Numerical treatment of the hyperboloidal initial value problem for the vacuum Einstein equations. I. The conformal field equations

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This is the first in a series of articles on the numerical solution of Friedrich's conformal field equations for Einstein's theory of gravity. We will discuss in this paper why one should be interested in applying the conformal method to physical problems and why there is good hope that this might even be a good idea from the numerical point of view. We describe in detail the derivation of the conformal field equations in the spinor formalism which we use for the implementation of the equations, and present all the equations as a reference for future work. Finally, we discuss the implications of the assumptions of a continuous symmetry.
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I. INTRODUCTION

Much of the current work in numerical and experimental relativity is devoted to obtaining information about the gravitational radiation that is emitted by astrophysical processes which are taking place in our universe. The goal is to obtain wave forms, i.e., the ''finger prints'' by which different processes can be identified when the gravitational waves are detected by laser interferometers such as the Laser Interferometric Gravitational Wave Observatory (LIGO) or VIRGO. The common way to describe such a system within Einstein's theory of gravity is by way of an idealization, where the system is considered as being so far away from the rest of the universe that the influence of the latter can be neglected (cf. [1] for a clear discussion of what is involved in this process). Then, intuitively, the fields far away from the source should decay so that the space-time becomes asymptotically flat. The detectors are then idealized as observers which are located ''at infinity,'' where they can gather the gravitational radiation coming from the system.

Isolated gravitating systems and the structure of their ''far fields'' have been investigated for a long time because of their importance for the interpretation of measurements. A series of articles which heavily influenced the way we look at the subject today was published in the early 1960s. In these articles various important contributions were made: the ''peeling property'' of the Weyl tensor [2], the idea of analyzing the vacuum field equations on outgoing null hypersurfaces resulting in the Bondi mass loss formula [3,4], the invention of the Newman-Penrose (NP) formalism and the proposal for considering the vacuum Bianchi identity as a field equation for the Weyl tensor [5] and the asymptotic solution of the Einstein vacuum equations [6]. Assuming that certain components of the Weyl curvature fall-off in a specific way, it was found by formal power series analysis of the asymptotic characteristic initial value problem that the fall-off behavior of the fields along null directions could be characterized in terms of certain special coordinate systems whose existence was presupposed. Finally, it was realized by Penrose [7] that these fall-off conditions as well as the peeling property could be understood in a purely geometric way. He introduced the notion of a conformal extension by which a Lorentz manifold $(\tilde{M},\tilde{g})$ is embedded into a bigger manifold (M,g) with isomorphic conformal structure, but with a Lorentz metric which differs from $\tilde{g}$ by a positive factor $g=\Omega^2\tilde{g}$. The idea was to study the global conformal properties of Minkowski space in order to obtain a criterion for what one should call an ''asymptotically flat'' space-time. Guided by the Minkowski situation, Penrose suggested that such space-times allow the attachment of a conformal boundary $\mathscr{J}$ which is characterized by the vanishing of the conformal factor Ω. This boundary is a regular null hypersurface in the ambient unphysical manifold. It can be interpreted as the points which are at infinity for the physical manifold along null directions.

The question arises as to what extent this geometric picture is compatible with the Einstein equations. Friedrich could derive a system of equations [8], the ''conformal field equations,'' which are defined on that larger unphysical manifold. Furthermore, a solution of the conformal field equations gives rise to a solution of the standard field equations on the physical space-time. This system is written in terms of geometric quantities of the unphysical manifold and the conformal factor Ω and it is regular everywhere even at points where Ω vanishes. In a usual $3+1$ decomposition, the conformal field equations split into constraint equations and evolution equations. Using this system of equations, Friedrich was able to reduce the asymptotic characteristic initial value problem for the Einstein equations, where data are given on a part of (past) null infinity and an ingoing null hypersurface which intersects null infinity in a two-dimensional surface to a characteristic initial value problem for a symmetric hyperbolic system [8].

In order to describe a physical situation, one would like to prescribe initial data for the conformal field equations on some initial space-like hypersurface and determine from them the future of the system. Ideally, the data should be given on an asymptotically flat space-like surface. It does turn out that the initial data for the conformal field equations on such a hypersurface are necessarily singular because the conformal structure of space-time is singular at i^0, whenever the Arnowitt-Deser-Misner (ADM) mass is nonzero (see [9] for a new approach towards the solution of this problem). Therefore, the initial data are given on a space-like hypersurface which intersects $\mathcal{J}$ transversely in a two-dimensional surface. Such hypersurfaces are called hyperboloidal surfaces because they behave like spaces of constant negative curvature in the neighborhood of $\mathcal{J}$. Friedrich [10] has shown that the Cauchy problem for data given on such hypersurfaces, the hyperboloidal initial value problem, is well posed, i.e., given smooth initial data which solve the constraints then there exists a solution of the evolution equations in some neighborhood of the initial surface. If the data are close enough to Minkowski data, then the future development is complete in the sense that there exists a regular point i^+ whose past light cone coincides with $\mathcal{J}$.

