Simulation of supersymmetric models with a local Nicolai map

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We study the numerical simulation of supersymmetric models having a local Nicolai map. The mapping can be regarded as a stochastic equation and its numerical integration provides an algorithm for the simulation of the original model. In this paper, the method is discussed in detail and applied to examples in 0+1 and 1+1 dimensions. [S0556-2821(98)00418-4]

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I. INTRODUCTION

A deep property of supersymmetric field theories is the existence of the Nicolai map [1] that is a nonlinear transformation of the bosonic field ϕ such that (i) the transformed action describes a Gaussian field ξ with unit covariance and (ii) the Jacobian determinant of the transformation exactly cancels the fermion determinant.

In some cases, the map is local: the new field ξ is expressed by a polynomial in ϕ and its derivatives. When this happens, it is convenient to regard ξ as a random field and the transformation $\xi = \xi(\phi)$ as a stochastic equation [2].

A (0+1)-dimensional example is supersymmetric quantum mechanics [3] which we shall adopt as a toy model in later discussions. In 1+1 dimensions, a class of models with a local Nicolai map is that of $N=2$ Wess-Zumino (WZ) models. Other examples are 1+3 dimensional $N=1$ super Yang-Mills theory in the light cone gauge [4,5] and d dimensional lattice linear gauge theory [6].

The existence of a local Nicolai map plays a major role in the lattice formulation of such models. Indeed, its discretization provides a recipe for the construction of a lattice field theory which retains most of the continuum symmetries. In particular, the doubling of the fermions is automatically implemented because of relations between the boson and fermion propagators [7,8]. Such lattice models can be studied by standard simulation techniques, but it is interesting to see if the underlying discrete stochastic structure can be useful for the purpose of numerical computations in a more direct way.

If one succeeds in solving the stochastic equation $\xi = \xi(\phi)$ then uncorrelated ϕ configurations may be obtained by generating ξ samples. Moreover, explicit fermion fields can be avoided because their correlation functions may be expressed by means of the so-called stochastic identities [5,9] in terms of statistical correlations of the solution $\phi(\xi)$ and the random field ξ.

The actual implementation of this program faces three difficulties: (i) specific boundary conditions must be imposed to preserve supersymmetry making it highly nontrivial to determine $\phi(\xi)$, (ii) there may be more than one solution, and (iii) the stochastic equation may admit no thermal equilibrium [10,9]. In other words, when regarded as an evolution equation for ϕ, unbounded solutions may appear with possible instability problems in the numerical simulation.

In this paper, we discuss the above three problems in (0+1)- and (1+1)-dimensional specific examples. We study general properties of the continuum and discretized Nicolai map and perform explicit numerical simulations to check the feasibility of the method.

II. STOCHASTIC EQUATIONS AND SUPERSYMMETRY

Let $\phi_\alpha(t)$, $t \in [0,\beta]$ be a time dependent field obeying the equation

$$\xi_\alpha(t) = \frac{d}{dt}\phi_\alpha(t) + W_\alpha(\phi), \quad W_\alpha(\phi) = \frac{\delta W}{\delta \phi_\alpha}, \tag{2.1}$$

where ξ_α is a Gaussian white noise

$$\langle \xi_\alpha(t)\xi_\beta(t')\rangle = \delta_{\alpha\beta}\delta(t-t'), \tag{2.2}$$

and $W(\phi)$ is an arbitrary potential function. For the moment we do not specify the boundary conditions in Eq. (2.1) that make the problem well posed. The field index α may include spatial variables which we assume to vary in a finite volume. Equation (2.1) may be regarded as a functional change of variable $\phi \to \xi$ which can be inverted in a certain range of ξ giving rise to many branches $\phi = \phi^{(n)}(\xi)$, $n = 1, \ldots, N(\xi)$.

As shown in [9], the field model with periodic boundary conditions and classical action

$$S = \int_0^\beta dt \left[\frac{1}{2}(\dot{\phi}_\alpha + W_\alpha)^2 + \bar{\psi}_\alpha \frac{\partial \xi_\alpha}{\partial \phi_\beta}\psi_\beta\right],$$

$$\phi(0) = \phi(\beta), \quad \psi(0) = \psi(\beta) \tag{2.3}$$

is $N=2$ supersymmetric and supertraces can be computed as stochastic averages

$$
\begin{aligned}
\mathrm{Str}[\Omega(\phi)e^{-\beta H}] &= \mathrm{Tr}[\Omega(\phi)e^{-\beta H}(-1)^F] \\
&= \int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}\; e^{-S}\Omega(\phi) \\
&= \sum_n \int \mathcal{D}\xi e^{-S(\xi)}\Omega(\phi^{(n)}(\xi)) \\
&\quad \times \mathrm{sgn}\,\det\frac{\partial\xi}{\partial\phi}\bigg|_{\phi=\phi^{(n)}(\xi)},
\end{aligned}
\tag{2.4}
$$

where F is the fermion number, H is the Hamiltonian, and $S(\xi)=\frac{1}{2}\int_0^\beta \xi^2 dt$.

Equation (2.4) may be exploited if one is able to solve numerically Eq. (2.1) with periodic boundary conditions $\phi(0)=\phi(\beta)$.

