General relativistic corrections to the Sagnac effect

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The difference in travel time of corotating and counterrotating light waves in the field of a central massive and spinning body is studied. The corrections to the special relativistic formula are worked out in a Kerr field. An estimation of numeric values for the Earth and satellites in orbit around it show that a direct measurement is on the order of concrete possibilities. [S0556-2821(98)06616-8]

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I. INTRODUCTION

The fact that the round trip time for a light ray moving along a closed path (thanks to suitably placed mirrors) when its source is on a turntable varies with the angular speed ω of the platform may be thought classically as obvious. Furthermore that time, for a given ω, will be different if the beam is corotating or counterrotating: longer in the former case, shorter in the latter. This difference in times, when superimposing the two oppositely rotating beams, leads to a phase difference with consequent interference phenomena or, in the case of standing waves, to a frequency shift and ensuing beats. According to Stedman [1] this phenomenon was anticipated by Lodge at the end of the 19th century and by Michelson at the beginning of the 20th. Experiments were actually performed by Harress [1,3], without being aware of what he observed, and by Sagnac [4] in 1913 and the interference effect we are speaking of was since named after him. Sagnac was looking for an ether manifestation and his approach was entirely classical, but a special relativistic explanation was soon found giving, to lowest order in ω, the same formula for the time lag between the two light beams

$$\delta\tau = 4\frac{S}{c^2}\omega. \tag{1}$$

S is the area of the projection of the closed path followed by the waves to contour the platform, orthogonal to the rotation axis, c is the speed of light, and ω is the rotational velocity of the source or receiver. The phenomenon is manifested for any kind of waves, including matter waves. The Sagnac effect has indeed been tested for light, x rays [5], and various types of matter waves, such as Cooper pairs [6], neutrons [7], Ca^{40} atoms [8], and electrons [3]. A lot of different deductions of Eq. (1) have been given all showing the universal character of the phenomenon; examples are Refs. [6,9–17]. Basically the Sagnac effect is a consequence of the break of the univocity of simultaneity in rotating systems [18]: this was recognized very early and has also had a direct experimental verification using identical atomic clocks slowly transported around the world [19].

The Sagnac effect has found a variety of applications both for practical purposes and fundamental physics, especially after the generalized introduction, after the 1960's, of lasers and ring lasers [2] allowing unprecedented precisions in interferometric and frequency shift measurements. The great accuracy of these measurements poses the problem of higher order corrections to Eq. (1), which have been sought for, usually in the special relativistic approach. It seems, however, not to be unreasonable to consider also general relativistic effects due to the fact that the "turntable" is massive or that the observer is orbiting a massive and rotating body. This is precisely the scope of the present paper. A previous work with an aim similar to this was published by Cohen and Mashhoon [20]; they worked in parametrized post-Newtonian (PPN) first order approximations and obtained results consistent with those presented in this paper.

Section II contains the derivation of the delay in returning to the starting point for a pair of oppositely rotating light beams in a Kerr field, in the case of an equatorial trajectory of the rotating observer. Both exact and approximated results are obtained. In Sec. III the case of a polar trajectory is treated. Section IV specializes the formulas for a freely falling observer (circular equatorial orbit). Section V presents some numerical estimates of the corrections to the usual Sagnac effect, due to the mass and angular momentum of the Earth. Finally Sec. VI contains a short discussion of the possibility to measure some of the calculated corrections.

II. SAGNAC EFFECT ON A MASSIVE ROTATING BODY

The Kerr metric describes a rotating black hole (actually a rotating ring singularity). We begin studying it because it allows for some exact results and, when suitably approximated, may be used to describe the gravitational field around a rotating massive body. The Kerr line element in Boyer-Lindquist space-time coordinates is [21]

$$\begin{aligned} ds^2 =& \frac{r^2 - 2G(M/c^2)r + a^2/c^2}{r^2 + (a^2/c^2)\cos^2\theta}\left(c\,dt - \frac{a}{c}\sin^2\theta\, d\phi\right)^2 \\ &- \frac{\sin^2\theta}{r^2+(a^2/c^2)\cos^2\theta}\left[\left(r^2+\frac{a^2}{c^2}\right)d\phi - a\,dt\right]^2 \\ &- \frac{r^2+(a^2/c^2)\cos^2\theta}{r^2-2G(M/c^2)r+a^2/c^2}dr^2 \\ &- \left(r^2+\frac{a^2}{c^2}\cos^2\theta\right)d\theta^2. \end{aligned}$$

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Here M is the (asymptotic) mass of the source and a is the ratio between the angular momentum J and the mass:

$$a=\frac{J}{M}.$$

Everything is seen and measured from its effects far away from the black hole, where space-time is practically flat.