The fact that the domain of dependence of the initial hyperboloidal hypersurface includes the complete physical space-time in the future allows the study of global phenomena like the behavior of horizons and the causal structure of singularities. But also, since one has access to null infinity, where gravitational radiation is registered, one can in principle ''extract'' the radiative information by purely local manipulations from the fields ''on $\mathcal{J}$.'' This suggests that the hyperboloidal initial value problem is an appropriate device for examining these issues. The goal of the present work is the investigation whether the conformal field equations, and in particular, the hyperboloidal initial value problem, can provide an effective numerical tool for analyzing the global structure of asymptotically flat space-times and for obtaining information about the gravitational radiation emitted by the system in question. The starting point for this investigation was the work by Hübner [11] who was able to demonstrate the feasibility of that approach in the spherically symmetric case of gravity coupled to a scalar field.

There are various other numerical approaches towards these problems based on the numerical solution of the standard field equations, either in terms of a Cauchy problem, a characteristic initial value problem or a combination of both. The first two alternatives both have some problems. The standard Cauchy problem cannot provide complete global information because one has to cut off the initial data surface and provide boundary data on a time-like hypersurface which intersects the initial surface in a two-dimensional surface. The boundary data change the solution in their domain of influence and hence, if the boundary data are unphysical, so will be the solution. Even if the boundary conditions are physical, the radiation data obtained are still only approximate because the boundary is not at infinity. Only there, radiation can unambiguously be defined. Therefore, as a matter of principle, the standard Cauchy problem can provide only approximate radiation information. In the hyperboloidal problem, there is only one idealization involved, namely, that of how an isolated system is to be described.

The characteristic initial value problem, on the other hand, can be put to good use in the neighborhood of $\mathcal{J}$. Space-time is foliated by outgoing null hypersurfaces and one can perform a conformal transformation to obtain a problem, where null infinity is at finite places. This is one boundary for the outgoing initial null hypersurface. The other boundary is located someplace not too far in the interior of the space-time. The numerical procedure for solving the characteristic initial value problem is relatively simple compared to the Cauchy problem which is due to the fact that the equations split into hypersurface equations, which are essentially ordinary differential equations and one evolution equation. Three further equations, the so called ''conservation equations,'' have to be satisfied on $\mathcal{J}$. The problem with this approach is the fact that null hypersurfaces invariably tend to form caustics, places where the hypersurface intersects itself so that it becomes impossible (or at least very difficult) to give unambiguous initial data. The stronger the fields are, the earlier the caustics will appear.

The last alternative, commonly called the Cauchy-characteristic matching (CCM) procedure (probably going back to [12]), has been intensely studied by various groups cf. [13–15]. The idea is to combine the two previous approaches without their respective disadvantages. The procedure is roughly to divide the physical space-time by a time-like world tube $\mathcal{T}$ and to evolve the inner part by solving a Cauchy problem. The exterior of $\mathcal{T}$ is evolved by solving the characteristic initial value problem based on outgoing null hypersurfaces connecting the world tube with null infinity. An initial hypersurface for the combined problem consists of a space-like hypersurface $\mathcal{S}$ with a boundary (which indicates the intersection of $\mathcal{S}$ and $\mathcal{T}$) together with the outgoing null hypersurface emanating from the boundary. Obviously, at the interface, where the initial hypersurface changes its causal character from space-like to null, there is a non-differentiable kink. The great challenge is to implement numerically the information exchange across that kink. This problem has been solved in various simpler circumstances, see, e.g. [16,17].

When viewed in the unphysical space-time, the initial surface for the CCM procedure intersects $\mathcal{J}$ in a two-dimensional ''cut.'' Now consider a space-like hypersurface $\mathcal{S}$ which also goes through that same cut. It is clear that this surface is a hyperboloidal hypersurface. Its domain of dependence is the same as that of the Cauchy-characteristic hypersurface. The region of $\mathcal{J}$ which can be described is the same in both cases. One advantage of evolving with the conformal field equations is certainly the fact that the causal character of the foliation does not change so that there is no interface and no need to change the evolution algorithm. Another advantage is that one can go smoothly through $\mathcal{J}$ which allows one to keep $\mathcal{J}$ in the interior of the grid in order to avoid numerical influences from the grid boundaries. There are more equations to solve in the case of the conformal field equations than there are in standard ADM-like formulations. In the particular formulation of the conformal field equations which is put forward here, there are fifty-three variables for the full three dimensional case. This might be considered as a drawback. However, there are recent formulations of the Einstein equations as hyperbolic systems [18] which have to introduce many additional variables so that the resulting system is comparable in size to the system of conformal equations. Furthermore, the quantities in the latter system which are evolved in addition to the spatial metric and the extrinsic curvature in the standard case have a geometric meaning (Ricci- and Weyl tensor components) and any code which aspires to analyze the space-time structure needs to compute those quantities anyway.

