A point mass in an isotropic universe: Existence, uniqueness, and basic properties

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Criteria which a space-time must satisfy to represent a point mass embedded in an open Robertson-Walker (RW) universe are given. It is shown that McVittie's solution in the case $k=0$ satisfies these criteria, but does not in the case $k=-1$. The existence of a solution for the case $k=-1$ is proven and its representation in terms of an elliptic integral is given. The following properties of this and McVittie's $k=0$ solution are studied; uniqueness, the behavior at future null infinity, the recovery of the RW and Schwarzschild limits, the compliance with energy conditions, and the occurrence of singularities. The existence of solutions representing more general spherical objects embedded in a RW universe is also proven. [S0556-2821(98)05716-6]

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I. INTRODUCTION

This paper deals with the embedding of massive objects in Robertson-Walker (RW) universes. There are three ways in which the physical embedding may be modelled and thus treated mathematically. First, one can treat the body as a test body whose dynamics are described by a suitable set of equations of motion (e.g., geodesic equations for a test particle, equations derived from the Nambu action for cosmic strings). Secondly, the history of the surface of the object can be treated as a boundary Σ^-, which is then matched with a diffeomorphic surface Σ^+ in the ''exterior'' RW geometry. The usual matching conditions are the continuity of the first and second fundamental forms of $\Sigma \cong \Sigma^{\pm}$ [1]. This technique has been used to study the formation and evolution of voids in cosmology [2], and to study domain walls [3].

The third method is to solve Einstein's field equations, exactly or approximately, in such a way that the resulting solution can be interpreted as an embedding of some massive object in a RW background. Two landmark papers in this vein are those of McVittie [4], who gave solutions of Einstein's equations with perfect fluid source which have been claimed to represent the embedding of the Schwarzschild field in the three families ($k=0,\pm 1$) of RW space-times, and of Hawking [5], who studied gravitational radiation from a bound source in the $k=-1$ dust filled RW space-time. There also exist several papers dealing with the *superposition* of the Kerr-Newman and RW space-times (see [6], or [7] for a summary). We stress that we are considering the embedding of extended objects of finite size, so that the extensive studies (see, e.g., [8]) of perturbations which occur throughout the universe do not concern us here.

These three approaches incorporate various degrees of coupling between the mass-energy of the extended body and the geometry of the universe at large. In particular physical situations, one of the three provides an appropriate model. For example, in cosmology, the galactic source of some observable effect is treated as a test body moving on a timelike geodesic of the RW geometry; the Einstein-Straus vacuole [9] provides a description of the effect of the cosmic expansion on the gravitational field of the sun, but McVittie's solution is a more appropriate description of the gravitational field outside a supermassive spherical body in an otherwise uniform RW space-time.

An interesting result was derived recently by Senovilla and Vera regarding the cylindrical analogue of these models [10]. String dynamics in an RW universe deals with extended bodies which are limits of cylindrical objects and is well understood. However, moving to the next level of coupling, the aforementioned authors showed that no static cylindrical region can be matched continuously to a RW universe. Similiar results have also been obtained for the axially symmetric case [11]. Since real strings have internal structure, this implies that at this level, strings cannot be embedded in a RW universe. The same is true for any static locally cylindrical objects; coins, bottles and (true) cylinders. This begs the question: Can the third type of embedding be carried out for cylindrical objects? That is, can we find an exact solution of Einstein's equation representing a cylindrical object embedded in a RW space-time?

We will not attempt to answer this question here, but by examining carefully the spherical case, suggest how the problem may be approached. Thus we readdress the problem first discussed by McVittie, but from a modern point of view.

We find that McVittie's solution in the case $k=0$ satisfactorily describes a massive particle embedded in a RW universe, but that his $k=-1$ solution does not. Motivated by the differences between these two solutions, we lay down *a priori* conditions that a space-time (V,g) must satisfy to represent a massive particle embedded in a RW space-time. We provide a solution in the case $k=-1$ and discuss uniqueness in each case, which has not been done before. We emphasize that we have not found a new solution of Einstein's equations, but have determined the physical significance of a certain class of shear-free spherically symmetric perfect fluid solutions (see Chap. 14 of [12]). Furthermore, we discuss the following properties (mathematical and physical) of the solutions: (a) representation of the $k=-1$ solution by an elliptic integral, (b) recovery of the Schwarzschild solution in the vacuum limit, (c) behavior at future null infinity, (d) compliance with energy conditions and (e) existence and nature of the central singularity. A central tool in this analysis is Hawking's quasilocal mass [5]. We show by this example

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how such quasilocal constructions can be used to obtain boundary conditions for Einstein's equations useful for obtaining solutions in particular situations. Since we are dealing with asymptotics in open space-times, the case $k=+1$ is excluded from our discussion.

The structure of the paper is as follows. In the next section we review McVittie's solution and the Hawking mass and point out problems with the interpretation of the former in the case $k=-1$. In Sec. III, we set out conditions for a space-time (V,g) to represent a massive particle embedded in a RW universe. Using these conditions and Einstein's field equations, we show how the problem reduces to finding a solution, with a certain asymptotic behavior, of a nonlinear second order differential equation. In Sec. IV, we prove the existence of such a solution, and discuss uniqueness. Using this solution, the properties listed above are discussed in Sec. V, and we make some concluding comments in Sec. VI. The global struture of these space-times is to be analyzed in a subsequent paper.