A. Equatorial effect

Let us now assume that the source or receiver of two oppositely directed light beams is moving around the rotating black hole which generates the gravitational field, along a circumference on the equatorial plane. Suitably placed mirrors send back to their origin both beams after a circular trip about the central hole.

In this case $r=R=\text{const}$ and $\theta=\pi/2$; the line element is

$$ds^2=\frac{R^2-2G(M/c^2)R+a^2/c^2}{R^2}\left(c\,dt-\frac{a}{c}d\phi\right)^2-\frac{1}{R^2}\left[\left(R^2+\frac{a^2}{c^2}\right)d\phi-a\,dt\right]^2.$$

Let us then assume that the rotation is uniform, so that the rotation angle of the source or observer is

$$\phi_0=\omega_0 t. \tag{2}$$

Then

$$ds^2=\left\{\frac{R^2-2G(M/c^2)R+a^2/c^2}{R^2}\left(1-\frac{a}{c^2}\omega_0\right)^2-\frac{1}{R^2}\left[\left(R^2+\frac{a^2}{c^2}\right)\frac{\omega_0}{c}-\frac{a}{c}\right]^2\right\}(c\,dt)^2. \tag{3}$$

For light moving along the same circular path it must be $ds=0$ which happens when

$$\frac{R^2-2G(M/c^2)R+a^2/c^2}{R^2}\left(1-\frac{a}{c^2}\omega\right)^2-\frac{1}{R^2}\left[\left(R^2+\frac{a^2}{c^2}\right)\frac{\omega}{c}-\frac{a}{c}\right]^2=0. \tag{4}$$

Now ω is an unknown; solving Eq. (4) for it one finds two values:

$$\Omega_\pm=\frac{1}{a^2/c^2+2G(M/c^4R)a^2+R^2}\times\left(2G\frac{M}{c^2R}a\pm c\sqrt{\frac{a^2}{c^2}+R^2-2G\frac{M}{c^2}R}\right). \tag{5}$$

Ω_- is actually negative when R exceeds the Schwarzschild limit $2G(M/c^2)$.

The rotation angles for light are then

$$\phi_\pm=\Omega_\pm t. \tag{6}$$

Eliminating t between Eqs. (2) and (6),

$$\phi_\pm=\frac{\Omega_\pm}{\omega_0}\phi_0.$$

Now we proceed by applying the geometrical four-dimensional approach that may be found in Refs. [11,22,18]. The first intersection of the world lines of the two light rays with the one of the orbiting observer after the emission at time $t=0$ is when

$$\phi_+=\phi_0+2\pi,$$

$$\phi_-=\phi_0-2\pi,$$

i.e.,

$$\frac{\Omega_\pm}{\omega}\phi_0=\phi_0\pm 2\pi.$$

Solving for ϕ_0,

$$\phi_{0\pm}=\mp\frac{2\pi\omega_0}{\Omega_\pm-\omega_0}=\mp\frac{2\pi\omega_0}{\{1/[a^2/c^2+2G(M/c^4R)a^2+R^2]\}\{[2G(M/c^2R)a\pm c\sqrt{a^2/c^2+R^2-2G(M/c^2)R}]-\omega_0\}}. \tag{7}$$

The proper time of the rotating observer is deduced from Eq. (3) calling in Eq. (2):

$$d\tau=\sqrt{\left(R^2-2G\frac{M}{c^2}R+\frac{a^2}{c^2}\right)\left(1-\frac{a}{c^2}\omega_0\right)^2-\left[\left(R^2+\frac{a^2}{c^2}\right)\frac{\omega_0}{c}-\frac{a}{c}\right]^2}\,\frac{d\phi_0}{R\omega_0}.$$