III. THE NICOLAI MAP IN THE CONTINUUM

A. 0+1 Dimensions, supersymmetric quantum mechanics

In the case of SUSY quantum mechanics [3], Eq. (2.1) is simply

$$
\dot{q}=f(q)+\xi,
$$
$$
q(0)=q(\beta), \tag{3.1}
$$

and the Hamiltonian H appearing in Eq. (2.4) is

$$
H=\frac{1}{2}p^2+\frac{1}{2}f(q)^2-\frac{1}{2}f'(q)\sigma_3,
$$
$$
p=-i\frac{d}{dq}, \quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{3.2}
$$

Some properties of Eq. (3.1) do not depend on ξ being a random field. For this reason, we begin by regarding Eq. (3.1) as an inner map in the space $C_p^\infty([0,\beta])$ of periodic smooth functions in $[0,\beta]$.

The Jacobian of Nicolai map can be computed explicitly [11] and it is

$$
\det\left(\frac{d}{dt}+f'(q)\right)=c\cdot\sinh\left[\frac{1}{2}\int_0^\beta dt\, f'(q(t))\right]. \tag{3.3}
$$

The number $N(\xi)$ of solutions ϕ for a given ξ is an integer valued function of ξ with possible jumps across the critical manifold

$$
\mathcal{M}_c=\left\{q(t)\in C_p^\infty([0,\beta])\,\middle|\,\int_0^\beta f'(q(t))dt=0\right\}. \tag{3.4}
$$

The geometry of $\mathcal{M}_c$ provides information on $N(\xi)$; for instance, suppose that $C_p^\infty([0,\beta])/\mathcal{M}_c$ is simply connected, then $N(\xi)$ must be constant. To evaluate it, we choose ξ constant and deduce from

$$
\dot{q}=f(q)+\xi\Rightarrow(\dot{q})^2=f(q)\dot{q}+\xi\dot{q}\Rightarrow\int_0^\beta(\dot{q})^2dt=0, \tag{3.5}
$$

so that q is also constant and, therefore, given by the roots of the equation

$$
f(q)+\xi=0. \tag{3.6}
$$

A particularly interesting case is that of $f'>0$, which implies $\mathcal{M}_c=\varnothing$. In this case, in the open problem

$$
\frac{\partial}{\partial t}q(t,q_0)=f(q(t,q_0))+\xi(t),
$$
$$
q(0,q_0)=q_0, \tag{3.7}
$$

we have

$$
\frac{\partial q(t,q_0)}{\partial q_0}=\exp\int_0^t f'(q(z,q_0))dz>1, \tag{3.8}
$$

so that $\Delta(t,q_0)=q(t,q_0)-q_0$ is a monotone function of q_0 and we conclude that if a solution exists then it must be unique.

To see this argument working in particular cases, let us consider what happens when $f(q)=\mu q^n$ with $\mu>0$ and $n=1,2,3$.

$n=1$. This is the simplest case. Indeed, $q(\beta,q_0)$ is a linear function of q_0 and the problem

$$
\dot{q}=\mu q+\xi,
$$
$$
q(0)=q(\beta), \tag{3.9}
$$

has the unique periodic solution

$$
q(t)=e^{\mu t}q(0)+e^{\mu t}\int_0^t e^{-\mu z}\xi(z)dz,
$$
$$
q(0)=\frac{1}{e^{-\mu\beta}-1}\int_0^\beta e^{-\mu t}\xi(t)dt. \tag{3.10}
$$

$n=2$. In this case, $C_p^\infty([0,\beta])/\mathcal{M}_c$ is not simply connected and, moreover, we cannot solve explicitly the problem

$$
\dot{q}=\mu q^2+\xi,
$$
$$
q(0)=q(\beta). \tag{3.11}
$$

Motivated by what happens when q and ξ are constant we guess that, in general, there cannot be solutions for all ξ and that, when there are solutions, they come in pairs. This follows also from the constraint on ξ,

$$
\int_0^\beta \xi(t)dt=-\mu\int_0^\beta q^2(t)dt\leqslant 0, \tag{3.12}
$$

which excludes some ξ and from the fact that Eq. (3.11) is a Riccati equation. If $q_1(t)$ is one particular solution for a given $\xi(t)$ then another solution is $q_2(t)$ where

$$q_2(t)=q_1(t)+\frac{1}{w(t)}, \tag{3.13a}$$

$$\dot{w}+2q_1(t)w=-1, \quad w(0)=w(\beta). \tag{3.13b}$$

In terms of

$$F(t)=\exp\left(2\int_0^t q_1(\alpha)d\alpha\right), \tag{3.14}$$

the function w satisfies

$$\frac{\partial}{\partial t}(F(t)w(t))=-F(t), \tag{3.15}$$

and the periodicity condition $w(\beta)=w(0)$ gives

$$w(0)=-\frac{1}{F(\beta)-1}\int_0^\beta F(\alpha)d\alpha. \tag{3.16}$$

Hence, a second solution q_2 can always be found except when $F(\beta)=1$, namely, on the critical manifold,

$$\int_0^\beta q(t)dt=0, \tag{3.17}$$

of the map.

$n=3$. This is a case where $f'(q)=3\mu q^2$ is always positive. The difference $\Delta(\beta,q_0)$ is a monotonically increasing function and it is easy to prove that

$$\lim_{q_0\to\pm\infty}\Delta(\beta,q_0)=\pm\infty. \tag{3.18}$$

(Strictly speaking this holds only at the discrete level where blowing solutions do not appear.) From this remarks it follows that there is a unique periodic solution for each periodic $\xi(t)$.

In the above analysis, ξ was assumed to be smooth. When ξ is a white noise it gives fluctuations δq around the $\xi=0$ solutions, namely, $q=q^*$ with $f(q^*)=0$. The size of δq depends on $f'(q^*)$ and β. Because of periodicity, we expect it to approach a constant when $\beta\to\infty$ and diverge as $\beta\to 0$ with no regards to the stability of the fixed point $q=q^*$.

