbf{x}\cdot\mathbf{p}-x^-p^+)]. \tag{33}$$

The ones in the first line, $[\mathbf{x},\mathbf{p}],[x^-,p^+]$ are the true canonical operators for the relativistic particle, which are quantized according to the usual canonical rules:

$$[\mathbf{x}^i,\mathbf{p}^j]=i\delta^{ij},\quad [x^-,p^+]=i\eta^{+-}=-i. \tag{34}$$

On the other hand, $x^+=0,x^{+'}=1,p^{+'}=0$ are gauge choices and $p^-,p^{-'},x^{-'}$ are dependent operators which must be replaced by the given expressions in all gauge-invariant observables.

Recall that the Lorentz generators L^{MN} are gauge independent and commute with the $\mathrm{Sp}(2,R)$ generators. Therefore they can be expressed in any gauge, consistently with the constraints, by simply replacing our gauge choice (31), (32) into Eq. (5). Thus, we obtain

$$L^{ij}=\mathbf{x}^i\mathbf{p}^j-\mathbf{x}^j\mathbf{p}^i, \tag{35}$$

$$L^{+i}=-\mathbf{x}^ip^+,\quad L^{-i}=x^-\mathbf{p}^i-\frac{\mathbf{p}^j\mathbf{x}^i\mathbf{p}^j}{2p^+}, \tag{36}$$

$$L^{+-}=-\frac{1}{2}(x^-p^++p^+x^-),\quad L^{-'+}=\frac{1}{2}\mathbf{x}^2p^+, \tag{37}$$

$$L^{+'+}=p^+,\quad L^{+'-}=\frac{\mathbf{p}^2}{2p^+},\quad L^{+'i}=\mathbf{p}^i, \tag{38}$$

$$L^{+'-'}=\frac{1}{2}(\mathbf{x}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{x}-x^-p^+-p^+x^-), \tag{39}$$

$$L^{-'-}=\left[\begin{array}{l}\frac{1}{8p^+}(\mathbf{x}^2\mathbf{p}^2+\mathbf{p}^2\mathbf{x}^2-2\alpha)\\ -\frac{x^-}{2}(\mathbf{x}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{x})+x^-p^+x^-\end{array}\right], \tag{40}$$

[2]There is another trivial state that satisfies the physical state conditions with some modification in the weak condition. This is the Fock vacuum if one uses a harmonic oscillator representation with $X_M=(a_M+a_M^\dagger)/\sqrt{2}$ and $P_M=(a_M-a_M^\dagger)/\sqrt{2}i$. Taking into account operator ordering, then one finds $J_0=\frac{1}{2}a^\dagger\cdot a+\frac{1}{4}(d+2)$ and computes that the Fock vacuum has $j(j+1)=-1+d^2/4$ and $m_0=\frac{1}{4}(d+2)$. The physical state condition gets modified to $J_0=\frac{1}{4}(d+2)$ instead of zero. This state is the lowest state of the nontrivial discrete series representation of $\mathrm{Sp}(2)$. However, it is the trivial singlet state from the point of view of $\mathrm{SO}(d,2)$ since $L_{MN}=a_M^\dagger a_N-a_N^\dagger a_M$ annihilates it. This is the only state that would exist in the theory if the signature were $(d+2,0)$ or $(d+1,1)$.

$$L^{-'i}=\begin{bmatrix}\frac{1}{2}\mathbf{x}^j\mathbf{p}^i\mathbf{x}^j-\frac{1}{2}\mathbf{x}\cdot\mathbf{p}\mathbf{x}^i\\ -\frac{1}{2}\mathbf{x}^i\mathbf{p}\cdot\mathbf{x}+\frac{1}{2}\mathbf{x}^i(x^-p^++p^+x^-)\end{bmatrix}, \tag{41}$$

where operators are ordered to ensure that all components of L^{MN} are Hermitian. All possible ordering constants are uniquely fixed by Hermiticity except for the parameter α in $L^{-'-}$. Our aim is to show that these operators form the correct commutation rules for SO(d,2), namely,

$$[L_{MN},L_{PQ}]=i\,\eta_{MP}L_{NQ}+i\,\eta_{NQ}L_{MP}-i\,\eta_{NP}L_{MQ} -i\,\eta_{MQ}L_{NP}. \tag{42}$$

This requirement fixes the parameter $\alpha=-1$. In particular,

$$[L^{-'i},L^{-j}]=i\delta^{ij}L^{-'-}, \quad \rightarrow \quad \alpha=-1. \tag{43}$$