Finally, integrating between ϕ_{0-} and ϕ_{0+}, we obtain the Sagnac delay

$$\delta\tau=\sqrt{\left(R^2-2G\frac{M}{c^2}R+\frac{a^2}{c^2}\right)\left(1-\frac{a}{c^2}\omega_0\right)^2-\left[\left(R^2+\frac{a^2}{c^2}\right)\frac{\omega_0}{c}-\frac{a}{c}\right]^2}\,\frac{\phi_{0+}-\phi_{0-}}{R\omega_0}$$

or explicitly [using Eq. (7)]

$$\delta\tau = \frac{4\pi}{c^6 R} \frac{(a^2Rc^2 + 2GMa^2 + R^3c^4)\omega_0 - 2c^2GMa}{\sqrt{1-(2/R)G(M/c^2) + 4G(M/c^4R)a\omega_0 - [a^2/c^4 + 2G(M/c^6R)a^2 + (R^2/c^2)]\omega_0^2}}. \tag{8}$$

This result has some features which are typical of a Kerr geometry. We see, for instance, that the delay is zero when the angular speed of the orbiting observer is

$$\omega_n = \frac{2c^2GMa}{a^2Rc^2 + 2GMa^2 + R^3c^4} = 2\frac{(GM/c^2R)(a/R^2)}{1+2(GM/c^2R)(a^2/c^2R^2) + a^2/c^2R^2}$$

and provided $a \neq 0$.

This is the velocity of the ''locally nonrotating observers'' of the Kerr geometry [23]. these are equivalent to the static (with respect to distant stars) observers of the Schwarzschild geometry for which no Sagnac effect would be present either.

Vice versa when the observer keeps a fixed position with respect to distant stars ($\omega_0 = 0$) a time lag, hence a Sagnac effect, is still present, again under the condition that $a \neq 0$. The time lag is

$$\delta\tau_{(\omega=0)} = \delta\tau_0 = -8\pi\frac{GM}{c^4R}\frac{a}{\sqrt{1-2(GM/c^2R)}} = -8\pi\frac{G}{c^4R}\frac{J}{\sqrt{1-2(GM/c^2R)}}. \tag{9}$$

Cohen and Mashhoon [20] found the first order approximation of this same result, which they actually calculated for a static observer sending a pair of light beams in opposite directions along a closed triangular circuit, rather than along a circumference.

The delay (9) is nothing else than the gravitational analogue of the Bohm-Aharonov effect [24]. In fact the Sagnac effect is a sort of inertial Bohm-Aharonov effect [10,25] and what we found is an exact expression for a rotating ring singularity, whereas Ref. [26] gives an approximated but not simpler result.

Now recalling the Lense-Thirring effect one has a precession velocity [17,2,27] which, in our geometry and notation, for an equatorial observer is

$$\omega_{LT} = -\frac{GJ}{c^2R^3}.$$

We see that

$$\delta\tau_0 = 8\frac{\omega_{LT}}{c^2}\frac{\pi R^2}{\sqrt{1-2(GM/c^2R)}}.$$

The quantity $\delta\tau_0$ doubles the Sagnac delay due to the Lense and Thirring precession, i.e., to the pure drag by the rotating mass.

B. Approximations

As we have seen, the deduction of exact results in a Kerr metric, at least in the special conditions we assumed, is rather straightforward, but of course in most cases many terms in the equations are very small. This means that a series of approximations are in order, though it is not necessary to introduce them from the very beginning as others did [28,29].

Let us first assume that $\beta = \omega_0 R/c \ll 1$, consequently developing Eq. (8) in powers of β and retaining only terms up to the second order. The result is

$$\begin{aligned}\delta\tau \simeq & -8\frac{\pi}{c^4R}GM\frac{a}{[1-(2/R)G(M/c^2)]^{1/2}} \\ & + \frac{4\pi R}{c[1-(2/R)G(M/c^2)]^{3/2}}\left(1+\frac{a^2}{R^2c^2} - 2\frac{GM}{c^2R}\right)\beta \\ & -12\pi\frac{GMa}{c^4R}\frac{1+a^2/c^2R^2 - (2/R)(GM/c^2)}{[1-(2/R)G(M/c^2)]^{5/2}}\beta^2\end{aligned}$$

or

$$\begin{aligned}\delta\tau \simeq & \delta\tau_0 + \frac{4\pi}{c[1-(2/R)G(M/c^2)]^{3/2}}\left(1+\frac{a^2}{R^2c^2} - 2\frac{GM}{c^2R}\right) \\ & \times\left(R\beta - \frac{GMa}{c^3R}\frac{3}{1-(2/R)G(M/c^2)}\beta^2\right).\end{aligned}$$