An illustrative solvable example is that of $f(q)=\mu q$. From

$$q(0)=\frac{1}{e^{-\mu\beta}-1}\int_0^\beta e^{-\mu t}\xi(t)dt, \tag{3.19}$$

we obtain

$$\langle q(0)^2\rangle=\frac{1}{2\mu}\,\frac{1-e^{-2\mu\beta}}{(1-e^{-\mu\beta})^2}, \tag{3.20}$$

and indeed we find

$$\langle q(0)^2\rangle \overset{\beta\to 0}{\sim} \frac{1}{\mu^2\beta}, \quad \langle q(0)^2\rangle \overset{\beta\to+\infty}{\to} \frac{1}{2|\mu|}. \tag{3.21}$$

Here $q^*=0$ and the parameter μ is just $f'(q^*)$, namely, what we can call the tree level mass. In a numerical simulation, in the $\beta\to\infty$ limit, we expect to have one solution for each fixed point with large deviations depressed exponentially by potential barriers controlled by parameters like μ. In these regimes [assuming $f'(q^*)\neq 0$] we can compute the sign of the Jacobian $\det\partial\xi/\partial q$ at each $q=q^*$ and use it in a neighborhood of that point.

B. 1+1 Dimensions, WZ models

In 1+1 dimensions, let $z=x_1+ix_2$ and $\phi(z,\overline{z})$ be a complex field. Equation (2.1) takes the form

$$2\frac{\partial\phi}{\partial z}=\overline{f(\phi)}+\eta, \tag{3.22}$$

where $\eta=\eta_1+i\eta_2$ with η_1 and η_2, real independent white noises, and $f(\phi)=u(\phi)+iv(\phi)$, an arbitrary holomorphic function of ϕ. The reason for the peculiar structure of Eq. (3.22) is that it guarantees that the associated field model (the so-called WZ model) is Lorentz covariant [9]. The explicit form of Eq. (3.22) is

$$\partial_1\phi_1+\partial_2\phi_2=u(\phi_1,\phi_2)+\eta_1, \tag{3.23a}$$

$$\partial_1\phi_2-\partial_2\phi_1=-v(\phi_1,\phi_2)+\eta_2. \tag{3.23b}$$

The numerical integration of Eq. (3.23) is difficult because even in the simplest cases the associated random flow does not admit an equilibrium distribution. To see this, it is enough to consider a homogeneous (independent on x_2) solution of Eq. (3.23) without noise. It must satisfy

$$\frac{d\phi}{dx_1}=\overline{f(\phi)}. \tag{3.24}$$

If $F(\phi)$ is a primitive of $f(\phi)$ then the quantity $H=\mathrm{Im}\,F(\phi)$ is a constant of motion since

$$\frac{dH}{dx_1}=\mathrm{Im}\left(f(\phi)\frac{d\phi}{dx_1}\right)=\mathrm{Im}|f|^2=0. \tag{3.25}$$

If we consider a function $f(z)$ with asymptotic power behavior $\sim z^n$ we see that the level curves of H are not closed and equilibrium cannot be reached.

As in the (0+1)-dimensional case, for a constant η we have (by periodicity)

$$\begin{aligned}2\frac{\partial\phi}{\partial z}=\overline{f(\phi)}+\eta \Rightarrow 2\left|\frac{\partial\phi}{\partial z}\right|^2 &= \overline{f(\phi)\frac{\partial\phi}{\partial z}}+\eta\overline{\frac{\partial\phi}{\partial z}} \Rightarrow \int_0^\beta dx_1\int_0^\sigma dx_2\left|\frac{\partial\phi}{\partial z}\right|^2=0,\end{aligned} \tag{3.26}$$

which implies that ϕ is a constant obtained from the equation

$$f(\phi)+\overline{\eta}=0. \tag{3.27}$$

In particular, when $\eta\to 0$, ϕ tends to one of the zeroes of $f(\phi)$. As in the (0+1)-dimensional case, the Nicolai map is not singular at $\phi=\phi^*$ with $f(\phi^*)=0$ and $f'(\phi^*)\neq 0$. Actually, an infinitesimal periodic zero mode λ such that $\phi=\phi^*+\lambda$ is a solution of Eq. (3.22) must satisfy

$$2\frac{\partial\lambda}{\partial z}=\overline{f'(\phi^*)\lambda}, \tag{3.28}$$

and therefore

$$(4\partial\overline{\partial}-|f'(\phi^*)|^2)\lambda=0\Rightarrow\int dx_1dx_2(|\nabla\lambda|^2+|f'(\phi^*)|^2|\lambda|^2)=0, \tag{3.29}$$

giving $\lambda\equiv 0$.

IV. THE DISCRETE NICOLAI MAP

When time is discretized the interval $[0,\beta]$ is divided into N subintervals and the Nicolai map is a change of variables

$$(\xi_0,\ldots,\xi_{N-1})\leftrightarrow(q_0,\ldots,q_{N-1}), \tag{4.1}$$

where q_n and ξ_n are the values of q and ξ at times $t_n=n\beta/N$. The next step is the choice of a definite discretization scheme of Eq. (2.1). Then, one has (i) a discrete stochastic equation whose numerical properties must be studied and (ii) a precise fermion action to be analyzed from the point of view of the fermion doubling problem. In this section we address these problems as well as the use of the so-called stochastic identities to compute fermion correlation functions without introducing Grassmann fields at all.