II. MCVITTIE'S SOLUTION AND THE HAWKING MASS

In 1933, McVittie [4] found solutions of Einstein's field equations for a perfect fluid energy-momentum tensor, representing a Schwarzschild field embedded in the RW space-times. His solutions can be written [13] (using units in which $c=G=1$)

$$ds^2=-\left(\frac{1-m/2w}{1+m/2w}\right)^2 dt^2+e^{\beta}\left(1+\frac{m}{2w}\right)^4 \times\{dr^2+h^2(d\theta^2+\sin^2\theta d\phi^2)\}, \tag{2.1}$$

where

$$m=m(t), \quad \beta=\beta(t), \quad \dot{\beta}=-2(\dot{m}/m), \tag{2.2}$$

and here and throughout, an overdot indicates partial differentiation with respect to t (a prime will be used for differentiation with respect to the variable r). The functions $h(r),w(r)$ depend on a choice of $k(=-1,0,+1)$, the Riemannian curvature of the surfaces of homogeneity $t=$const in the background RW universe:

$$h(r)=\begin{cases}\sinh r, & k=-1;\\ r, & k=0;\\ \sin r, & k=+1;\end{cases} \qquad w(r)=\begin{cases}2\sinh\frac{r}{2}, & k=-1;\\ r, & k=0;\\ 2\sin\frac{r}{2}, & k=+1.\end{cases}$$

The isotropic pressure $p_{\rm mv}$ and the energy density $\rho_{\rm mv}$ obtained from Einstein's field equations are given by

$$8\pi p_{\rm mv}=-\frac{3}{4}\dot{\beta}^2-\ddot{\beta}\left(\frac{1+\frac{m}{2w}}{1-\frac{m}{2w}}\right)-\frac{ke^{-\beta}}{\left(1-\frac{m}{2w}\right)\left(1+\frac{m}{2w}\right)^5}, \tag{2.3}$$

$$8\pi\rho_{\rm mv}=\frac{3}{4}\dot{\beta}^2+\frac{3ke^{-\beta}}{\left(1+\frac{m}{2w}\right)^5}. \tag{2.4}$$

The properties of this solution have been summarized by Raychaudhuri as follows (cf. [14], p. 97): ''The McVittie solution follows uniquely under the following conditions: (i) The line element is spherically symmetric with a singularity at the center; (ii) the energy-stress tensor is that of a perfect fluid; (iii) the fluid motion is shear free; (iv) the metric must asymptotically go over to the isotropic cosmological form.'' (It should be noted that neither a proof of this statement, nor a reference to one is given; McVittie's *ad hoc* approach does not include such a proof.)

The function $m=m_0e^{-\beta/2}$ for some constant m_0 by Eq. (2.2), and is interpreted as the mass at the singularity. When this is set equal to zero, the line element (2.1) is that of a RW space-time.

The characterization of McVittie's solution quoted above is unsatisfactory, as points (i) and (iv) refer to properties which are deduced simply by looking at the metric tensor components relative to the line element (2.1). We can show that point (iv) in particular is misleading. This point seems to imply that the solution corresponds to a point mass embedded in the RW geometry, so that the gravitational field is asymptotically that of a RW space-time.

Hawking [5] has made this notion precise with a renormalized (against the RW background) quasilocal mass measured at future null infinity $\mathcal{I}^+$. Since we are dealing with asymptotic regions of the space-time, we restrict our attention to the cases $k=-1,0$.

The Hawking mass is defined by analogy with the Bondi mass [15] of a bound source of gravitation in an asymptotically flat space-time; it measures the mass of a bound source of gravitation in an asymptotically RW universe. The additions to the total (infinite) mass from the RW background are subtracted away in a gauge invariant manner, as we describe now. The construction is valid in any space-time.

We use the null tetrad $\{l^a,n^a,m^a,\overline{m^a}\}$, where l^a is chosen to be an outgoing null vector, and take v to be an affine parameter along the integral curves of l^a, so that

$$l^a=\frac{dx^a}{dv}. \tag{2.5}$$

Taking S to be a spacelike 2-sphere orthogonal to l^a and n^a (so that $v=$const on S), the quasilocal mass surrounded by S is defined to be

$$M(S)=\kappa\int(-\Psi_2-\sigma\lambda+\Phi_{11}+\Lambda)dS, \tag{2.6}$$

where

$$\kappa = \frac{1}{(4\pi)^{3/2}} \left(\int dS \right)^{1/2}$$

and the terms in the integrand have their usual meanings in Newman-Penrose notation. In the appropriate limits, M yields the Bondi mass and the ADM mass, and is the spherical version of Hayward's improved quasilocal mass [16].

The renormalization is carried out by subtracting the local (fluid) matter which manifests itself in the Ricci tensor terms, and to leave the nonlocal gravitational terms. To do this, Hawking [5] has defined

$$M_1(S) = \kappa \int (\Phi_{11} + \Lambda) dS, \tag{2.7}$$

and

$$M_2(S) = \kappa \int (-\Psi_2 - \sigma\lambda) dS. \tag{2.8}$$

In order that the bound source mass is measured at $\mathcal{I}^+$,

$$\tilde{M}_2 = \lim_{v \to \infty} M_2, \tag{2.9}$$

is defined to be the mass of the model.

Using a suitable null-tetrad, we can evaluate Eq. (2.8) for McVittie's space-time (notice that due to spherical symmetry $\lambda = \sigma = 0$). We find

$$M_2(S) = m_0 \frac{h^5}{w^5}. \tag{2.10}$$

Thus for $k=0$, wherein $h=w=r$, we have $M_2 = m_0$, and so the Hawking mass is

$$\tilde{M}_2 = m_0, \tag{2.11}$$

which verifies the interpretation of m_0 as being the mass of a point particle embedded in the RW cosmos in this case. However in the case $k=-1$, since $r \to \infty$ as $v \to \infty$, we find that

$$M_2(S) \to \infty, \quad \text{as} \quad v \to \infty \tag{2.12}$$

and so the renormalized mass is infinite. Thus the $k=-1$ McVittie solution *does not* represent a point mass embedded in a RW space-time.

The main aim of this paper, then, is to provide a solution of Einstein's equation which does represent a point mass embedded in the $k=-1$ RW space-time.

III. DESCRIPTION OF SPACE-TIME REPRESENTING A POINT MASS EMBEDDED IN AN RW UNIVERSE

In this section, we give three conditions **(C1–C3)** on space-time (V,g) which, if these conditions are satisfied, we postulate to represent a point mass embedded in a RW universe. These conditions are motivated by the discussion above. For convenience, we will use the symbol (M,g) to refer to such a space-time.