Now assume also that $\epsilon = GM/c^2R \ll 1$. To first order in ϵ it is

$$\begin{aligned}\delta\tau \simeq & -8\frac{\pi}{c^2}a\epsilon + 4\pi\frac{R}{c}\left(1+\frac{a^2}{R^2c^2}\right)\beta \\ & + \left[-8\pi\frac{R}{c} + 12\pi\frac{R}{c}\left(1+\frac{a^2}{R^2c^2}\right)\right]\epsilon\beta \\ & -12\pi\frac{a}{c^2}\left(1+\frac{a^2}{R^2c^2}\right)\epsilon\beta^2.\end{aligned}$$

If a/Rc is at least as small as ϵ,

$$\delta\tau \simeq -8\frac{\pi}{c^2}a\epsilon + 4\pi\frac{R}{c}(1+\epsilon)\beta - 12\pi GM\frac{a}{c^4R}\beta^2$$

explicitly and calling $\delta\tau_S$ the usual Sagnac effect

$$\delta\tau \simeq -8\pi a\frac{GM}{c^4 R}+4\pi\frac{R}{c}\left(1+\frac{GM}{c^2 R}\right)\beta-12\pi\frac{GM}{c^4 R}a\beta^2$$
$$=\delta\tau_S-8\pi a\frac{GM}{c^4 R}+4\pi\frac{R}{c^2}\frac{GM}{c^2}\omega_0-12\pi R\frac{GM}{c^4}\frac{a}{c^2}\omega_0^2 \tag{10}$$

evidencing the angular momentum

$$\delta\tau \simeq \delta\tau_S-8\pi\frac{GJ}{c^4 R}+4\pi\frac{R}{c^2}\frac{GM}{c^2}\omega_0-12\pi R\frac{GJ}{c^6}\omega_0^2. \tag{11}$$

The usual Sagnac effect is recovered when the terms containing GM and J are negligible. On the other side, a second order correction in ω_0^2 (β^2) is present only if the angular momentum of the source is considered.

In these approximations the terms containing J coincide with the first order (in J) corrections to the Schwarzschild field. This fact allows us to apply the formulas to the simple case of a rotating spherical object whose radius is R_0. Now the angular momentum may be expressed as $J=I\Omega_0$, where Ω_0 is the rotational velocity of the sphere and I is its moment of inertia. If, just to fix ideas, we assume the object to have uniform density ρ, one has

$$I=\frac{8}{15}\rho\pi R_0^5=\frac{2}{5}MR_0^2.$$

Hence the value for a is approximately

$$a\simeq\frac{2}{5}R_0^2\Omega_0.$$

Then for a fixed observer looking at the Earth from the distance R it comes out that

$$\delta\tau_0\simeq-\frac{64}{15}\pi^2\frac{G\rho}{c^4}\frac{R_0^5\Omega_0}{R}=-\frac{16}{5}\pi\frac{GM}{c^4}\frac{R_0^2}{R}\Omega_0.$$

III. POLAR (CIRCULAR) ORBIT

It may be interesting to study a circular trajectory contouring the central mass passing over the poles also. In this case it is again $r=R$, but now $\phi=$const and, retaining uniform motion, $\theta=\omega_0 t$. Then

$$ds^2=\frac{R^2-2G(M/c^2)R+a^2/c^2}{R^2+(a^2/c^2)\cos^2(\omega_0 t)}c^2dt^2-\frac{\sin^2(\omega_0 t)}{R^2+(a^2/c^2)\cos^2(\omega_0 t)}a^2dt^2-\left[R^2+\frac{a^2}{c^2}\cos^2(\omega_0 t)\right]\omega_0^2dt^2. \tag{12}$$

For light it is of course $ds=0$ which happens when

$$\left(R^2-2G\frac{M}{c^2}R+\frac{a^2}{c^2}\right)c^2-a^2\sin^2\theta-\left(R^2+\frac{a^2}{c^2}\cos^2\theta\right)^2\left(\frac{d\theta}{dt}\right)^2=0.$$

Solving for the angular speed we find that it is no longer constant:

$$\frac{d\theta}{dt}=\pm\frac{\sqrt{[R^2-2G(M/c^2)R+(a^2/c^2)]c^2-a^2\sin^2\theta}}{R^2+(a^2/c^2)\cos^2\theta}.$$