A. Choice of the discretization procedure

For simplicity, let us consider a single component field $q(t)$ in 0+1 dimensions. The Langevin equation

$$\dot{q}=f(q)+\xi,$$

$$\langle\xi(t)\xi(t')\rangle=D(q(t))\delta(t-t'), \tag{4.2}$$

where we allowed a q dependent covariance, may be discretized according to

$$q_{n+1}=q_n+\epsilon[\alpha f(q_n)+(1-\alpha)f(q_{n+1})]+\sqrt{\epsilon}D^{1/2}[\alpha q_n+(1-\alpha)q_{n+1}]\xi_n, \tag{4.3}$$

where the constant parameter α takes into account the ambiguity in the evaluation of f and D. We recall that $\alpha=1$ and $\alpha=1/2$ are usually referred to as the Ito and Stratonovich discretization schemes [12]. For nonconstant $D(q)$, the value of α is important. Actually, in the $\epsilon\to 0$ limit we can replace Eq. (4.3) by

$$q_{n+1}=q_n+\epsilon[f(q_n)+(1-\alpha)D'(q_n)\xi_n^2]+\sqrt{\epsilon}D^{1/2}(q_n)\xi_n+\cdots, \tag{4.4}$$

and read the associated Fokker-Planck equation

$$\frac{\partial P}{\partial t}=\frac{1}{2}\frac{\partial}{\partial q}\left(D^{1-\alpha}\frac{\partial}{\partial q}(D^{\alpha}P)\right)-\frac{\partial}{\partial q}(fP). \tag{4.5}$$

In our case, however, $D\equiv 1$ is a constant and the same continuum limit is obtained for each value of α. However, the equivalence of the two discretizations is not obvious at first sight because the Jacobian of the change of variables (4.1) depends on α. Actually, the Jacobian of the transformation $\{q\}\to\{\xi\}$ expressed by Eq. (4.3) with periodic boundary conditions $\xi_N=\xi_0$, $q_N=q_0$ is proportional to

$$J=\det\begin{pmatrix}-1-\alpha\epsilon f_0' & 1-(1-\alpha)\epsilon f_1' & 0 & \cdots & 0\\ 0 & -1-\alpha\epsilon f_1' & 1-(1-\alpha)\epsilon f_2' & \cdots & 0\\ \cdots & & & & \\ 1-(1-\alpha)\epsilon f_0' & 0 & 0 & \cdots & -1-\alpha\epsilon f_{N-1}'\end{pmatrix},\quad f_n\equiv f(q_n), \tag{4.6}$$

which we can compute

$$J=\prod_{k=0}^{N-1}(-1-\alpha\epsilon f'_k)+(-1)^{N-1}\prod_{k=0}^{N-1}(1-(1-\alpha)\epsilon f'_k)$$

$$\overset{\epsilon\to 0}{\sim} e^{\alpha\int f'(q(t))dt}[1-e^{-\int f'(q(t))dt}]. \tag{4.7}$$

On the other hand, in the discrete Gaussian action

$$\frac{1}{2}\sum_n \xi_n^2=\frac{1}{2\epsilon}\sum\{q_{n+1}-q_n-\epsilon[\alpha f_n+(1-\alpha)f_{n+1}]\}$$

$$=\frac{1}{2\epsilon}\sum_n\{(q_{n+1}-q_n)^2+\epsilon^2 f_n^2\}$$

$$-\sum_n(q_{n+1}-q_n)(\alpha f_n+(1-\alpha)f_{n+1}). \tag{4.8}$$

We need some care to take the limit $\epsilon\to 0$ in the last term. However, if we write

$$\sum_n (q_{n+1}-q_n)[\alpha f_n+(1-\alpha)f_{n+1}]$$

$$=\sum_n(q_{n+1}-q_n)\left[\frac{1}{2}(f_n+f_{n+1})+\frac{1-2\alpha}{2}(f_{n+1}-f_n)\right], \tag{4.9}$$

then the correct continuum limit can be read, it depends on α and is

$$\int f(q)dq+\left(\frac{1}{2}-\alpha\right)\int f'(q(t))dt. \tag{4.10}$$

Putting this result together with J we find a final expression which is independent from α and reproduces precisely Eq. (3.3). In numerical simulations, the most convenient choice of α is $\alpha=1$ because in this case q_{n+1} may be evaluated directly from q_n without solving any equation.

B. Doubling problem

As is well known, naive discrete fermion actions are plagued by the doubling problem [13]. In a formulation based on a discrete Nicolai map the fermion action is constrained by the bosonic one (another manifestation of supersymmetry) and should be free from doublers. In this section we briefly explain how the discrete Nicolai map accomplishes this task by implementing naturally Wilson type lattice fermions. To see this, let us adopt the standard notation

$$\nabla^+ f_n=f_{n+1}-f_n,\quad \nabla^- f_n=f_n-f_{n-1},$$

$$\nabla^S f_n=\frac{1}{2}(\nabla^++\nabla^-)f_n,$$

$$\nabla^A f_n=\frac{1}{2}(\nabla^+-\nabla^-)f_n.$$

The doubling problem is related to the kinetic term only and in this section we set $f\equiv 0$. We begin with the $(0+1)$-dimensional case: the Ito discretization of equation $\dot{q}=\xi$ is based on the replacement $\dot{q}\to\nabla^+ q$. This fixes the fermion matrix $\partial\xi/\partial q$ and leads to the fermion action

$$S_F=\sum_n \bar{\psi}_n(\psi_{n+1}-\psi_n). \tag{4.11}$$

On the other hand, since $\nabla^+=\nabla^S+\nabla^A$, we can identify in Eq. (4.11) the naive fermion action plus a Wilson term:

$$S_F=\frac{1}{2}\sum_n\bar{\psi}_n(\psi_{n+1}-\psi_{n-1})+\frac{1}{2}\sum_n\bar{\psi}_n(\psi_{n+1}-2\psi_n+\psi_{n-1}). \tag{4.12}$$