Condition C1. (M,g) *is spherically symmetric with a shear-free perfect fluid energy-momentum tensor.*

Demanding a perfect fluid energy-momentum tensor and spherical symmetry are obvious requirements for the space-time we seek to describe; the requirement that the fluid flow lines be shear-free is a convenience that eases integration of the field equations without ruling out the existence of a solution. There does not seem to be any *a priori* reason why the shear should be set equal to zero.

Under these assumptions, the line element can be written as [12]

$$ds^2 = -e^{\nu} dt^2 + e^{\mu}\{dr^2 + h^2(r) d\omega^2\}, \tag{3.1}$$

where $\nu = \nu(r,t)$, $\mu = \mu(r,t)$ and h is an arbitrary function of r, which we may always assume is one of the functions $h(r)$ of Sec. II. Doing so maintains the connection with the corresponding forms of the RW space-times.

The density and pressure obtained from Einstein's field equations are given by

$$8\pi\rho = \frac{3}{4}\dot{\mu}^2 e^{-\nu} - e^{-\mu}\left\{ \mu'' + \frac{1}{4}\mu'^2 + 2\frac{h'}{h}\mu' + 3\frac{h''}{h} \right\}, \tag{3.2}$$

$$8\pi p = e^{-\mu}\left\{ \frac{1}{2}(\mu'' + \nu'') + \frac{1}{4}\nu'^2 + \frac{1}{2}\frac{h'}{h}(\nu' + \mu') + \frac{h''}{h} \right\} - e^{-\nu}\left\{ \ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{\dot{\mu}\dot{\nu}}{2} \right\}. \tag{3.3}$$

The remaining field equations reduce to [12]

$$e^{\nu} = \dot{\mu}^2 e^{-g(t)}, \tag{3.4}$$

$$\mu'' - \frac{1}{2}\mu'^2 - \frac{h'}{h}\mu' = F(r) e^{-\mu/2}, \tag{3.5}$$

where $g(t)$ and $F(r)$ are arbitrary functions of their arguments.

The Weyl tensor is Petrov type D, and on a naturally occurring null tetrad, the only nonzero Newman-Penrose component is

$$\Psi_2 = -\frac{1}{6} F(r) e^{-3\mu/2}. \tag{3.6}$$

Then the Hawking mass enclosed by any $t=$ const $r=$ const surface S is

$$M_2(S) = \frac{1}{6} h^3(r) F(r). \tag{3.7}$$

This surface S is a metric sphere of radius $R(r,t)$: $= h(r) e^{\mu/2}$. In the solutions we examine, $\partial R / \partial r > 0$, and so since $M_2(S)$ is independent of t, the limit $r \to \infty$ of Eq. (3.7) yields the Hawking mass at infinity.

The second condition achieves two things. First, it identifies the RW background against which the Hawking mass is measured, and secondly, it ensures that the limit $r\to\infty$ has physical significance. This condition is a minimal requirement that (M,g) ‘‘looks like’’ a RW space-time near infinity.

Condition C2:

$$\lim_{r\to\infty}\mu(r,t)=\beta(t)$$

for all t *in the range of* μ, *and where* β *is the function appearing in the line element of the RW universe,*

$$ds^2=-dt^2+e^{\beta(t)}\{dr^2+h^2(r)d\omega^2\}. \tag{3.8}$$

As we shall see below, it is not necessary to give the corresponding condition for ν ($\nu\to 0$ as $r\to\infty$), as this is ensured by a naturally arising choice of function of integration.

We turn now to the Hawking mass and consider how this may be used in our description. We wish to embed a finite nonzero mass in the RW universe, indicating that $M_2(S)$ should yield a finite positive number when measured at infinity, i.e.,

$$\lim_{r\to\infty}M_2(S)=m_0 \tag{3.9}$$

for some positive constant m_0. However this allows the possibility of surplus Hawking mass arbitrarily close to infinity. We will therefore consider the stronger condition,

$$M_2(S)=m_0, \tag{3.10}$$

for some positive constant m_0. This condition suggests that there is an isolated body of mass m_0 at the center of the space-time. In order to present a more general discussion in Sec. IV, we will deal with the condition (3.9), but focus on the special case of Eq. (3.10).

We see from the above that Eq. (3.9) implies

$$F(r)=O(h^{-3}) \quad \text{as} \quad r\to\infty,$$

while Eq. (3.10) gives

$$F(r)=\frac{6m_0}{h^3(r)}. \tag{3.11}$$

This last equation is interesting, as it is a necessary and sufficient condition for the energy density to be spatially homogeneous, i.e.,

$$F(r)=\text{const}\times h^{-3}(r)\Leftrightarrow\rho=\rho(t). \tag{3.12}$$

Thus to obtain the constant Hawking mass condition (3.11), or equivalently to express the fact that there is no ‘‘extra’’ energy density in the universe outside the embedded mass, we take the third condition to be as follows.

Condition C3. *The energy density* (3.2) *obeys* $\rho=\rho_0(t)$, *where* $\rho_0(t)$ *is the energy density calculated via Einstein's field equations of the RW space-time with line element* (3.8). *Furthermore, the constant quantity* $m_0:=h^3(r)F(r)/6$ *is positive.*

We can show now how to obtain $\nu\to\infty$ as $r\to\infty$. From the above, the energy-density of (M,g) is given by

$$8\pi\rho=\frac{3}{4}e^{g(t)}+3ke^{-\beta(t)},$$

while that of the RW background is given by

$$8\pi\rho_0=\frac{3}{4}\dot{\beta}^2+3ke^{-\beta(t)}.$$

[ρ_0 and p_0, the energy-density and pressure of the RW background may be read off from Eqs. (2.4) and (2.3) respectively by taking $m=0$.] Thus $\rho(t)=\rho_0(t)$ requires $e^{g(t)}=\dot{\beta}^2$. We will see below that there exists a solution satisfying the three conditions above for which μ is differentiable with respect to t, and which obeys $\lim_{r\to\infty}\dot{\mu}=\dot{\beta}$. Thus

$$\lim_{r\to\infty}e^{\nu}=\lim_{r\to\infty}\dot{\mu}^2e^{-g(t)}=1,$$

as claimed.