This differential equation is easily solvable when $a^2/c^2R^2\ll 1$. To first order and assuming $t=0$ when $\theta=0$,

$$t\simeq\frac{R}{c[1-2G(M/c^2R)]^{1/2}}\theta+\frac{a^2[1-4G(M/c^2R)]}{2c^3R[1-2G(M/c^2R)]^{3/2}}\int_0^\theta\cos^2\theta'\,d\theta',$$

i.e.,

$$t\simeq\frac{R}{c[1-2G(M/c^2R)]^{1/2}}\theta+\frac{a^2[1-4G(M/c^2R)]}{4c^3R[1-2G(M/c^2R)]^{3/2}}(\cos\theta\sin\theta+\theta),$$

and finally

$$t\simeq\left[\frac{R}{c[1-2G(M/c^2R)]^{1/2}}+\frac{a^2[1-4G(M/c^2R)]}{4c^3R[1-2G(M/c^2R)]^{3/2}}\right]\theta+\frac{a^2[1-4G(M/c^2R)]}{8c^3R[1-2G(M/c^2R)]^{3/2}}\sin(2\theta).$$

At the same time the rotating observer describes the angle θ_0 while light travels an angle $2\pi\pm\theta_0$ (+ for the corotating beam, − for the counterrotating one):

$$\frac{\theta_0}{\omega_0}=\left[\frac{R}{c[1-2G(M/c^2R)]^{1/2}}+\frac{a^2[1-4G(M/c^2R)]}{4c^3R[1-2G(M/c^2R)]^{3/2}}\right]\times(2\pi\pm\theta_0)\pm\frac{a^2[1-4G(M/c^2R)]}{8c^3R[1-2G(M/c^2R)]^{3/2}}\sin(2\theta_0).$$

Assume, as we did already, a low speed observer and we expect $2\theta_0$ to be little enough for $\sin(2\theta_0)\simeq 2\theta_0$. Then

$$\frac{\theta_0}{\omega_0}=\left[\frac{R}{c[1-2G(M/c^2R)]^{1/2}}+\frac{a^2[1-4G(M/c^2R)]}{4c^3R[1-2G(M/c^2R)]^{3/2}}\right](2\pi\pm\theta_0)\pm\frac{a^2[1-4G(M/c^2R)]}{4c^3R[1-2G(M/c^2R)]^{3/2}}\theta_0$$

Solving for θ_0 one obtains two results,

$$\theta_{0\pm}=\pi\frac{2c^2R^2[1-2G(M/c^2R)]+\frac{1}{2}a^2[1-(4GM/c^2R)]}{(c^3R/\omega_0)[1-2G(M/c^2R)]^{3/2}\mp c^2R^2[1-2G(M/c^2R)]\mp\frac{1}{2}a^2[1-(4GM/c^2R)]}.$$

Finally the difference in round trip times as seen from an inertial reference frame (recalling the approximation already used for the solution of this case) results in

$$t_+-t_-=\frac{\theta_{0+}-\theta_{0-}}{\omega_0}$$

$$\simeq\pi\frac{R^2}{c^2}\frac{4[1-2(GM/c^2R)]^2+\dfrac{[3+7\beta^2-6(GM/c^2R)]}{[1+\beta^2-2(GM/c^2R)]}[1-6(GM/c^2R)+8(G^2M^2/c^4R^2)](a^2/c^2R^2)}{[1-2(GM/c^2R)]^3+[1-2(GM/c^2R)]^2\beta^2}\omega_0. \tag{13}$$

For $a=0$ the usual relativistic Sagnac effect is recovered. To first order in ϵ Eq. (13) becomes

$$t_+-t_-\simeq\pi\frac{R^2}{c^2}\frac{\omega_0}{1+\beta^2}\left(4+\frac{3+7\beta^2}{1+\beta^2}\frac{a^2}{c^2R^2}+\frac{8}{1+\beta^2}\frac{GM}{c^2R}\right)$$

and finally to first order in β,

$$t_+-t_-\simeq\pi\frac{R^2}{c^2}\left(4+3\frac{a^2}{c^2R^2}+8\frac{GM}{c^2R}\right)\omega_0. \tag{14}$$

The correction for the moment of inertia of the source is interestingly independent from R. It is indeed

$$3\pi\frac{a^2}{c^4}\omega_0,$$

which for a sphere in nonrelativistic approximation is

$$\frac{12}{25}\pi\frac{R_0^4}{c^4}\Omega_0\omega_0.$$

In order to obtain what the rotating observer sees the result must be expressed in terms of his proper time. This is done on the basis of Eq. (12):

$$\tau=\int\left\{\frac{R^2-2G(M/c^2)R+(a^2/c^2)}{R^2+(a^2/c^2)\cos^2(\omega_0t)}-\frac{\sin^2(\omega_0t)}{R^2+(a^2/c^2)\cos^2(\omega_0t)}\frac{a^2}{c^2}-\left[R^2+\frac{a^2}{c^2}\cos^2(\omega_0t)\right]\frac{\omega_0^2}{c^2}\right\}^{1/2}dt.$$