A similar mechanism happens in 1+1 dimensions. For instance, the discrete Nicolai map proposed in [8] for the WZ models is a cubic symmetric version of $\dot{q}\to\nabla^+ q$. The continuum map

$$\partial_1\phi+i\sigma_2\partial_2\phi=\xi,\quad \phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\quad \xi=\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix},$$

$$\sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \tag{4.13}$$

may be discretized as[1]

$$(\nabla_1^++\nabla_2^A)\phi_1+\nabla_2^S\phi_2=\xi_1, \tag{4.14a}$$

$$(\nabla_1^--\nabla_2^A)\phi_2-\nabla_2^S\phi_1=\xi_2, \tag{4.14b}$$

which, using $\nabla^\pm=\nabla^S\pm\nabla^A$, leads to the fermion action

$$S_F=\sum_n\bar{\psi}_n[\nabla_1^S+i\sigma_2\nabla_2^S+\sigma_3(\nabla_1^A+\nabla_2^A)]\psi, \tag{4.15}$$

namely, (after a redefinition of ψ) the naive discrete action plus a Wilson term.

C. Some exact properties of the discrete maps

In Sec. III A and Sec. III B we discussed some analytical tools for the study of the existence of periodic solutions of the Langevin equation as well as for the determination of their number $N(\xi)$.

Some of those conclusions hold also in the discrete case. Let us begin with the $(0+1)$-dimensional case. The Ito discretization

$$q_{n+1}=q_n+\epsilon f(q_n)+\sqrt{\epsilon}\xi_n, \tag{4.16}$$

of the open problem (3.7) gives

[1]With this choice of ∇^+ and ∇^- the cross terms in $\xi_1^2+\xi_2^2$ cancel.

$$\frac{dq_N}{dq_0}=\prod_n (1+\epsilon f'(q_n)). \tag{4.17}$$

Hence, in the case $f'(q)>0$ we have

$$\frac{d}{dq_0}(q_N-q_0)>0, \tag{4.18}$$

and (at least for asymptotic $f\sim q^{2n+1}$ with positive n) it is easy to prove the existence of a unique periodic sequence $\{q_n\}$ for each periodic $\{\xi_n\}$.

Concerning the WZ models, we now show that also in the discrete equations there are not zero modes when ϕ is constant. The equation for the zero mode λ is [see Eq. (3.28)]

$$[\nabla_1^S+i\sigma_2\nabla_2^S+\sigma_3(\nabla_1^A+\nabla_2^A-u_1)-\sigma_1 u_2]\lambda=0,$$

$$u_i=\frac{\partial u}{\partial \phi_i}\bigg|_{\phi=\phi_0}. \tag{4.19}$$

Expanding the periodic λ in the Fourier series

$$\lambda_i(x_1,x_2)=\sum_{k_1,k_2} c_{i,k_1,k_2}e^{i(k_1x_1+k_2x_2)}, \tag{4.20}$$

we find the determinant of the operator in Eq. (4.19):

$$\det[\nabla_1^S+i\sigma_2\nabla_2^S+\sigma_3(\nabla_1^A+\nabla_2^A-u_1)-\sigma_1 u_2]$$
$$=\prod_{k_1,k_2}\left[-\sin^2 k_1-\sin^2 k_2-u_2^2 -\left(u_1+2\sin^2\frac{k_1}{2}+2\sin^2\frac{k_2}{2}\right)^2\right], \tag{4.21}$$

which is not zero unless $u_1=u_2=0$.

D. Stochastic identities

The stochastic identities relate fermion correlation functions to stochastic averages involving the solution q of the Langevin equation and the noise ξ. In this paper we shall need only the simplest of them. To prove it at the discrete level we write

$$\langle\Omega(q)\xi_\alpha\rangle$$
$$=\int\frac{d\xi_0}{\sqrt{2\pi}}\cdots\frac{d\xi_{N-1}}{\sqrt{2\pi}}e^{-1/2(\xi_0^2+\cdots+\xi_{N-1}^2)}\Omega(q(\xi))\xi_\alpha$$
$$=-\int\frac{d\xi_0}{\sqrt{2\pi}}\cdots\frac{(de^{-1/2\xi_\alpha^2})}{\sqrt{2\pi}}\cdots\frac{d\xi_{N-1}}{\sqrt{2\pi}}\Omega(q(\xi))$$
$$=\int\frac{d\xi_0}{\sqrt{2\pi}}\cdots\frac{d\xi_{N-1}}{\sqrt{2\pi}}e^{-1/2(\xi_0^2+\cdots+\xi_{N-1}^2)}\frac{\partial}{\partial\xi_\alpha}\Omega(q(\xi))$$
$$=\left\langle\frac{\partial\Omega}{\partial q_\beta}\frac{\partial q_\beta}{\partial\xi_\alpha}\right\rangle. \tag{4.22}$$

However, in terms of the fermion matrix,

$$J_{\alpha\beta}=\frac{\partial\xi_\alpha}{\partial q_\beta}, \tag{4.23}$$

this means

$$\langle\Omega(q)\xi_\alpha\rangle=\langle\partial_\lambda\Omega(q)(J^{-1})_{\lambda\alpha}\rangle, \tag{4.24}$$

and, in particular,

$$\langle q_\alpha\xi_\beta\rangle=\langle(J^{-1})_{\alpha\beta}\rangle=\langle\bar\psi_\alpha\psi_\beta\rangle, \tag{4.25}$$

which expresses the fermion propagator in terms of the q-ξ correlation.