This present article is meant to be the first in a series of papers on the numerical treatment of the conformal vacuum field equations. In this paper, we derive the conformal field equations in a formalism using space spinors. Although this has been done previously [19], we present the equations here in a form suitable for our immediate purposes, the main reason being to establish a common notation and for reference. The space spinor formalism has the advantage that the equations can easily be decomposed into evolution and constraint parts and that the evolution part comes out automatically in symmetric hyperbolic form. Furthermore, the equations can be written in a more compact form and the possibility of decomposing spinor fields into their irreducible parts can be used to remove any redundancy from the set of unknowns.

In the second article [20], we present the numerical treatment of the evolution part with the additional assumption of a symmetry and in the third, we want to discuss the solution of the constraint equations. The conventions used throughout this work are those of Penrose and Rindler [21].

II. THE CONFORMAL FIELD EQUATIONS

In this section we want to give a derivation of the conformal field equations and a brief discussion of their properties. Apart from introducing the necessary background on the hyperboloidal initial value problem, this section also serves as a reference to the actual equations used in the code.

An essential ingredient in this approach towards the examination of global properties of space-times is the notion of a conformal transformation. Let $(\tilde{M},\tilde{g})$ be a Lorentz manifold with vanishing Einstein tensor (we will assume throughout that the cosmological constant vanishes). Assume that this ''physical space-time'' is such that the following conditions hold: there exists a Lorentz manifold (M,g) with boundary $\mathcal{J}$ and a function Ω on M such that $\Omega \geqslant 0$ on M and $\Omega=0$, $d\Omega \neq 0$ on $\mathcal{J}$, the physical manifold can be identified with the interior of M and there the equation $g=\Omega^2\tilde{g}$ holds.

These conditions state that the physical manifold is conformal to the interior of the ''unphysical'' manifold M. The points on the boundary $\mathcal{J}$ can be thought of as representing the points of $\tilde{M}$ which are ''at infinity'' with respect to the physical metric $\tilde{g}$. With the vanishing of the cosmological constant, it follows that $\mathcal{J}$ is a regular null hypersurface in M on which the Weyl curvature vanishes (although this is only strictly proven in the case, where $\mathcal{J}$ has the topology $S^2 \times \mathbf{R}$ [22]).

The conformal field equations can now be obtained from the basic geometric equations on M and $\tilde{M}$, the Einstein equation which holds on $\tilde{M}$, and the conformal transformation properties of the geometric fields. In view of the numerical application, it is advantageous to have a first order system. This is easily achieved by using a frame formalism. The system then consists of the following equations.

The first of Cartan's structure equations, which expresses the fact that the connection on M is torsion free. It can be viewed as an equation for the components of the chosen tetrad with respect to the chosen coordinates.

The second structure equation which relates the Ricci rotation coefficients of the connection to the curvature components. It can be viewed as an equation for the connection components with respect to the chosen tetrad.

The Bianchi identity for $\tilde{g}$. This is an identity which relates the derivatives of the physical Ricci and the Weyl curvature. Since $\tilde{M}$ is a vacuum space-time, this yields an equation for the physical Weyl curvature. Expressing this equation in terms of the unphysical connection yields an equation for the rescaled Weyl curvature $D_{abcd} = \Omega^{-1}C_{abcd}$, which looks formally like the familiar spin-2 zero rest-mass equation.

The Bianchi identity for g. Again, this is a relation between the derivatives of the Ricci and the Weyl curvature, but now on the unphysical space-time. Using the equation of the rescaled Weyl curvature, this identity yields an equation for the unphysical Ricci curvature.

Equations for the conformal factor Ω and its derivatives obtained from the conformal transformation law for the Ricci curvature.

An equation for the function $S := \frac{1}{4}\Box\Omega$, which is a consequence of the earlier equations.

Because of the geometric origin of these equations, there is gauge freedom in this system. Several variables can be chosen freely. Apart from the coordinates, this is true also for the tetrad which is fixed by the metric only up to Lorentz transformations and for the conformal factor, which is fixed by the conditions above up to multiplication with a strictly positive function $\Omega \mapsto \theta\Omega$, where $\theta > 0$ on M. This allows the free choice of eleven functions, the gauge source functions, which can be fixed in numerous ways. It is here where the development of a code to evolve space-times turns into an art.

The essential property of the above system is the following: With the gauge source functions fixed as arbitrary functions of the coordinates, the system can be decomposed by a usual $3+1$ splitting into two separate systems with respect to a given foliation of space-like hypersurfaces. The first of those, the constraints, is intrinsic to the space-like hypersurfaces and therefore it restricts the values of the variables there. The second part, the evolution equations, can be written as a quasi-linear symmetric hyperbolic system. This has the consequence that the Cauchy problem for this system is well posed: given initial data for the unknown functions on a space-like hypersurface $\mathcal{S}$, then in a neighborhood of $\mathcal{S}$, there will exist a solution of the system acquiring the prescribed values on $\mathcal{S}$. It turns out, that once the constraints are satisfied on the initial hypersurface, they will be satisfied everywhere by virtue of the evolution equations, they are propagated by the evolution. Therefore, given initial data which satisfy the constraints, then they will evolve into a solution of the conformal field equations.