For short enough time intervals the integrand may be approximated as

$$\left[\frac{1-2G(M/c^2R)+a^2/c^2R^2}{1+a^2/c^2R^2}-\left(1+\frac{a^2}{c^2R^2}\right)\frac{R^2\omega_0^2}{c^2}\right]^{1/2}+O(t^2)$$

and, after integration,

$$\tau\simeq\left[\frac{1-2G(M/c^2R)+a^2/c^2R^2}{1+a^2/c^2R^2}-\left(1+\frac{a^2}{c^2R^2}\right)\frac{R^2\omega_0^2}{c^2}\right]^{1/2}t.$$

Adopting the usual approximations,

$$\tau\simeq\sqrt{1-2G\frac{M}{c^2R}-R^2\frac{\omega_0^2}{c^2}}\,t.$$

Then

$$\delta\tau_p\simeq\sqrt{1-2G\frac{M}{c^2R}-R^2\frac{\omega_0^2}{c^2}}\,(t_+-t_-)$$

and explicitly (first order in β and ϵ)

$$\delta\tau_p\simeq\pi\frac{R^2}{c^2}\left(4+3\frac{a^2}{c^2R^2}+4\frac{GM}{c^2R}\right)\omega_0$$

$$=\delta\tau_S+\frac{\pi}{c^4}(3a^2+4RGM)\omega_0. \tag{15}$$

Comparing with the ‘‘equatorial’’ situation one has

$$\delta\tau-\delta\tau_p \simeq -8\pi a G\frac{M}{c^4 R}-3\pi\frac{a^2}{c^4}\omega_0. \tag{16}$$

IV. GEODESICS

Now we specialize the previous results to a freely falling observer: his orbit will then be geodesic. If u^μ is the velocity four-vector and $\Gamma^\mu_{\nu\lambda}$ the Christoffel symbols, the equation of the geodetics is $\partial u^\mu/\partial s+\Gamma^\mu_{\nu\alpha}u^\alpha u^\nu=0$, where s coincides with the observer's proper time τ.

Continuing to use Boyer-Lindquist coordinates (generalization of Schwarzschild coordinates) we are interested in constant radius orbits for which

$$r=R,$$

$$u^r=0.$$

From the geodesic equations and applying these conditions one obtains the angular speed of the motion about the symmetry axis $\omega=u^\phi/u^0$; actually there are two different values for the two possible choices of the rotation with respect to the orientation of the angular momentum of the source. These angular velocities are in general complicated functions of θ; this is no problem as long as $\theta=$const, i.e., $u^\theta=0$. Considering this simplified situation and introducing the Christoffel symbols appropriate to the Kerr metric, the rotation speeds turn out to be

$$\omega_\pm=\frac{2aGMc^2\pm c^2\sqrt{3a^2G^2M^2+GMc^4R^3}}{a^2GM-c^4R^3}. \tag{17}$$

Recalling now Eq. (8) and using Eq. (17) it is possible to find an exact expression for the time lag for a freely falling object in circular equatorial orbit.

It is, however, simpler to develop Eq. (17) up to first order in a/cR:

$$\omega_\pm\simeq\mp\frac{c}{R}\sqrt{G\frac{M}{c^2R}}-\frac{2}{R^2}\frac{GM}{c^2R}a. \tag{18}$$

Recalling Eq. (10) and introducing Eq. (18) we end up with

$$\delta\tau_\pm\simeq 8\pi a\frac{GM}{c^4R}\pm 4\pi\frac{R}{c}\left(1+\frac{GM}{c^2R}\right)\left(\sqrt{\frac{GM}{c^2R}}+2\frac{GM}{c^2R}\frac{a}{cR}\right)$$
$$\simeq\mp 4\pi\frac{R}{c}\sqrt{\frac{GM}{c^2R}}+16\pi a\frac{GM}{c^4R}.$$