V. NUMERICAL SIMULATION

From the very existence of a local Nicolai map and previous discussions it follows an algorithm for the numerical computation of supertraces. The first step is the extraction of the Gaussian random numbers $\{\xi_n\}$. Then, let ϕ_n be the field obeying a discretized version of Eq. (2.1); we must find the initial condition ϕ_0 such that

$$\Delta(\phi_0)=\|\phi_N(\phi_0)-\phi_0\|=0. \tag{5.1}$$

In 0+1 dimensions it is easy to identify all solutions of Eq. (5.1) as well as the sign of the Jacobian determinant of the Nicolai map. For the WZ models in 1+1 dimensions the problem is harder. However, as we have seen, at least for large separations of the zeroes ϕ^* of $f(\phi)$ we can expect to have one solution for each ϕ^*. Therefore we can use $\{\phi^*\}$ as starting guesses and take for $\det\partial\xi/\partial q$ its value at ϕ^*. In the simplest case $f(\phi)=\mu\phi+g\phi^2$ we have $\phi^*=0,-\mu/g$ and the above regime is obtained for large μ/g.

If an operator $\mathcal{O}(\phi)$ is averaged as in Eq. (2.4) over the realizations of $\{\xi\}$ we obtain an estimate of the supertrace $\mathrm{Str}[\mathcal{O}\exp(-\beta H)]$ and when $\beta\to+\infty$ we obtain $\langle 0|\mathcal{O}|0\rangle$ in the case of unbroken SUSY. In the free (0+1)-dimensional case $f(q)=\mu q$ (and similarly in the free WZ models) we can solve analytically the discrete equations and determine the correlation functions with the stochastic algorithm or in field theory.

The discrete Langevin equation is

$$q_{n+1}=\omega q_n+\sqrt{\epsilon}\ \xi_n,\quad \omega=1+\epsilon\mu. \tag{5.2}$$

The linear system

$$\begin{pmatrix}\omega & -1 & 0 & \cdots & 0\\ 0 & \omega & -1 & \cdots & 0\\ & & \cdots & & \\ -1 & 0 & 0 & \cdots & \omega\end{pmatrix}\begin{pmatrix}q_0\\ q_1\\ \cdots\\ q_{N-1}\end{pmatrix}=-\sqrt{\epsilon}\begin{pmatrix}\xi_0\\ \xi_1\\ \cdots\\ \xi_{N-1}\end{pmatrix} \tag{5.3}$$

is easily solved. The inverse of the first factor is

$$\frac{1}{\omega^N-1}\cdot\begin{pmatrix} \omega^{N-1} & \cdots & \omega & 1 \\ 1 & \cdots & \omega^2 & \omega \\ & \cdots & & \\ \omega^{N-2} & \cdots & 1 & \omega^{N-1} \end{pmatrix} \tag{5.4}$$

and therefore the initial value is

$$q_0 = -\sqrt{\epsilon}\frac{1}{\omega^N-1}(\omega^{N-1}\xi_0+\cdots+1\cdot\xi_{N-1}). \tag{5.5}$$

The two-point function is

$$C_k = \langle q_0 q_k\rangle = \frac{\epsilon}{(\omega^N-1)^2}\sum_{i=0}^{N-1}\omega^{i \bmod N+(i+k) \bmod N}, \tag{5.6}$$

and after a straightforward algebra

$$C_k = \frac{2\epsilon\ \omega^{N/2}}{(\omega^2-1)(\omega^N-1)}\cosh\left(\left(k-\frac{N}{2}\right)\log\omega\right). \tag{5.7}$$

If we take the limit $N\to\infty$ with $\epsilon=\beta/N$ and introduce the time variable $\tau=\epsilon k$ we obtain

$$\mathrm{Str}(qe^{-\tau H}qe^{-\beta H}) = \lim_{N\to\infty}\langle q_0 q_k\rangle = \frac{1}{2\mu}\frac{\cosh(\mu(\tau-\beta/2))}{\sinh(\mu\beta/2)} \tag{5.8}$$

and

$$\lim_{\beta\to+\infty}\mathrm{Str}(qe^{-\tau H}qe^{-\beta H}) = \langle 0|q(0)q(\tau)|0\rangle = \frac{1}{2\mu}\exp(-\mu\tau). \tag{5.9}$$

From the field theoretical point of view, we are computing $\mathrm{Str}(qe^{-\tau H}qe^{-\beta H})$ with the free action

$$S=\frac{1}{2}\int_0^\beta dt(\dot{q}^2+\mu^2 q^2), \tag{5.10}$$

and periodic boundary conditions. The generating functional is

$$Z = \left\langle \exp\left(\int\ d\tau Jq\right)\right\rangle = \exp\left(\int d\tau d\tau' J(\tau)G(\tau-\tau')J(\tau')\right), \tag{5.11}$$

where

$$G(\tau) = \frac{1}{2\beta}\sum_{k=-\infty}^{\infty}\frac{1}{\mu^2+\dfrac{4\pi^2k^2}{\beta^2}}\exp\left(-\frac{2\pi i k}{\beta}\tau\right), \tag{5.12}$$

and

$$\mathrm{Str}(qe^{-\tau H}qe^{-\beta H}) = 2G(\tau). \tag{5.13}$$

If we use the summation formula

$$\sum_{k=-\infty}^{\infty}\frac{\exp(-i\alpha k)}{k^2+\lambda^2} = \frac{\pi}{\lambda\ \sinh(\pi\lambda)}\times\cosh(\lambda(\alpha-\pi)),\quad 0<\alpha<2\pi, \tag{5.14}$$