If M is such that the initial hypersurface $\mathcal{S}$ and $\mathcal{J}$ intersect transversely in a regular compact two-dimensional surface, then one can talk about the hyperboloidal initial value problem. The standard example for such ''hyperboloidal'' surfaces are the conformal images of the space-like hyperboloids in Minkowski space in the usual conformal picture. The fact that $\mathcal{S}$ and $\mathcal{J}$ intersect in the unphysical space-time means in physical terms that the space-like surface extends out to null infinity. Thus, such surfaces are not Cauchy surfaces for the standard Cauchy problem for the Einstein equations in the physical space-time.

In summary, the conformal field equations allow a well-posed initial value problem on space-like hypersurfaces in the unphysical space-time whose physical ''pre-images'' extend asymptotically towards null infinity. Initial data which satisfy the constraints evolve into a solution of the complete system.

III. SPACE SPINORS

In this section we briefly introduce the basic formalism used to write down the equations and to separate them into the evolution and constraint part. This can very conveniently be achieved using the space spinor formalism [23], which in addition allows writing (and coding) the equations in a more compact form.

The essential ingredient in the space spinor formalism is a time-like vector field t^a which is normalized by $t_a t^a = 2$ (note, that we use throughout the conventions of [21]). In terms of spinors, we have $t^a = t^{AA'}$ and $t_{AA'} t^{BA'} = \epsilon_A{}^B$. The existence of this vector field allows the conversion of all primed spinor indices to unprimed ones by extension of the map $\pi_{A'} \mapsto \pi_{A'} t^{A'}{}_A$ to the full spinor algebra. E.g., any covector $v_a = v_{AA'}$ is mapped to $v_{AB} = v_{AA'} t^{A'}{}_B$. Note, that this spinor can be decomposed into irreducible parts: $v_{AB} = 1/2\, \epsilon_{AB} v + \tilde{v}_{AB}$, where $\tilde{v}_{AB} = \tilde{v}_{BA}$. In terms of the original covector, these parts correspond to the components along t^a, $v = t^a v_a$, and orthogonal to t^a.

The vector $t^{AA'}$ can be used to define a complex conjugation map on the algebra of unprimed spinors by extension of the map $\pi_A \mapsto \hat{\pi}_A := t_A{}^{A'} \overline{\pi_{A'}}$ to the full algebra. Note, that $\hat{\hat{\pi}}_{A_1 \dots A_n} = (-1)^n \pi_{A_1 \dots A_n}$. An even valence spinor $\pi_{A_1 \dots A_{2n}}$ is called real, if $\hat{\pi}_{A_1 \dots A_{2n}} = (-1)^n \pi_{A_1 \dots A_{2n}}$.

The derivative operator ∇ on M can be decomposed as follows:

$$\nabla_{AA'} = \tfrac{1}{2} t_{AA'} D - t_{A'}{}^B D_{AB}, \tag{3.1}$$

or, equivalently,

$$t^{A'}{}_B \nabla_{AA'} = \tfrac{1}{2} \epsilon_{AB} D + D_{AB}, \tag{3.2}$$

where $D := t^a \nabla_a$ and $D_{AB} = t^{A'}{}_{(B} \nabla_{A)A'}$ are the parts which act along and perpendicular to t^a, respectively. Thus, the general procedure we will follow is to write the equations in spinorial form, then convert to space spinors, and finally decompose them into irreducible parts.

The derivative of $t^{AA'}$ gives rise to two important spinor fields, $K_{AB} = t^{A'}{}_B D t_{AA'}$ and $K_{ABCD} = t^{C'}{}_D D_{AB} t_{CC'}$. Note the symmetry and reality properties of these fields: $K_{AB} = K_{(AB)} = -\hat{K}_{AB}$ and $K_{ABCD} = K_{(AB)(CD)} = \hat{K}_{ABCD}$. Geometrically, K_{AB} corresponds to the acceleration vector of t^a, while K_{ABCD} is related to the geometry of the distribution defined by $t_a = 0$. This distribution is integrable if and only if $K^A{}_{(BD)A} = 0$. Then t^a is hypersurface orthogonal and K_{ABCD} corresponds to the extrinsic curvature of the orthogonal surfaces.