The physical problem has been modelled using the conditions **C1**–**C3** above. We see that the remaining mathematical analysis is to find a solution $\mu(r,t)$ of

$$e^{\mu/2}\left(\mu''-\frac{1}{2}\mu'^2-\frac{h'}{h}\mu'\right)=\frac{6m_0}{h^3}, \tag{3.13a}$$

with the boundary condition

$$\lim_{r\to\infty}\mu(r,t)=\beta(t), \tag{3.13b}$$

for all t in the range of μ.

More generally, we look for a solution of Eq. (3.13a) with the right hand side replaced by $O(h^{-3}(r))$ as $r\to\infty$. Our analysis below is based on this version of the equation, which corresponds to the condition (3.9).

We note at this stage that McVittie's solution [4] in the case $k=0$ satisfies conditions **C1**–**C3**. We focus henceforth on the case $k=-1$, and work under this assumption.

We conclude this section by giving a useful transformation which will make the problem (3.13) more manageable. Defining $\gamma=e^{-\mu/2}$ and $x=w^2$ (cf. Sec. II), these become respectively

$$\gamma_{xx}=G(x)\gamma^2, \tag{3.14a}$$

$$\lim_{x\to\infty}\gamma=e^{-\beta/2}, \tag{3.14b}$$

where

$$G(x)=-24m_0(x^2+4x)^{-5/2}=O(x^{-5}) \quad \text{as} \quad x\to\infty,$$

and the subscript indicates partial differentiation with respect to x. Using $G(x)=O(x^{-5})$ corresponds to the general case

(3.9). We now proceed to prove existence of a solution of the boundary value problem (3.14).

IV. EXISTENCE OF A SOLUTION AND UNIQUENESS CONSIDERATIONS

Writing $\gamma(x,t)=a(t)(1+Y(x,t))$ where $a(t):=e^{-\beta(t)/2}$ allows us to restate the problem (3.14) as

$$Y_{xx}=aG+2aGY+aGY^2,$$

$$Y(x,t)=o(1),\quad x\to\infty.$$

We treat x as a complex variable and solve the equation in a neighborhood of infinity which includes $\{x\in\mathcal{C}:\ \Im x=0,\ \Re x>x_0>0\}$ where x_0 is some real constant.

Consider the equation

$$Y_{xx}=aG+\delta(2aGY+aGY^2),\tag{4.1}$$

where $\delta>0$ is a small parameter. We look for a solution of this equation of the form

$$Y(x,t)=\sum_{n=0}^{\infty}\delta^n Y_n(x,t),\tag{4.2}$$

and having found one, show that for sufficiently large values of x, this converges in the limit $\delta=1$ and obeys $Y=o(1)$ as $x\to\infty$. This will prove existence of the required solution. We will use the ansatz $Y(x,t)=a^{n+1}(t)y_n(x)$, and throughout this section, a prime on y_n indicates differentiation with respect to argument.

To proceed, we fill out Eq. (4.1) using Eq. (4.2), and equate powers of δ. This leads to the system of equations

$$y_0''=G(x),\tag{4.3a}$$

$$y_n''=2G(x)y_{n-1}+G(x)\sum_{m=0}^{n-1}y_m y_{n-1-m},\quad n\geqslant 1.\tag{4.3b}$$

Clearly, this system admits solutions obeying

$$y_n(x)=o(1),\quad y_n'(x)=o(x^{-1}),\quad x\to\infty.$$

Then we can write

$$y_n(x)=\int_\infty^x y_n'(s)ds$$

$$=\int_\infty^x (x-s)y_n''(s)ds.$$

Transforming the integral to one over a finite contour via $z=s^{-1}$ and using basic bounds for integrals, we obtain

$$|y_n(x)|\leqslant 2|x^{-1}|\sup_{\{|x|<|s|\}}|s^3y_n''(s)|.$$

Then taking $\mathcal{D}$ to be a neighborhood of infinity contained in the intersection of $\{x\in\mathcal{C}:\ |x|>x_0>0\}$ and a sector containing the positive real axis, we have

$$|y_n(x)|\leqslant C\|W_n\|_{\mathcal{D}}$$

for all $x\in\mathcal{D}$, where $C:=2x_0^{-1}$, $W_n(x):=x^3y_n''(x)$ and $\|\cdot\|_{\mathcal{D}}$ is the supremum norm restricted to $\mathcal{D}$. So

$$\|y_n\|_{\mathcal{D}}\leqslant C\|W_n\|_{\mathcal{D}}.\tag{4.4}$$

Applying these definitions to Eq. (4.3b), we obtain

$$\|W_n\|_{\mathcal{D}}\leqslant 2B\|y_{n-1}\|_{\mathcal{D}}+B\sum_{m=0}^{n-1}\|y_m\|_{\mathcal{D}}\|y_{n-1-m}\|_{\mathcal{D}},$$

where $B=B(x_0)=\|x^3G(x)\|_{\mathcal{D}}$, and so using Eq. (4.4),

$$\|y_n\|_{\mathcal{D}}\leqslant 2A\|y_{n-1}\|_{\mathcal{D}}+A\sum_{m=0}^{n-1}\|y_m\|_{\mathcal{D}}\|y_{n-1-m}\|_{\mathcal{D}},$$

with $A=BC$.