we obtain again the result of Eq. (5.8). A similar computation can be carried out for the fermionic propagator. From the solution of Eq. (5.2) we have

$$\langle q_k\xi_0\rangle\frac{1}{\sqrt{\epsilon}} = \frac{\omega^{k-1}}{1-\omega^N},\quad \langle q_0\xi_k\rangle\frac{1}{\sqrt{\epsilon}} = \frac{\omega^{N-k-1}}{1-\omega^N}, \tag{5.15}$$

which can be combined to give

$$\langle q_k\xi_l\rangle\frac{1}{\sqrt{\epsilon}} = \omega^{k-l-1}\left[\frac{1}{1-\omega^N}-\theta(l-k)\right], \tag{5.16}$$

which has the correct continuum limit

$$\langle q(\tau)\xi(\tau')\rangle = e^{\mu(\tau-\tau')}\left[\frac{1}{1-e^{\mu\beta}}-\theta(\tau'-\tau)\right] = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{1}{\dfrac{2\pi i n}{\beta}-\mu}\exp\left(\frac{2\pi i n}{\beta}(\tau-\tau')\right), \tag{5.17}$$

associated to the fermion propagator in $0+1$ dimensions and confirming the stochastic identity.

In the interacting case, ϕ_0 must be determined by some iterative algorithm. In $1+1$ dimensions it is fundamental to start from a good guess. We use the Newton-Raphson algorithm [14] to solve iteratively the set of nonlinear equations

$$\Delta_i = \phi_i^{(N)} - \phi_i^{(0)} = 0, \tag{5.18}$$

where $\phi_i^{(k)}$ is the ith spatial component of ϕ at the kth time slice. The correction $\delta\phi_i^{(0)}$ in

$$\phi_i^{(0)}\to\phi_i^{(0)}+\delta\phi_i^{(0)}, \tag{5.19}$$

is given by

$$\frac{\partial\Delta_i}{\partial\phi_j^{(0)}}\ \delta\phi_j^{(0)} = -\Delta_i. \tag{5.20}$$

The scheme is made more robust by introducing a relaxation parameter ω_R in the update of $\phi_i^{(0)}$:

$$\phi_i^{(0)}\to\phi_i^{(0)}+\omega_{\mathrm{R}}\ \delta\phi_i^{(0)},\quad 0<\omega_{\mathrm{R}}<1. \tag{5.21}$$

Another help against numerical instabilities is to require

TABLE I. Lightest boson (m_B) and fermion (m_F) masses as functions of g at $\mu=4$, $\beta=5$ on a $T=200$ lattice in the (0+1)-dimensional model with drift $f(q)=-\mu q-gq^3$.

g	m_B	m_F	$m_{\rm pert}$
0.0	4.00(3)	4.01(3)	4.0000
0.1	4.04(3)	4.04(3)	4.0368
0.2	4.07(3)	4.08(3)	4.0750
0.3	4.11(3)	4.11(3)	4.1062
0.4	4.14(3)	4.14(3)	4.1388
0.5	4.17(3)	4.18(3)	4.1699
0.6	4.20(3)	4.21(3)	4.1997
0.7	4.23(3)	4.24(3)	4.2281
0.8	4.26(3)	4.27(3)	4.2550

$$\frac{|\delta\phi^{(0)}|}{|\phi^{(N)}|}<\rho \quad \text{or simply} \quad |\delta\phi^{(0)}|<\rho, \tag{5.22}$$

where ρ is a minimum correction threshold. The choice of the optimal $\omega_{\rm R}$ and ρ must be done empirically, but we did not find it to be critical.

Finally, another general trick which is useful to improve numerical stability is to follow a bootstrap procedure and solve the problem on a $L\times(T-1)$ lattice to provide a guess for the $L\times T$ problem.

As a numerical test of the algorithm we measure the boson and fermion propagators in simple (0+1)- and (1+1)-dimensional cases. In 1+1 dimensions, to gain statistics, we average over the spatial dimension and sum over all pairs of time slices with fixed temporal separation. Moreover, we explicitly symmetrize the propagators under $\tau\to\beta-\tau$.

The simplest interacting system in 0+1 dimensions is

$$f(q)=-\mu q-gq^3, \quad \mu,g>0, \tag{5.23}$$

where dynamical breaking of supersymmetry does not occur. In Table I we show the lightest mass as a function of g at $\mu=4$, $\beta=5$ on a $T=200$ lattice. It is obtained by fitting the boson and fermion propagators where the latter is computed by means of the stochastic identity (4.25). We also show the $\mathcal{O}(g^2)$ perturbative value

$$E_1=\mu+\frac{3}{2\mu}g-\frac{9}{2\mu^3}g^2+O(g^3). \tag{5.24}$$

In Fig. 1, just to give an example at the critical point $\mu=0$ we plot the two propagators at $g=1$ and $\mu=0$ computed with $\beta=10$ on a $T=100$ lattice. As expected, the slope of the logarithmic plots is the same.