We will assume, henceforth, that t^a is hypersurface orthogonal. Hence, the covector t_a is proportional to the conormal of the space-like hypersurfaces given by $t_a = 0$. Then, the derivative D_{AB} is the so called Sen-Witten connection which plays an important role in various areas of general relativity. It is not completely intrinsic to the hypersurfaces, but contains information about the embedding of the surfaces in space-time. This is reflected in the fact that the connection thus defined possesses torsion which is proportional to the extrinsic curvature of the surfaces. Therefore, to obtain a completely intrinsic covariant derivative operator on the hypersurfaces, we define for an arbitrary spinor π_C the operator

$$\partial_{AB} \pi_C := D_{AB} \pi_C + \tfrac{1}{2} K_{ABC}{}^D \pi_D. \tag{3.3}$$

The connection defined by this derivative operator is torsion free and respects the intrinsic metric of the hypersurfaces. Thus, it is the $SU(2)$ spin connection of the intrinsic metric. In complete analogy, we define the operator

$$\partial \pi_C := D \pi_C + \tfrac{1}{2} K_C{}^D \pi_D. \tag{3.4}$$

This operator defines a connection along the integral curves of t^a which turns out to be the spinorial equivalent of the Fermi-Walker connection. E.g., a vector $v^a = v^{AB}$ is Fermi-Walker transported along the t^a curves if $\partial v^{AB} = 0$.

These two operators are real in the sense that they map real fields into real fields, which is obvious from the relations

$$\widehat{\partial \pi_C} = \partial \hat{\pi}_C, \tag{3.5}$$

$$\widehat{\partial_{AB} \pi_C} = -\partial_{AB} \hat{\pi}_C. \tag{3.6}$$

In order to phrase the structure equations in this formulation, we need to know about the commutators of these operators because these define the torsion and the curvature. The commutators are given by the formula, valid for an arbitrary spinor α_C,

$$[\partial, \partial_{AB}] \alpha_C = \tfrac{1}{2} K_{AB} \partial \alpha_C + K_{AB}{}^{EF} \partial_{EF} \alpha_C - \Box_{AB} \alpha_C + \hat{\Box}_{AB} \alpha_C + \tfrac{1}{2} \{ \partial_{AB} K_{CD} - \partial K_{ABCD} - K_{AB}{}^{EF} K_{CDEF} + K_{(C}{}^E K_{D)EAB} + \tfrac{1}{2} K_{AB} K_{CD} \} \alpha^D \tag{3.7}$$

and

$$2\partial_{E(A}\partial^{E}{}_{B)}\alpha_C = \Box_{AB}\alpha_C + \hat{\Box}_{AB}\alpha_C - \{\partial_{E(A}K^{E}{}_{B)CD} - \tfrac{1}{2}KK_{ABCD} + \tfrac{1}{2}K_{EFAB}K^{EF}{}_{CD} + \tfrac{1}{4}\epsilon_{C(A}\epsilon_{B)D}(K_{EFGH}K^{EFGH} - K^2)\}\alpha^D, \tag{3.8}$$

where we have defined the trace of the extrinsic curvature $K := K^{AB}{}_{AB}$ and introduced the curvature derivations

$$\Box_{AB}\alpha_C = -\Psi_{ABC}{}^{D}\alpha_D + 2\Lambda\epsilon_{C(A}\alpha_{B)}, \tag{3.9}$$

$$\hat{\Box}_{AB}\alpha_C = -\hat{\Phi}_{ABC}{}^{D}\alpha_D. \tag{3.10}$$

Here, we have used the curvature spinors, i.e., the Weyl spinor Ψ_{ABCD}, the (space spinor equivalent of the) Ricci spinor $\Phi_{ABCD} := \Phi_{ABA'B'}t^{A'}{}_{C}t^{B'}{}_{D}$ and the scalar curvature Λ.

In view of the reality properties of the derivative operators, we find that the commutation relation can be split into various parts. This yields equations for the spinor fields K_{AB} and K_{ABCD},

$$\begin{aligned}\partial K_{ABCD} = {} & \tfrac{1}{2}(\partial_{AB}K_{CD} + \partial_{CD}K_{AB}) + K_{AB}{}^{EF}K_{EFCD} \\ & + \tfrac{1}{2}K_{AB}K_{CD} + (\Phi_{ABCD} + \Phi_{CDAB}) \\ & - (\hat{\Psi}_{ABCD} + \Psi_{ABCD}) \\ & + 4\Lambda\epsilon_{C(A}\epsilon_{B)D},\end{aligned} \tag{3.11}$$

$$\partial_{C(A}K^{C}{}_{B)} = 0, \tag{3.12}$$

$$\partial_{E(A}K^{E}{}_{B)CD} = \tfrac{1}{2}(\Psi_{ABCD} - \hat{\Psi}_{ABCD}) - \tfrac{1}{2}(\Phi_{ABCD} - \Phi_{CDAB}) \tag{3.13}$$

and simplified commutation relations

$$\begin{aligned}[\partial, \partial_{AB}]\alpha_C = {} & \tfrac{1}{2}K_{AB}\partial\alpha_C + K_{AB}{}^{EF}\partial_{EF}\alpha_C \\ & + \tfrac{1}{2}K^{E}{}_{(C}K_{D)EAB}\alpha^D - \tfrac{1}{2}(\Psi_{ABCD} - \hat{\Psi}_{ABCD})\alpha^D \\ & - \tfrac{1}{2}(\Phi_{ABCD} - \Phi_{CDAB})\alpha^D\end{aligned} \tag{3.14}$$

$$\begin{aligned}2\partial_{E(A}\partial^{E}{}_{B}\alpha_C = {} & \{\tfrac{1}{2}KK_{ABCD} - \tfrac{1}{2}K_{EFAB}K^{EF}{}_{CD} - \tfrac{1}{4}\epsilon_{C(A}\epsilon_{B)D}(K^{EFGH}K_{EFGH} - K^2)\}\alpha^D - 2\Lambda\epsilon_{C(A}\epsilon_{B)D}\alpha^D \\ & + \tfrac{1}{2}(\Psi_{ABCD} + \hat{\Psi}_{ABCD})\alpha^D + \tfrac{1}{2}(\Phi_{ABCD} + \Phi_{CDAB})\alpha^D.\end{aligned} \tag{3.15}$$