Using this inequality, we can derive a geometric bound for the $\|y_n\|_{\mathcal{D}}$, which will suffice to prove the convergence properties of Σy_n required to show that this *formal* solution is a (convergent) solution. To see how, define the sequence of positive reals $\{b_n\}_{n=0}^{\infty}$ by

$$b_0=\|y_0\|_{\mathcal{D}},$$

$$b_n=2b_{n-1}+\sum_{m=0}^{n-1}b_m b_{n-1-m}.$$

Then we see from the last inequality that

$$\|y_n\|_{\mathcal{D}}\leqslant A^n b_n,\quad n\geqslant 0.\tag{4.5}$$

Consider the formal power series

$$P(X):=\sum_{n=0}^{\infty}b_nX^n.$$

From the recurrence relation for the b_n, we find that $P(X)$ obeys

$$P=b_0+X(2P+P^2),$$

the solution of which consistent with the definition of P is

$$P(X)=\frac{1-2X-[(1-2X)^2-4b_0X]^{1/2}}{2X}.\tag{4.6}$$

This is an analytic function of X in a neighborhood of the origin, and so for any $0<\lambda\in\mathcal{R}$ with $|\lambda|<$ radius of convergence of $P(X)$,

$$\sum_{n=0}^{\infty}b_n\lambda^n$$

is convergent. Then each term in this series must be bounded, i.e., there exists some positive real constant K such that

$$b_n\lambda^n<K,\quad n\geqslant 1,$$

and so

$$\|y_n\|_{\mathcal{D}}\leqslant K\left(\frac{A}{\lambda}\right)^n, \quad n\geqslant 1.$$

Then for x_0 sufficiently large, this last inequality will read

$$\|y_n\|_{\mathcal{D}}\leqslant \kappa^n, \tag{4.7}$$

for some $0<\kappa<1$. To see this, notice that as x_0 increases, the region $\mathcal{D}$ gets smaller, so that $A=2x_0^{-1}\|x^3G(x)\|_{\mathcal{D}}$ decreases [recall that $G(x)=O(x^{-5})$]. $b_0=\|y_n\|_{\mathcal{D}}$ is nonincreasing, which by Eq. (4.6) indicates that the radius of convergence of $P(X)$ in nondecreasing, allowing the use of nondecreasing values of λ.

The condition (4.7) is sufficient to imply that

$$\sum_{n=0}^{\infty} y_n(x)$$

converges uniformly on $\mathcal{D}$ (see, e.g., [17]). Hence

$$Y(x,t)=\sum_{n=0}^{\infty} a^{n+1}(t)y_n(x)$$

converges uniformly on some subset of $\mathcal{D}\times\mathcal{R}$ $(x\in\mathcal{D},\ t\in\mathcal{R})$. By our construction, this is a solution of Eq. (4.1) in the case $\delta=1$ obeying $Y(x,t)=o(1)$ as $x\to\infty$.

Furthermore, for each fixed value of x, the series

$$\sum y_n(x)a^{n+1}(t), \quad \sum y_n(x)(n+1)a^n(t)\frac{da}{dt}$$

are both uniformly convergent on some interval of the real t-axis (which will contain the set $\{t\in\mathcal{R}:|a(t)|<1\}$). Hence by standard results [18], $\partial Y/\partial t$ exists on this interval, and

$$\frac{\partial Y}{\partial t}=\sum_{n=0}^{\infty}\frac{\partial Y_n}{\partial t}(x,t).$$

We summarize and extend as follows.

Theorem 1 (Existence and uniqueness). *There exists* $0<x_0\in\mathcal{R}$ *and a nonempty subset* $A\subseteq\mathcal{R}$ *such that on* $\{x\in\mathcal{R}:|x|>x_0\}\times A$, *there exists a solution of* $\gamma_{xx}=G(x)\gamma^2$ *where* $G(x)=O(x^{-5})$ *as* $x\to\infty$, *obeying* $\gamma(x,t)=e^{-\beta(t)/2}+o(1)$ *as* $x\to\infty$. *This solution is differentiable with respect to t. Furthermore, if* $G(x)$ *is analytic in a neighborhood of infinity, then this solution is the unique analytic solution.*

For the proof of the last statement, note first of all that if $G(x)$ is analytic in a neighborhood of infinity, then the existence proof of the theorem gives the construction of an analytic solution of Eq. (3.14). For each term in the series Eq. (4.2) is found by integrating analytic functions, and is therefore analytic; uniform convergence guarantees analyticity of the sum. Next, we argue that if $G(x)$ and a solution $\gamma(x,t)$ of Eq. (3.14) are analytic functions of x in a neighborhood of infinity, then this solution is unique. For let γ_1,γ_2 be two such solutions, and define

$$\Gamma:=\gamma_1-\gamma_2, \quad H(x,t):=(\gamma_1+\gamma_2)G(x)=O(x^{-5}).$$

Then H is analytic in x in a neighborhood of infinity, and Γ obeys

$$\Gamma_{xx}=H(x,t)\Gamma, \tag{4.8a}$$

$$\Gamma=o(1), \quad x\to\infty. \tag{4.8b}$$

By analyticity, the solutions of this linear differential equation, which has a regular singular point at infinity, can be written in the form

$$\Gamma=\sum_{n=0}^{\infty} a_n(t)x^{-n+p}$$

for some real p. Using a Taylor series expansion about $x^{-1}=0$ for $H(x,\cdot)$ and filling out the equation (4.8a), we find that the two independent solutions are described by

(i) $p=0$, a_0 arbitrary, $a_1=a_2=0$, $a_n, n\geqslant 3$ determined by recurrence relations, and proportional to a_0;

(ii) $p=1$, a_0,a_1 arbitrary, $a_2=0$, $a_n, n\geqslant 3$ determined by recurrence relations given by linear combinations of a_0 and a_1.

In either case, we see that nonzero solutions do not obey $\Gamma=o(1)$ as $x\to\infty$, so that $\Gamma\equiv 0$ is the only solution of Eq. (4.8), and so $\gamma_1=\gamma_2$, proving uniqueness.