In a laborious calculation it can be verified that our construction satisfies the correct commutation rules. The structure of the algebra may be described as follows. First note that $L^{\mu\nu}=(L^{ij},L^{\pm i},L^{+-})$ form the SO($d-1$,1) Lorentz algebra, and that $p^{\mu}=(L^{+'+},L^{+'-},L^{+'i})$ are the generators of translations. The set $(L^{\mu\nu},p^{\mu})$ forms the Poincaré algebra ISO($d-1$,1) in the massless sector $p^2=0$. The operators $K^{\mu}=(L^{-'+},L^{-'-},L^{-'i})$ are the special conformal transformations and finally $D=L^{+'-'}$ is the dilatation operator.

It is also useful to note that the subset $(L^{\pm'\pm},L^{\pm'\mp},L^{+'-'},L^{+-})$ forms the algebra of SO(2,2). Since SO(2,2)$=$SL(2,R)$_L\otimes$SL(2,R)$_R$, it is convenient to identify the SL(2,R)$_L\otimes$SL(2,R)$_R$ combinations as

$$G_2^L=\frac{1}{2}(L_{+'-'}+L_{+-}), \quad G_0^L\pm G_1^L=L_{\pm'\pm}, \tag{44}$$

$$G_2^R=\frac{1}{2}(L_{+'-'}-L_{+-}), \quad G_0^R\pm G_1^R=L_{\pm'\mp}, \tag{45}$$

which satisfy $[G_a^L,G_b^R]=0$ and

$$[G_0^{L,R},G_1^{L,R}]=iG_2^{L,R},[G_0^{L,R},G_2^{L,R}]=-iG_1^{L,R}, \tag{46}$$

$$[G_1^{L,R},G_2^{L,R}]=-iG_0^{L,R}. \tag{47}$$

In our case we found the representation

$$G_2^L=\frac{1}{4}(\mathbf{x}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{x})-\frac{1}{2}(x^-p^++p^+x^-), \tag{48}$$

$$G_0^L+G_1^L=p^+, \tag{49}$$

$$G_0^L-G_1^L=\begin{bmatrix}\frac{1}{8p^+}(\mathbf{x}^2\mathbf{p}^2+\mathbf{p}^2\mathbf{x}^2-2\alpha)\\ -\frac{x^-}{2}(\mathbf{x}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{x})+x^-p^+x^-\end{bmatrix}, \tag{50}$$

and

$$G_2^R=\frac{1}{4}(\mathbf{x}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{x}), \tag{51}$$

$$G_0^R+G_1^R=\frac{\mathbf{p}^2}{2p^+}, \tag{52}$$

$$G_0^R-G_1^R=\frac{1}{2}\mathbf{x}^2p^+. \tag{53}$$

These structures do indeed correctly form the Lie algebras of SO(2,2)$=$SL(2,R)$_L\otimes$SL(2,R)$_R$. We compute the quadratic Casimir operator of each SL(2,R) by $j(j+1)=G_0^2-G_1^2-G_2^2$. We find

$$j_R(j_R+1)=\frac{1}{4}\mathbf{L}^2+\frac{1}{16}(d-2)^2-\frac{1}{4}(d-2), \tag{54}$$

$$j_L(j_L+1)=j_R(j_R+1)-\frac{1+\alpha}{4}, \tag{55}$$

where $\mathbf{L}^2=\frac{1}{2}L_{ij}L^{ij}$ is the Casimir operator for the orbital rotation subgroup SO($d-2$):

$$\mathbf{L}^2=\mathbf{p}^i\mathbf{x}^2\mathbf{p}^i-\mathbf{p}\cdot\mathbf{x}\mathbf{x}\cdot\mathbf{p}. \tag{56}$$

Note that for $\alpha=-1$ we have $j_L=j_R$. The overall quadratic Casimir operator for SO(d,2) of Eq. (28) takes the form

$$C_2=\{L^{+'+},L^{-'-}\}+\{L^{+'-},L^{-'+}\}-(L^{+'-'})^2-(L^{+-})^2 -\{L^{+'i},L^{-'i}\}-\{L^{+i},L^{-i}\}+\frac{1}{2}L_{ij}L^{ij}, \tag{57}$$

$$=\mathbf{L}^2+\frac{1}{4}(d-2)^2-(d-2)-2\mathbf{L}^2-\frac{1}{2}(d-2)^2+\mathbf{L}^2, \tag{58}$$

$$=-\frac{d^2}{4}+1. \tag{59}$$

As expected, the ‘‘orbital’’ part involving the canonical pairs (x^-,p^+) and $(\mathbf{x},\mathbf{p})$ dropped out. By comparison to the covariant form (28) we have verified that $C_2(\mathrm{Sp}(2))=0$. This makes sense since we have enforced the constraints at the classical level and thereby guaranteed that the Sp(2,R) generators vanish in the physical sector.