When acting on functions, these commutators yield

$$[\partial, \partial_{AB}]f = \tfrac{1}{2}K_{AB}\partial f + K_{AB}{}^{EF}\partial_{EF}f, \tag{3.16}$$

$$2\partial_{E(A}\partial^{E}{}_{B)}f = 0. \tag{3.17}$$

IV. THE EQUATIONS

We are now in a position to give the derivation of the conformal field equations on the unphysical manifold M. To this end, we need to introduce coordinates and a spin frame with respect to which we express all the spinor fields involved. Since we have already assumed the existence of a family of space-like hypersurfaces, we may now introduce a time coordinate t by requiring that it be constant with non-vanishing differential on the hypersurfaces. Then, necessarily, we have $t_a \propto \nabla_a t$. Now, we choose arbitrary coordinates $\{x^1, x^2, x^3\}$ on M. These coordinates can be characterized by their change along the integral curves of t^a, thus defining lapse function and shift vector.

The choice of frame is probably best described in terms of orthonormal tetrads. On the initial surface, we choose the time-like leg of the tetrad to be proportional to t^a. Then the other members are tangent to the surface and we choose them arbitrarily. To propagate the tetrad off the hypersurface, we could use the Fermi-Walker transport along the integral curves of t^a which has the property that it leaves the angles between vectors constant, while keeping the tangent vector along the curve fixed. However, this is not the most general propagation law with these properties. Any other one with the above properties differs from the Fermi-Walker transport law by the addition of a term which involves an infinitesimal rotation. Thus, we fix an arbitrary transport law which leaves angles invariant and fixes the tangent vector to propagate the frame into the full space-time. The infinitesimal rotation involved in the transport law determines some of the Ricci rotation coefficients of the tetrad thus obtained.

In terms of spinors, this choice of frame is expressed as follows. On the initial surface, we choose a normalized spinor field o_A, i.e., we have $\hat{o}^A o_A = 1$. To complete the spin frame, we define $\iota_A = \hat{o}_A$. Then the orthonormal tetrad constructed from this spin frame has the above properties on the initial surface. Now we impose the transport equation in the form

$$\partial o_A = F_{AB}o^B, \tag{4.1}$$

where F_{AB} is an arbitrary purely imaginary and symmetric spinor field. It corresponds exactly to the infinitesimal rotation mentioned above. It was already pointed out that ∂ corresponds to the Fermi-Walker transport, which is, consequently, selected by choosing $F_{AB} = 0$.

The connection is defined by the spinor fields K_{AB} and K_{ABCD}, the field F_{AB} and finally by the field Γ_{ABCD}, defined by

$$\partial_{AB}o_C = \Gamma_{ABCD}o^D. \tag{4.2}$$

It satisfies the relations $\Gamma_{ABCD} = \Gamma_{(AB)(CD)} = -\hat{\Gamma}_{ABCD}$, thus being purely imaginary.

The frame components with respect to a coordinate basis are usually obtained by applying the vector fields which make up the tetrad to the coordinates. Similarly, in the present case: we apply the operators ∂ and ∂_{AB} to the coordinates

$$\partial t = \frac{1}{N}, \quad \partial_{AB}t = 0, \tag{4.3}$$

$$\partial x^i = -T^i, \quad \partial_{AB}x^i = C^i_{AB}. \tag{4.4}$$

This defines several additional fields on M which fix the frame in terms of the chosen coordinates. The second equation reflects the fact that we have chosen the time-like leg of the tetrad to be the unit normal of the surfaces throughout. The function N is the lapse, while the three functions T^i are closely related to the shift vector which appear in all variants of $3+1$ decompositions.