In particular, in the case of most interest to us, $G(x)=-24m_0(x^2+4x)^{-5/2}$ is such a function, and so the solution produced by the theorem is the unique analytic solution. Notice also that when $m_0=0$ in this solution, we obtain the RW line element (3.8). It will be useful to have the first few terms of this solution. These terms are obtained by integrating Eq. (4.3a) with the appropriate choice of integration constants, and yield

$$\gamma(x,t)=e^{-\beta/2}\left\{1-m\left(\frac{2}{(x^2+4x)^{1/2}}-(x+2)+(x^2+4x)^{1/2}\right)\right\}+O(x^{-6}), \tag{4.9}$$

where $m=m_0e^{-\beta/2}$. Converting to the original coordinates using the transformation given prior to Eq. (3.14), this leads to

$$e^{\mu}=e^{\beta(t)}(1-4m_0e^{-\beta/2}e^{-3r})+O(e^{-5r}). \tag{4.10}$$

We note that the corresponding first order term in McVittie's $k=-1$ solution is

$$e^{\mu}=e^{\beta}(1+2m_0e^{-\beta/2}e^{-r/2})+O(e^{-3r/2}),$$

from which we can identify the problem with this solution; the metric coefficients do not tend to those of the RW metric rapidly enough.

We now turn our attention briefly to the case $k=0$. The existence proof of this section is of course not needed for this case, because as we have seen already, McVittie's $k=0$ solution satisfies the conditions **C1–C3**. The uniqueness result of this section *does* apply, and so we see that if Raychaudhuri's conditions quoted in Sec. II are replaced by the conditions of Sec. III, then under the added hypothesis of analyticity, the solution is indeed unique.

V. PROPERTIES OF THE SOLUTION

In this section, we discuss various properties of the unique analytic space-time (M,g) found in the previous section which obeys the conditions of Sec. III with $k=-1$ and $G(x)=-24m_0(x^2+4x)^{-5/2}$. Throughout this section, terms such as "the solution," "the line element," etc. refer to this solution and its line element, etc. unless otherwise specified.

A. Representation with an elliptic integral

We show here that the function e^{μ} can be represented intrinsically by an elliptic integral; the basic results are from [12].

Writing $\gamma=(x^2+4x)^{1/2}u$, we can obtain the following first integral of Eq. (3.14a):

$$(x^2+4x)^2u_x^2=4(u^2-4m_0u^3)+A(t), \tag{5.1}$$

where $A(t)$ is a function of integration. Using Eq. (4.9) above, we can compare powers of x^{-1} in this last equation to obtain

$$A(t)=e^{-\beta(t)}.$$

A straightforward calculation shows that

$$(x^2+4x)u_x=-\frac{\gamma^2}{h}\frac{\partial R}{\partial r},$$

where $R(r,t):=h(r)e^{\mu/2}$ is the radius of metric 2-spheres in the space-time, so to ensure $\partial R/\partial r>0$, we take the negative square root of Eq. (5.1), which leads to an intrinsic elliptic integral representation for u:

$$\int\frac{du}{(4u^2-16m_0u^3+e^{-\beta})^{1/2}}=-\frac{1}{4}\ln\left(\frac{x}{x+4}\right)+B(t),$$

and $B(t)$ is the sole remaining function of integration. This term may involve m_0, and so cannot be determined from the other relevant limit, $m_0=0$. The integral on the left hand side does not have a representation in terms of elementary functions for $m_0\neq 0$, and so nor does our solution.

We note that it may be possible to determine the functions $A(t),B(t)$ in the elliptic integral which yield the correct asymptotic behavior for μ *without* prior knowledge of the solution. The advantage is that we would not need the existence proof of the previous section. However, this would be a rather difficult problem involving inversion of asymptotic formulas for elliptic integrals. We feel that the chosen method is the most direct, and has the advantage of dealing with the general case, $G(x)=O(x^{-5})$.

B. Energy conditions

Using the field equation (3.4) and the first integral of the main equation (3.13a) as found in Sec. V A above, we can write the pressure (3.3) as

$$8\pi p=-\frac{3}{4}\dot\beta^2+3e^{-\beta}-\dot\mu^{-1}\dot\beta(\ddot\beta+2e^{-\beta}). \tag{5.2}$$

Notice that by the main theorem of Sec. IV, in the limit $r\to\infty$, this coincides with $8\pi p_0$, the pressure of the background RW universe. Thus all of the curvature tensor terms match up with those of the RW background in this limit.

Using Eq. (4.9), we find that

$$8\pi p=8\pi p_0+(\ddot\beta+2e^{-\beta})m_0e^{-\beta/2}e^{-3r}+O(e^{-5r}). \tag{5.3}$$

Recall also that the energy density obeys $\rho(t)=\rho_0(t)$. We see then that the question of whether or not ρ and p obey appropriate energy conditions [19] is, for sufficiently large values of r, equivalent to the same question regarding ρ_0 and p_0. Notice that if the weak energy condition is satisfied in the RW background, then $\rho_0+p_0\geqslant 0$, leading to $\ddot\beta+2e^{-\beta}\leqslant 0$. Thus according to Eq. (5.3), to first order, the presence of a central mass in a $k=-1$ RW universe causes a *decrease* in the fluid pressure, contrary to what one would expect. This is in distinction to the situation in the $k=0$ model, where according to Eq. (2.3) we can write

$$8\pi p=8\pi p_0-\ddot\beta m_0e^{-\beta/2}r^{-1}+O(r^{-2}).$$

The weak energy condition in the RW background implies $\ddot\beta\leqslant 0$, and so the first order perturbation of the pressure is positive, as expected. Indeed this behavior is continued at all orders; $8\pi p\geqslant 8\pi p_0$ in the $k=0$ model, provided the weak energy condition holds in the RW background.

This latter situation is in line with what happens in the analogous situation in Newtonian cosmology. The potential describing the physical scenario under consideration is found (in this linear theory) by adding the potentials $\phi_m=-mr^{-1}$ and $\phi_c=\rho(t)r^2/12$ for respectively a point particle of mass m situated at $r=0$ and an isotropic cosmological model with density $\rho(t)$ (see [20] for the latter). We take the potential to be

$$\phi=\phi_m+\phi_c=-\frac{m}{r}+\frac{1}{12}\rho(t)r^2.$$

Then the pressure across the surface $\{S:r=\text{const}\}$ at time t is

$$p(S)=\int_S-\vec\nabla\phi\cdot\vec n d_2S,$$

where $\vec n$ is the unit inward normal to S. This yields

$$p(S)=4\pi m+\frac{2}{3}\pi\rho r^3.$$

Thus we see that in Newtonian theory, the central mass makes a positive contribution to the pressure.