In 1+1 dimensions, we simulate the WZ model with $f(\phi)=\mu\phi+g\phi^2$. In Fig. 2, we show the boson and fermion propagators evaluated at $\mu=4$, $g=0.1$ on a 20×90 lattice with $\epsilon_s=0.1$ and $\epsilon_t=0.01$, where ϵ_s and ϵ_t are the space and time discretization steps. The continuous line is the fit with the Ansatz

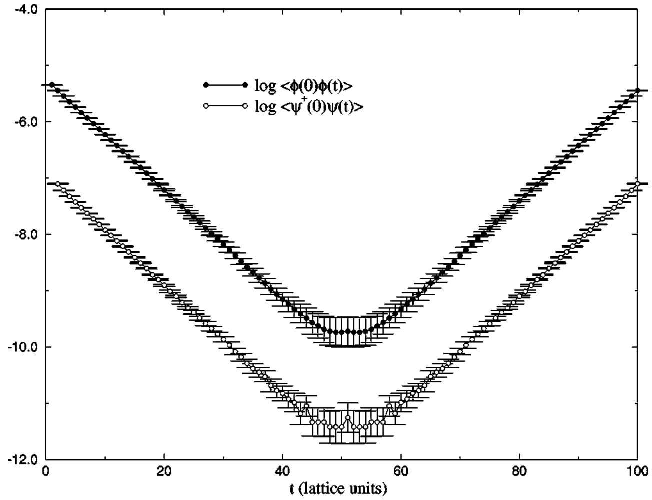

FIG. 1. Boson and fermion propagators for the (0+1)-dimensional model with drift $f(q)=-q^3$ at $\beta=10.0$ on a $T=100$ lattice. Apart from the different normalization the slopes of the two logarithmic plots are equal.

$$C(\tau)=A_0\cosh\left[\mu\left(\tau-\frac{1}{2}\beta\right)\right]. \tag{5.25}$$

On a 20×50 lattice we have varied g with the results reported in Table II together with the one loop value of the doublet mass which is

$$m(g)=\mu-\frac{2}{3\sqrt{3}}\frac{g^2}{\mu}. \tag{5.26}$$

Finite time step errors can be investigated in a first approximation by studying numerically the finite lattice propagator integrated over space. This, in standard notation, is

$$D(k)=\sum_{n_t}\exp(k \;\; \epsilon_t p_t)\frac{1}{\hat{p}_t^2+\mu^2}, \tag{5.27}$$

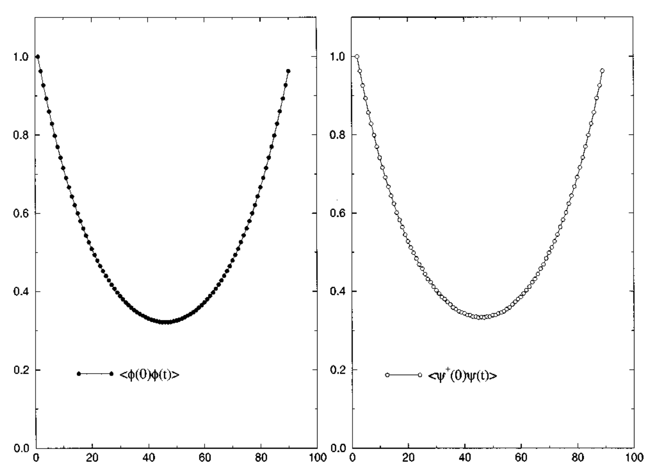

FIG. 2. Boson and Fermion propagators for the WZ model with drift $f(\phi)=\mu\phi+g\phi^2$ at the point $\mu=4$, $g=0.1$ on a 20×90 lattice with space and time steps $\epsilon_s=0.1$, $\epsilon_t=0.01$.

TABLE II. Lightest boson (m_B) and fermion (m_F) masses as functions of g at $\mu=4$, $\beta=0.5$ on a 20×50 lattice for the WZ model associated to $f(\phi)=\mu\phi+g\phi^2$.

g	m_B	m_F	m_{pert}
0.0	4.00(5)	4.00(5)	4.000
0.1	4.00(5)	3.99(5)	3.999
0.4	3.99(5)	3.99(5)	3.985
0.6	3.98(5)	3.98(5)	3.965
1.2	3.96(5)	3.97(5)	3.859

where

$$p_t=\frac{2\pi}{\beta}n_t, \quad n_t=0\cdots T-1, \quad \text{and} \quad \hat{p}_t=\frac{2}{\epsilon_t}\sin\frac{1}{2}\epsilon_t p_t. \tag{5.28}$$

One can study at fixed β, the difference in the propagator as ϵ_t is varied. We checked that at $\mu=4$, $\epsilon_t=0.01$ and $T=50$ the finite step effects are negligible.

We also remark that with this particular choice of $f(\phi)$ we can sample a single zero ($\phi^*=0$) without violating supersymmetry. The reason is that the shifted field $\tilde{\phi}=(\phi_1-\mu/g,\phi_2)$ obeys the same equations as ϕ but with $\mu\to-\mu$. For $\langle\phi\phi\rangle$ and the symmetrized $\langle\overline{\psi}\psi\rangle$ this change has no consequences.

VI. REMARKS AND CONCLUSIONS

In this paper we have shown that the existence of a local Nicolai map in supersymmetric models has useful consequences for numerical computations. It allows the formulation of a simulation algorithm which generates statistically independent field configurations by solving a Langevin equation with periodic boundary conditions. The so-called stochastic identities can be exploited to avoid Grassmann fields. The method is feasible and consistent numerical results are obtained in 0+1 dimensions and also in 1+1 dimension WZ models even if with some constraint in parameter space. Further developments are possible in the direction of more robust integration schemes for the Langevin equation as well as in the application to more realistic models. In particular, work is in progress on cases where the Nicolai map is determined perturbatively [15] and in four-dimensional QCD [4,5] where the Jacobian of the local Nicolai map is constant.

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