To make the relationship between the frame as defined above and the coordinate basis somewhat more precise, let us introduce the 1-forms θ, θ^{AB} which are dual to the operators ∂, ∂_{AB} considered as vector fields on M. I.e., for any spinor field α^{AB}, we have the relations

$$\langle \theta, \alpha^{AB}\partial_{AB}\rangle = 0, \quad \langle \theta^{AB}, \alpha^{CD}\partial_{CD}\rangle = \alpha^{AB}, \tag{4.5}$$

$$\langle \theta, \partial\rangle = 1, \quad \langle \theta^{AB}, \partial\rangle = 0. \tag{4.6}$$

Then the metric g on M when expressed in terms of the present formalism is simply

$$g = 2\,\theta\otimes\theta - \theta_{AB}\otimes\theta^{AB}. \tag{4.7}$$

We have the following relations between the coordinate differentials and the forms θ, θ^{AB}:

$$dt = \frac{1}{N}\theta, \quad dx^i = C^i_{AB}\theta^{AB} - T^i\theta, \tag{4.8}$$

$$\theta = N dt, \quad \theta^{AB} = D^{AB}_k(dx^k + T^k N dt), \tag{4.9}$$

with D^{AB}_k being the inverse of C^i_{AB}, i.e., $D^{AB}_k C^i_{AB} = \delta^i_k$. From these we get the metric expressed in terms of the frame components

$$g = 2N^2dt^2 - D^{AB}_i D_{ABk}(dx^i + NT^i dt)\otimes(dx^k + NT^k dt). \tag{4.10}$$

We now come to the system of equations. To express the first structure equation in terms of the fields defined above, we apply the commutators to the coordinates which gives the equations

$$\partial_{AB}N = \tfrac{1}{2}NK_{AB}, \tag{4.11}$$

$$\partial_{(A}{}^C K_{B)C} = 0, \tag{4.12}$$

$$\partial C^i_{AB} + \partial_{AB}T^i = -\tfrac{1}{2}K_{AB}T^i + K_{AB}{}^{EF}C^i_{EF}, \tag{4.13}$$

$$\partial^C{}_{(A}C^i_{B)C} = 0. \tag{4.14}$$

The second structure equation is obtained analogously by applying the commutators to o_C yielding the evolution equation for Γ,

$$\begin{aligned}\partial\Gamma_{ABCD} = {}& \partial_{AB}F_{CD} - 2\Gamma_{ABE(D}F^E{}_{C)} + \tfrac{1}{2}K_{AB}F_{CD} \\ & + K_{AB}{}^{EF}\Gamma_{EFCD} + \tfrac{1}{2}K^E{}_{(C}K_{D)EAB} \\ & - \tfrac{1}{2}\{\Psi_{ABCD} - \hat{\Psi}_{ABCD}\} - \tfrac{1}{2}\{\Phi_{ABCD} - \Phi_{CDAB}\}\end{aligned} \tag{4.15}$$

and the constraint equation

$$\begin{aligned}2\partial_{E(A}\Gamma^E{}_{B)CD} = {}& 2\Gamma^E{}_{(B}{}^F{}_{|C|}\Gamma_{A)EDF} + \tfrac{1}{2}\{KK_{ABCD} - K_{EFAB}K^{EF}{}_{CD} - \tfrac{1}{2}\epsilon_{C(A}\epsilon_{B)D}(K^{EFGH}K_{EFGH} - K^2)\} - 2\Lambda\epsilon_{C(A}\epsilon_{B)D} \\ & + \tfrac{1}{2}\{\Psi_{ABCD} + \hat{\Psi}_{ABCD}\} + \tfrac{1}{2}\{\Phi_{ABCD} + \Phi_{CDAB}\}.\end{aligned} \tag{4.16}$$

The derivation of the equation for the curvature and the conformal factor will be given first using primed and unprimed spinors. At the end of this section, we will collect all the equations in the space spinor formalism. To obtain equations for the curvature components, we need to look at the Bianchi identities. These have the following spinorial form:

$$\nabla_{B'}{}^A\Psi_{ABCD} = \nabla^{A'}{}_{(B}\Phi_{CD)A'B'} \tag{4.17}$$

$$\nabla^{BB'}\Phi_{ABA'B'} + 3\nabla_{AA'}\Lambda = 0, \tag{4.18}$$

which is valid in any space-time. Since the Weyl curvature is conformally invariant, the first of these equations when viewed in the physical space-time takes the form

$$\tilde{\nabla}_{B'}{}^A\Psi_{ABCD} = 0, \tag{4.19}$$

with $\tilde{\nabla}$ being the physical connection. Here we have used the Einstein equation and the fact that the energy-momentum tensor in the physical space-time vanishes. The conformal transformation behavior of the connection implies that this equation is conformally invariant provided we rescale

Ψ_{ABCD} with the conformal factor in the appropriate way. Thus, if we define the rescaled Weyl spinor

$$\psi_{ABCD}=\Omega^{-1}\Psi_{ABCD} \tag{4.20}$$

then we have the equation

$$\nabla_{B'}{}^{A}\psi_{ABCD}=0 \tag{4.21}$$

holding on M. Note, that ψ_{ABCD} is well behaved on M, because the Weyl curvature vanishes on $\mathcal{J}$, where also $\Omega=0$.