C. Lorentz-covariant quantization and field theory

We may choose the gauge for the free particle partially to the following form in the basis $M=(+',-',\mu)$:

$$X^M(\tau)=\left(1,\frac{x^2(\tau)}{2},x^\mu(\tau)\right),$$

$$P^M(\tau)=(0,p(\tau)\cdot x(\tau),p^\mu(\tau)). \tag{60}$$

There remains the gauge degree of freedom that corresponds to τ reparametrization $\omega^{22}(\tau)$ and the corresponding constraint $p^2(\tau)=0$. The independent canonical pairs are quantized as $[x^\mu,p^\nu]=i\eta^{\mu\nu}$, which is Lorentz covariant. Physical states $|\phi\rangle$ must satisfy the $p^2|\phi\rangle=0$ condition weakly. The well-known solution may be given in x space, $\phi(x)=\langle x|\phi\rangle$, where $\langle x|p_\mu=-i(\partial/\partial x^\mu)\langle x|$, for which the constraint takes the form of the Klein-Gordon equation for a massless particle:

$$\Box\phi(x)=0. \tag{61}$$

The field theory ''effective action'' that gives this equation is

$$S_{eff}=\frac{1}{2}\int d^dx\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu}\eta^{\mu\nu}. \tag{62}$$

The solutions of the constraint are well known:

$$\phi(x)=\int\frac{d^dk\,\theta(k^0)}{(2\pi)^{d-1}}\delta(k^2)[a(k)e^{ik\cdot x}+a^\dagger(k)e^{-ik\cdot x}]. \tag{63}$$

The states $|\phi\rangle$ have the Lorentz-invariant positive norm defined by

$$\langle\phi|\phi\rangle=-\frac{i}{2}\int d^{d-1}x(\phi^*\partial_0\phi-\partial_0\phi^*\phi),$$

$$=\int d^{d-1}k a^\dagger(k)a(k), \tag{64}$$

which is independent of the time component x^0 even though x^0 is not integrated. The Lorentz invariance of this norm is well known from the study of the Klein-Gordon equation, and can be seen by writing it in the form $\int dx\wedge\cdots\wedge dx\wedge J$ where $J_\mu=(-i/2)(\phi^*\partial_\mu\phi-\partial_\mu\phi^*\phi)$. If one wishes one may rewrite the norm by choosing to fix the light cone time x^+ instead of the ordinary time x^0.

Since SO$(d,2)$ is not manifest, we must check that the gauge-invariant conserved symmetry generators for SO$(d,2)$ have the correct commutation rules (42). We must first compute the gauge-invariant L_{MN} in terms of (x^μ,p^μ) by inserting our gauge choice and solutions of constraints given in Eqs. (60). We find

$$L^{+'-'}=\frac{1}{2}(p\cdot x+x\cdot p)+i, \tag{65}$$

$$L^{+'\mu}=p^\mu, \tag{66}$$

$$L^{-'\mu}=\frac{1}{2}x_\lambda p^\mu x^\lambda-\frac{1}{2}x^\mu p\cdot x-\frac{1}{2}x\cdot p x^\mu-ix^\mu, \tag{67}$$

$$L^{\mu\nu}=x^\mu p^\nu-x^\nu p^\mu, \tag{68}$$

where operators are ordered. The commutation rules for SO$(d,2)$ are satisfied. All ordering ambiguities are uniquely determined by Hermiticity,

$$\langle\phi_1|L^{MN}\phi_2\rangle=\langle L^{MN}\phi_1|\phi_2\rangle, \tag{69}$$

relative to the nontrivial Lorentz-invariant norm in Eq. (64). This is the reason for the appearance of the anomalous corrections proportional to i in $L^{+'-'},L^{-'\mu}$. Without these anomaly pieces the generators are not Hermitian. As a check that we have correctly ordered our operators we compute the dimension of the scalar field by applying $L^{+'-'}$ on it:

$$iL^{+'-'}\phi(x)=\langle x|iL^{+'-'}|\phi\rangle=\frac{1}{2}x\cdot\partial\phi+\frac{1}{2}\partial\cdot(x\phi)-\phi$$

$$=x\cdot\partial\phi+\left(\frac{d}{2}-1\right)\phi. \tag{70}$$

The dimension $(d/2-1)$ is the correct dimension of the scalar field in the effective field theory action (62). We also see that by replacing $p_\mu=-i\partial/\partial x^\mu$ we arrive at the well-known construction of the conformal group in terms of differential operators as known in field theory. The effective field theory S_{eff} and the dot product are invariants under these SO$(d,2)$ conformal transformations applied on the field:

$$\delta\phi(x)=i\varepsilon_{MN}L^{MN}\phi(x),\quad \delta S_{eff}=0=\delta(\langle\phi_1|\phi_2\rangle). \tag{71}$$

Thus $L^{+'-'}$ is the dimension operator, $L^{+'\mu}$ is the translation operator, $L^{-'\mu}$ is the generator of special conformal transformations, and $L^{\mu\nu}$ is the generator of Lorentz transformations.

We can now compute the quadratic Casimir operator for SO$(d,2)$ in this gauge. As we have argued in the previous section, its value is gauge invariant; therefore it can be computed in any gauge. We find that it reduces to a number

$$C_2=-(L^{+'-'})^2-\{L^{+'\mu},L^{-'\nu}\}\eta_{\mu\nu}+\frac{1}{2}L_{\mu\nu}L^{\mu\nu} \quad (72)$$

$$=-\frac{d^2}{4}+1, \quad (73)$$

where all (x,p) dependence has dropped out.[3] The value of the gauge invariant quadratic Casimir is again the same. This fixes the Sp(2,R) representation uniquely to $C_2(\mathrm{Sp}(2))=0$ in agreement with the previous sections.

One may be puzzled by questions such as follows: Originally the operator $L^{+'-'}$ was a transformation that acted purely in the extra dimensions $X^{\pm'}$ while leaving Minkowski space x^μ untouched; how can it now act like the scale transformations in Minkowski space? The answer is that we chose the gauge $X^{+'}=1$ that fixed a scale. However linear transformation in global SO(d,2) transforms X^M out of this gauge slice. To come back to the same gauge one must apply also a duality gauge transformation on $X^M(\tau)$. The duality gauge transformation that corresponds to a rescaling of $X^{+'}$ also rescales the rest of the components. This is precisely what the operator $L^{+'-'}$ does on Minkowski space. The structure of the *gauge invariant* operator $L^{+'-'}$ ''knows'' that this gauge transformation must be performed on $X^M(\tau)$.

Through our construction, the conformal group of massless field theories has now acquired the new meaning of being the Lorentz-like group in an actual spacetime with two timelike dimensions X^M. The conformal field theory S_{eff} has been expressed in a fixed gauge of the larger $(d+2)$-dimensional space. There should exist a fully covariant effective field theory corresponding to the SO(d,2) $\otimes$Sp(2,R) covariant quantization. The fully covariant action in $d+2$ dimensions would collapse to the effective action of a massless particle given above upon gauge fixing. Such a field theory may be formulated in terms of a bilocal field $\phi(X_1^M,X_2^M)$.

V. OUTLOOK

We have seen that the familiar free massless particle in d-dimensional Minkowski spacetime may be viewed as residing in a larger spacetime of $d+2$ dimensions. The higher spacetime includes gauge degrees of freedom, but in their presence the full SO(d,2) conformal invariance takes the new meaning of being the linear ''Lorentz symmetry'' in a spacetime that includes two timelike dimensions. Which of the two ''times'' $x^{0'},x^0$ is *the* familiar time coordinate? For the gauge choice we have made, time is x^0 and with it we have described the dynamics a free particle. However, there are other choices of time as we have demonstrated in the classical solutions, here, and quantum solutions in another paper [1]. For other choices of time the Hamiltonian is different and the physics looks different (such as the H atom), even though we are describing the same overall system that corresponds to a single unique representation of the conformal group SO(d,2). So the concept of ''time'' seems to be more general, and both of our two times play a physical role. We may say that for the free massless particle the appearance of *conformal symmetry* is *the manifestation of a larger spacetime that includes two timelike coordinates*. Similarly, for the H atom and other dual systems, the presence of the conformal symmetry *is* part of the evidence of the presence of two timelike dimensions.