Now the traditional Sagnac effect is

$$\delta\tau_{S\pm}=\pm\frac{4\pi}{c^2}\sqrt{GMR} \tag{19}$$

so we may write

$$\delta\tau_\pm\simeq\delta\tau_{S\pm}+16\pi a\frac{GM}{c^4R}. \tag{20}$$

V. NUMERICAL ESTIMATES

It is interesting to estimate numerical values for the corrections in the case of the Earth as a central body. Now the relevant data are

$$R_\oplus=6.37\times10^6\ \text{m},$$

$$\Omega_\oplus=7.27\times10^{-5}\ \text{rad/s},$$

$$G\frac{M_\oplus}{c^2}=4.4\times10^{-3}\ \text{m},$$

$$a_\oplus=9.81\times10^8\ \text{m}^2/\text{s}.$$

On the surface of the Earth and if the circular path of the light rays were the equator, the usual Sagnac delay would be

$$\delta\tau_S=4.12\times10^{-7}\ \text{s}. \tag{21}$$

This quantity can be converted into a fringe shift multiplying by the frequency ν of the light as seen by the observer:

$$\Delta=\nu\,\delta\tau_S. \tag{22}$$

Considering that for visible light $\nu\sim10^{14}$ Hz one has a titanic shift of $\sim10^7$ fringes. This number makes sense only if the source has a coherence length as big as at least 123.6 m which is much but not impossible. What actually matters, however, is the value of Eq. (22) modulo an integer number, which is of course a fraction of a fringe. The problem is that the knowledge of Δ requires an accuracy of better than, say, 1 part in 10^8 and this in turn depends mainly on the accuracy and stability of the parameters entering the expression of $\delta\tau_S$.

The correction due to the pure mass contribution $4\pi(R_\oplus/c^2)(GM_\oplus/c^2)\Omega_\oplus$, is 2.84×10^{-16} s, nine orders of magnitude smaller than the main term. The corresponding fringe shift is $\sim10^{-2}$.

The correction calling in the moment of inertia of the planet at the lowest order in $\Omega_\oplus$, $-8\pi a(GM/c^4R)$, is -1.89×10^{-16} s. Again a $\sim10^{-2}$ fringe shift. These shifts are in principle observable, provided one could find the reference pattern from which they should be measured, i.e., the value of Δ modulo an integer number. Finally the last correction in Eq. (10) $-12\pi(GM_\oplus/c^6)R_\oplus a\Omega_\oplus^2$, is -6.76×10^{-28}: overwhelmingly small.

Let us now consider an orbiting geodetic observer and assume, just to fix numbers, that its orbit radius is $R=7\times10^6$ m. The main Sagnac term is (19), whose numeric value is

$$\delta\tau_S=7.35\times10^{-6}\ \text{s}. \tag{23}$$

The fringe shift is $\sim10^8$ and the necessary coherence length would be greater than ~1000 m. Considering that one is now able to emit light pulses as short as $\sim 10^{-9}$ s or less, both Sagnac delays (21) and (23) could be measured directly as such.

The first correction to Eq. (23) is $16\pi a_{\oplus}(GM_{\oplus}/c^4R)$ whose value is 4.16×10^{-16} s, i.e., $\sim 10^{-2}$ fringes. If the orbit is polar with the same radius and angular velocity $\omega_0 = 1/R\sqrt{(1/R)GM}$, the corrections are [see Eq. (15)] $\pi/c^4(3a^2+4RGM)\omega_0$, i.e., $(\pi/c^4)(3a_{\oplus}^2/R)\sqrt{G(M_{\oplus}/R)} + 4(\pi/c^4)GM_{\oplus}\sqrt{G(M_{\oplus}/R)}$. The value of the first term is 1.39×10^{-18} s ($\sim 10^{-4}$ fringes) and that of the second is 4.84×10^{-15} s ($\sim 10^{-1}$ fringes). Considering the mass contribution, the situation is a little bit better than for the equatorial orbit. Furthermore, when the difference (16) is evaluated we obtain precisely 1.39×10^{-18} s: this, as we said, is of the order of 10^{-4} fringes. It is a very small value, but it is obtained comparing two experimental fringe patterns without any reference to the basic Sagnac effect.

VI. DISCUSSION

Starting from the exact results for a Kerr metric and considering suitable approximations of them we have obtained the corrections to the Sagnac effect that the mass and angular momentum of a rotating object introduce. These are conceptually important, evidencing and strengthening the analogy between the Sagnac effect and the Bohm-Aharonov effect: particularly relevant to this purpose is the $\delta\tau_0$ of Eq. (9). Unfortunately, when considering the Earth as the source of the gravitational field the corrections are indeed very tiny, but per se in the range of what current optical interference measurements allow, provided a convenient zero (''pure'' Sagnac term) is experimentally fixed.