The negative first order contribution in the $k=-1$ case could be cancelled out by higher order terms, leaving a net positive contribution, but if not, it appears to be an interesting effect of the negative curvature of the spatial sections of the RW background.

C. Recovering the Schwarzschild space-time

We have seen above how the line element of the RW background is recovered by taking $m_0=0$. We show next how the line element of the exterior Schwarzschild field arises in a natural way as a limiting case of our solution.

In McVittie's $k=0$ solution, the Schwarzschild field is obtained by setting $\dot\beta$ equal to zero. Then following a constant rescaling of the coordinate r, the line element (2.1) with $k=0$ is the isotropic form of the Schwarzschild line element with mass parameter m_0. The procedure is quite natural; with $\dot\beta=0$, the energy density and pressure both vanish, yielding a spherical vacuum which is by necessity, the exterior Schwarzschild field. Note also that the expansion of the fluid flow lines $\theta=\frac{3}{2}\dot\beta$ is then also equal to zero.

Carrying out the same procedure in the $k=-1$ case leads to

$$\rho(t)=\rho_0(t)=\frac{3}{4}\dot\beta^2-3e^{-\beta}=0,$$

giving

$$e^{\beta}=(t+c)^2 \tag{5.4}$$

for some constant c. Calculating $p_0(t)$, the pressure of the RW background, we find $p_0(t)=0$, so that this space-time is (a portion of) Minkowski space-time. Similarly, calculating $p(r,t)$ for our solution using this form of β yields $p=0$, and so the Ricci tensor vanishes. From Eq. (3.6), the Weyl tensor remains nonzero, and so by Birkhoff's theorem, (M,g) is (a portion of) the exterior Schwarzschild field. The Hawking mass of the Schwarzschild field is the Schwarzschild mass parameter, and so m_0 in our solution is the Schwarzschild mass parameter.

Thus McVittie's solution for $k=0$, and our solution for the case $k=-1$, represents the Schwarzschild field embedded in a RW universe.

Another limiting case is of importance, namely when $\rho+p=0$, so that the space-time is an Einstein space. In both cases $(k=-1,0)$, we can explicitly verify that $\rho+p=0$ implies that p is constant. Again, the choice of $\beta(t)$ does not affect the value of the Weyl tensor, and so by the ''Birkhoff-with-a-cosmological-constant'' theorem, space-time is a portion of the Schwarzschild–de Sitter cosmos. Thus the solutions discussed here give genuinely cosmological (i.e., nonstationary) generalizations of this static space-time.

D. Behavior at future null infinity

As $r\to\infty$ on the spacelike hypersurfaces orthogonal to the fluid flow lines, the line element of our solution approaches that of a RW space-time. The question of how it behaves asymptotically along future null directions is more complicated, but the following argument indicates that the space-time tends to a RW universe in this limit. We show that the metric coefficients of our solution match up with those of the RW background as $r\to\infty$ along future null directions of the RW background, which are hence asymptotically future null directions of (M,g). We deal explicitly with the more complicated case $k=-1$; analogous results hold for $k=0$. We consider first the description of $\mathcal{I}^+$ in the RW background. The following relies heavily on [21].

Consider the RW background, whose line element may be written

$$\begin{aligned}ds^2&=-dt^2+e^{\beta(t)}\{dr^2+\sinh^2 r\,d\omega^2\}\\&=\Omega^2(\eta)\{-d\eta^2+dr^2+\sinh^2 r\,d\omega^2\},\end{aligned} \tag{5.5}$$

where $\Omega(\eta)=e^{\beta/2}(t)$, $dt=\Omega(\eta)d\eta$ and $d\omega^2$ is the line element of the unit 2-sphere. Define coordinates $u(0<u<\infty)$ and $\chi(0\leqslant\chi<\infty)$ by

$$u=e^{\eta-r}\leftrightarrow\eta=\frac{1}{2}\ln(u^2+2u\chi), \tag{5.6}$$

$$\chi=e^{\eta}\sinh r\leftrightarrow r=\frac{1}{2}\ln(1+2\chi u^{-1}). \tag{5.7}$$

In these coordinates, the line element (5.5) assumes the form

$$ds^2=F^2(u,\chi)\{-du^2-2du\,d\chi+\chi^2d\omega^2\},$$

where

$$F(u,\chi)=\Omega^2(\ln(u^2+2u\chi)^{1/2})(u^2+2u\chi)^{-1}.$$

Next, define $l=\chi^{-1}$ and introduce the nonphysical line element

$$\begin{aligned}d\tilde{s}^2&=H^2(u,l)ds^2\\&=-l^2du^2+2du\,dl+d\omega^2,\end{aligned} \tag{5.8}$$

where

$$H(u,l)=l\Omega^{-1}(\ln(u^2+2ul^{-1})^{1/2})(u^2+2ul^{-1})^{1/2}.$$

Then, in the usual way, future null infinity of the RW space-time is identified with the boundary $H=0$ of the space-time $(\tilde V,\tilde g)$ whose metric is given via the line element (5.8). If $H=0$ coincides with $l=0$, then a direct calculation shows that $\mathcal{I}^+$ is a shear-free null hypersurface. This depends upon $\Omega(\eta)$ being a sufficiently rapidly increasing function of its argument, which relates to the conditions required of $e^{\beta}(t)$ to ensure convergence of the solution in Sec. IV. We note that for a perfect fluid with equation of state $p=\alpha\rho$, we have

$$\Omega(\eta)=A\left(\sinh\left(\frac{3\alpha+1}{2}\eta\right)\right)^{2/(3\alpha+1)},$$

for some constant A, which leads to

$$H(u,l)=A^{-1}2^{2/(3\alpha+1)} \tag{5.9}$$

$$\times\frac{l(u^2l+2u)}{((u^2l+2u)^{(3\alpha+1)/2}-l^{(3\alpha+1)/2})^{2/(3\alpha+1)}}, \tag{5.10}$$

so that these conditions are satisfied if $3\alpha+1>0$.