Duality and the concept of two times are meshed together in our theory. The duality we found is in the same spirit of the duality symmetry of M theory, but its realization requires two timelike dimensions in target space $X^M(\tau)$. This is more in line with the ideas of S theory [6] and F theory [5]. In our case, we have actually constructed an action for a miniature s theory, which should serve as a guide for constructing a full fledged S theory in (10,2) and perhaps even in (11,3) dimensions [7].

It may be interesting to view our theory as *conformal gravity* on the world line as noted earlier in the paper. We may then regard the gauge fields (A^{22},A^{12},A^{11}) as the gauge fields for translations, dilatations, and special conformal transformations, respectively. Our theory may be used as a guide for generalizations from the world line to the world sheet or world volume for various p-branes. Although conformal gravity on the world sheet has been considered before [23], our approach in phase space is somewhat different and may yield a new and different action. Such a reformulation of p-brane actions would permit the introduction of two timelike dimensions in $X^M(\tau,\sigma_1,\ldots,\sigma_p)$ just as in the particle case $p=0$.

The present paper, as well as some of our previous papers [10–12], is an attempt to take the concept of two or more timelike dimensions seriously. We may ask, are there more observable effects of two timelike dimensions besides the conformal invariance and the duality connections we have suggested? To answer such a question it would be useful to study interacting theories that are consistent with the gauge duality symmetries. This is essential in order to avoid ghosts. As a first step one may explore the interacting theory S_G that would result from a curved background in $(d,2)$ dimensions. This is formulated by taking a curved metric $G_{MN}(X_1,X_2)$ instead of η_{MN} in the action (4). One way to maintain the local Sp(2,R) symmetry is to take G_{MN} as a function of only the gauge-invariant combination $X_i^M X_j^N \varepsilon^{ij}$. It is also possible to study interactions using S_A in the presence of background gauge fields $A_M^i(X_1,X_2)$ that couple in a gauge-invariant way to $D_\tau X_i^M(\tau)$ in $d+2$ dimensions. Here it would be in-

[3]The dropping out of the orbital part is a phenomenon that occurs more generally for any Casimir operator in a more general construction available for *any group* [22]. For example, a more general construction for SO(d,2) including the spin operator $s^{\mu\nu}$ and the anomalous dimension operator d_0,

$$J^{+'-'}=L^{+'-'}+id_0, \quad J^{+'\mu}=L^{+'\mu},$$
$$J^{-'\mu}=L^{-'\mu}-id_0x^\mu-s^{\mu\lambda}x_\lambda,$$
$$J^{\mu\nu}=L^{\mu\nu}+s^{\mu\nu},$$

also has the property that *all* Casimir operators do not depend on the ''orbital'' operators (x,p) contained in the L^{MN}. In particular the quadratic casimir is $C_2=-d^2/4+(d_0+1)^2+\frac{1}{2}s^{\mu\nu}s_{\mu\nu}$.

teresting to explore the possible relation between our Sp(2) doublet A_M^i and the electric-magnetic dual potentials of Maxwell's theory and its generalizations [14]. One thing that is becoming clearer is that bilocal fields $\phi(X_1,X_2)$ are probably going to be very useful for writing down the low energy effective theories consistently with the local Sp$(2,R)$ invariance.

The idea of bilocal fields also emerged before as a means of displaying the hidden timelike dimensions in certain Bogomol'nyi-Prasad-Sommerfield (BPS) sectors which provide short representations of the superalgebra of S theory [6]. It was emphasized that such BPS sectors, which reveal extra timelike dimensions in black holes [24], must be considered dual sectors to other BPS solutions of M theory. Progress along these and other directions for interacting theories will be reported in the future. We hope that such interacting theories would provide the means to discuss how to probe the higher hidden dimensions and perhaps find some additional measurable consequences and tests.

We would like to think that the presence of duality [4] and conformal symmetry [25] in M theory, as well as in special super Yang-Mills theories under current consideration, is also a sign of the presence of higher dimensions, and in particular of extra timelike dimensions. Indeed various signs that the mysterious theory may actually have 12 dimensions with signature (10,2) have been accumulating. It has also been argued that a fundamental theory that is *manifestly covariant* under both duality and supersymmetry requires 14 dimensions with signature (11,3) to display the covariance (in the spirit of the current paper), and it must have certain ''BPS'' constraints that are due to gauge invariances [7]. The various ideas outlined in this paper may be regarded as a small step toward a formulation of such a theory.

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