When considering devices such as ring lasers, where standing oppositely propagating waves form, the Sagnac time difference is automatically converted into a frequency shift and in general a fractional frequency shift may well be easier to measure than the equivalent fringe shift. Of course, here the difficulty is in stabilizing standing electromagnetic waves around the Earth, either in space or on the surface of the planet. However, what is hard for light might not be so using radiowaves, provided their Sagnac effect was not reduced too much.

Apparently there is also the possibility to exploit the difference between clockwise and counterclockwise rotating observers. In fact, considering Eqs. (19) and (20), we see that

$$\Delta(\delta\tau) = \delta\tau_+ - |\delta\tau_-| = 32\pi a \frac{GM}{c^4R}.$$

Numerically, for satellites orbiting the Earth at $R=7\times10^6$ m, one has $\Delta(\delta\tau)=5.8\times10^{-27}$, corresponding to a difference in the positions of the interference patterns of $\sim 10^{-13}$ fringes: absolutely unperceivable. Summarizing we conclude that experiments to test the existence of the lowest order general relativistic corrections to the basic Sagnac effect we computed are in the range of feasibility.

[1] R. Anderson and G. E. Stedman, Found. Phys. Lett. **7**, 273 (1994); A. A. Michelson, Philos. Mag. **8**, 716 (1904); both quoted in the review paper [2].

[2] G. E. Stedman, Rep. Prog. Phys. **60**, 615 (1997).

[3] F. Hasselbach and M. Nicklaus, Phys. Rev. A **48**, 143 (1993); **48**, 152 (1993).

[4] G. Sagnac, C. R. Hebd. Seances Acad. Sci. **157**, 708 (1913).

[5] A. H. Rostomyan and A. M. Rostomyan, Phys. Status Solidi A **126**, 29 (1991).

[6] J. E. Zimmermann and J. E. Mercereau, Phys. Rev. Lett. **14**, 887 (1965).

[7] S. A. Werner, J. L. Staudenmann, and R. Colella, Phys. Rev. Lett. **42**, 1103 (1979).

[8] F. Riehle, Th. Kisters, A. Witte, J. Helmcke, and Ch. J. Bordé, Phys. Rev. Lett. **67**, 177 (1991).

[9] M. Dresden and Chen Ning Yang, Phys. Rev. D **20**, 1846 (1979).

[10] J. J. Sakurai, Phys. Rev. D **21**, 2993 (1980).

[11] J. Anandan, Phys. Rev. D **24**, 338 (1981).

[12] H. C. Lefevre and H. J. Arditty, Appl. Opt. **21**, 1400 (1982).

[13] B. Prade and J.-Y. Vinet, Nuovo Cimento B **101**, 323 (1988).

[14] B. H. W. Hendriks and G. Nienhuis, Quantum Opt. **2**, 13 (1990).

[15] G. C. Scorgie, Eur. J. Phys. **12**, 64 (1991).

[16] D. Dieks, Eur. J. Phys. **12**, 253 (1991).

[17] B. Mashhoon, Phys. Lett. A **173**, 347 (1993).

[18] G. Rizzi and A. Tartaglia, Centro Vito Volterra, Università di Roma Tor Vergata Report. No. 299, Found. Phys. (to be published).

[19] D. W. Allan and M. A. Weiss, Science **228**, 69 (1985).

[20] J. M. Cohen and B. Mashhoon, Phys. Lett. A **181**, 353 (1993).

[21] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 877.

[22] A. A. Logunov and Yu. V. Chugreev, Sov. Phys. Usp. **31**, 861 (1988).

[23] Misner, Thorne, and Wheeler, *Gravitation* [21], p. 895.

[24] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[25] M. D. Semon, Found. Phys. **12**, 49 (1982).

[26] E. G. Harris, Am. J. Phys. **64**, 378 (1996).

[27] M. H. Soffel, *Relativity in Astrometry, Celestial Mechanics and Geodesy* (Springer-Verlag, Berlin, 1989).

[28] B. Mashhoon, Found. Phys. **15**, 497 (1985).

[29] R. N. Henriksen, Can. J. Phys. **64**, 175 (1985); B. Chaboyer and R. N. Henriksen, Phys. Lett. A **132**, 391 (1988).