This shows how to describe $\mathcal{I}^+$ in the RW backgrounds for a large class of such space-times. The importance of this for our situation is that it tells us that as $\chi\to\infty$ along u =const, we approach $\mathcal{I}^+$ in the RW universe. To conclude this section, we simply note that in this limit, the metric coefficients of our solution approach those of the RW background, and all the curvature tensor terms approach the background values. Thus our solution is asymptotically RW at future null infinity; we already know it to be asymptotic to the RW background at spacelike infinity.

E. Singularities

The solutions which we study here (McVittie's $k=0$ solution and our $k=-1$ version) have been shown to represent the Schwarzschild field embedded in a RW universe. It is therefore natural to ask if these solutions have a central singularity and event horizon, and if so, how they are affected by the cosmic expansion. We will treat this important issue in more depth elsewhere; we can give the following preliminary results here.

For any space-time, the quantity $I=\Psi_0\Psi_4-4\Psi_1\Psi_3+3\Psi_2^2$ is an invariant of the curvature. Here, we have for both $k=0,-1$

$$I=3\frac{m_0^2}{R^6}. \tag{5.11}$$

Thus we see that there is indeed an intrinsic curvature singularity at the center $R=0$. The coordinates we have used might not cover this region; this is immaterial as Eq. (5.11) derives from an invariant property of the curvature tensor, namely $M_2(S)=m_0$ for all metric 2-spheres S.

Hayward [22] has shown that the Misner-Sharp gravitational energy is a useful tool for investigating singularities in spherical symmetry. One of the equivalent definitions for this quantity is

$$E:=\frac{R}{2}(1-\chi),\quad \chi:=\nabla_aR\nabla^aR. \tag{5.12}$$

Carrying out a straightforward calculation which makes use of Eq. (3.4) and the first integral (5.1), we obtain the following nice results, which apply to both $k=0$ and $k=-1$:

$$\chi=-\frac{8\pi}{3}R^2\rho(t)+1-2m_0R^{-1}, \tag{5.13}$$

$$E=\frac{4\pi}{3}R^3\rho(t)+m_0. \tag{5.14}$$

These forms have the advantage of being coordinate independent; both ρ and R are invariantly defined quantities. Equation (5.14) is particularly satisfying; the effect on the gravitational energy of the presence of a particle of mass m_0 is an increase of exactly this amount.

Notice now that if x is any point in the boundary $R=0$, then

$$\lim_{\gamma\to x}E>0,$$

along any curve γ approaching x. Thus by a result of Hayward [22], the central singularity is spacelike and trapped, as in the Schwarzschild space-time.

In the case $k=0$, we see from Eq. (2.3) that there is also a curvature singularity at $r=m/2$, which, intriguingly, corresponds to $R=2m_0$, the gravitational radius of the central mass. We see from Eq. (5.13) that this is a spacelike hypersurface, and is surrounded by a trapped region. The existence of this singularity is fundamentally different to the vacuum case, and demands a thorough investigation of the singularity and horizon structure of this space-time. These issues are currently being studied.

F. Summary

The solution we have found represents a point mass m_0 embedded in a $k=-1$ RW universe. When $m_0=0$, we obtain this RW background. The energy-density is identical to that of the background, and the zero-density limit gives Schwarzschild's space-time with mass parameter m_0. The space-time is asymptotic to the RW universe at infinity and contains a spacelike singularity at the center.

VI. COMMENTS

We have given a prescription above for embedding the Schwarzschild field in an open RW universe. Consider the converse problem. Given a spherically symmetric shear-free perfect fluid space-time (V,g), how do we know if (V,g) represents a point mass in a RW universe and, if it does, how do we identify that RW universe? This presents us with a gauge problem. ''Suppose we consider the lumpy universe model S, not knowing how the (background) model $\overline{S}$ was used to make the construction; can we uniquely recover $\overline{S}$ from S?'' [8]. In fact the answer to this question is yes. Calculate the Hawking mass for an arbitrary metric 2-sphere of (V,g). If the result is not a constant, then (V,g) does not represent a point mass in a RW universe. If the result is a constant (m_0 say), and if further when $m_0=0$, (V,g) is a RW universe, we may proceed. This solves the gauge problem by identifying the background model. It remains then to check if (V,g) satisfies the remaining parts of conditions

C2,**C3** with respect to this well defined RW background.

In a previous paper [23], we interpreted certain space-times as being extended sources for the McVittie field in the three cases $k=0,\pm 1$. To further investigate the occurence of singularities and horizons in (M,g), it would be interesting to determine if a collapsing fluid can be used as a source. This would allow us to interpret (M,g) as the end state of the spherical collapse of a massive body in an expanding universe and may throw some light on the issue of what, if any, the effect of this expansion is on the collapse. This still leaves the problem of whether the space-time is of black-hole (collapsed object surrounded by an event horizon) or white-hole (lagging core of an expanding universe) type. This issue is to be addressed in a subsequent paper in which the horizon, singularity and asymptotic structure of these space-times is analyzed.

The nature of the solution we have found has opened up these interesting questions. However, our main purpose was to give a clear physical interpretation of some solutions of Einstein's equation. In particular, we hope to have given such for McVittie's solution, which in the case $k=0$ does indeed represent a point mass in an RW universe; some authors have contested this interpretation [24,25]. We have seen how Hawking's mass was a useful tool in this. Our aim now is to use this tool in an attempt to identify solutions representing the embedding of other objects (cosmic strings, the Reissner-Nordstrom and Kerr fields) in RW universes. We note that some of the solutions [6] given previously which it was claimed represent such do not reproduce McVittie's solution in the $k=0$ case or our solution in the $k=-1$ case in the appropriate limit (charge-free and nonrotating). This may be an inherent discontinuous feature of solutions of the field equations in such situations. However it leads to the suspicion that these solutions do not satisfy a set of conditions analogous to **C1**–**C3** which clearly determine their physical interpretation.

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