Expressing the Bianchi identities (4.17) in terms of the rescaled Weyl spinor and using the Eq. (4.21) together with the definition

$$\Sigma_{AA'}:=\nabla_{AA'}\Omega \tag{4.22}$$

we get an equation for the Ricci spinor

$$\nabla_{B}{}^{B'}\Phi_{CDA'B'}=\Sigma_{A'}{}^{E}\psi_{EBCD}+2\epsilon_{B(C}\nabla_{D)A'}\Lambda. \tag{4.23}$$

When these equations are expressed in terms of space spinors using the operators ∂ and ∂_{AB}, they become rather long. To simplify them, we decompose every spinor and every equation into their irreducible parts, thus obtaining a set of equations which can be further decomposed into real and imaginary parts. The resulting equations will be displayed below.

The next set of equations is the definition of Σ viewed as an equation for Ω and the equation which expresses the conformal transformation behavior of the Ricci spinor viewed as an equation for $\Sigma_{AA'}$:

$$\nabla_{BB'}\Sigma_{AA'}=-\Omega\Phi_{ABA'B'}+\epsilon_{AB}\epsilon_{A'B'}S \tag{4.24}$$

with $S:=1/4\Box\Omega$. The final equation is an equation for S which can be derived from the equations established so far. It reads

$$\nabla_{AA'}S=-\Phi_{ABA'B'}\Sigma^{BB'}+\Omega\nabla_{AA'}\Lambda+2\Lambda\Sigma_{AA'}. \tag{4.25}$$

The conformal transformation behavior of the scalar curvature implies an algebraic equation

$$2\Omega S-2\Omega^{2}\Lambda-\Sigma_{AA'}\Sigma^{AA'}=0. \tag{4.26}$$

This completes the derivation of the system of equations and we now want to present the full list of equations in the space spinor formalism. The variables for which we have an evolution equation are: N, C^{i}_{AB}, K_{AB}, K_{ABCD}, Γ_{ABCD}, $E_{ABCD}:=\frac{1}{2}(\psi_{ABCD}+\hat{\psi}_{ABCD})$, $B_{ABCD}:=(1/2i)(\psi_{ABCD}-\hat{\psi}_{ABCD})$, ϕ_{ABCD}, ϕ_{AB}, ϕ, Ω, Σ, Σ_{AB} and S. In this list we have included the irreducible parts of Φ_{ABCD} and $\Sigma_{AA'}$ defined by the decompositions

$$\Sigma_{AA'}=\tfrac{1}{2}t_{AA'}\Sigma-t_{A'}{}^{B}\Sigma_{AB}, \tag{4.27}$$

$$\Phi_{ABCD}=\phi_{ABCD}+\tfrac{1}{2}\epsilon_{A(C}\phi_{D)B}+\tfrac{1}{2}\epsilon_{B(C}\phi_{D)A}-\tfrac{1}{3}\epsilon_{A(C}\epsilon_{D)B}\phi. \tag{4.28}$$

We have included in this list the lapse N and K_{AB} for which we do not have an evolution equation yet. This can easily be achieved by computation of the ''harmonicity function'' $F:=2\Box t$. Expressing the wave operator in terms of ∂ and ∂_{AB} yields the equation for N, while commuting ∂ and ∂_{AB} on N gives the evolution equation for K_{AB}.

The evolution equations can be grouped together according to the geometric meaning of the variables:

The evolution of the frame components

$$\partial N=-KN-N^{2}F, \tag{4.29}$$

$$\partial C^{i}_{AB}=K_{AB}{}^{EF}C^{i}_{EF}-\partial_{AB}T^{i}-\tfrac{1}{2}K_{AB}T^{i}. \tag{4.30}$$

The evolution of the extrinsic curvature and the acceleration vector

$$\partial K_{AB}+2\partial^{CD}K_{ABCD}=K_{AB}{}^{EF}K_{EF}-K_{AB}K-4\phi_{AB}-2N\partial_{AB}F-NK_{AB}F, \tag{4.31}$$

$$\begin{aligned}\partial K_{ABCD}-\tfrac{1}{2}(\partial_{AB}K_{CD}+\partial_{CD}K_{AB})&=K_{AB}{}^{EF}K_{EFCD}+\tfrac{1}{2}K_{AB}K_{CD}+2\phi_{ABCD}\\&\quad-2\Omega E_{ABCD}+\tfrac{2}{3}(6\Lambda+\phi)\epsilon_{C(A}\epsilon_{B)D}.\end{aligned} \tag{4.32}$$

The evolution of the intrinsic connection

$$\begin{aligned}\partial\Gamma_{ABCD}&=\partial_{AB}F_{CD}-2\Gamma_{ABE(D}F^{E}{}_{C)}+\tfrac{1}{2}K_{AB}F_{CD}\\&\quad+K_{AB}{}^{EF}\Gamma_{EFCD}+\tfrac{1}{2}K^{E}{}_{(C}K_{D)EAB}-i\Omega B_{ABCD}\\&\quad-\epsilon_{(A(C}\phi_{D)B)}.\end{aligned} \tag{4.33}$$

The evolution of the Ricci curvature

$$\begin{aligned}\partial\phi_{ABCD}-\partial_{(AB}\phi_{CD)}&=K_{(AB}\phi_{CD)}+K_{(AB}{}^{EF}\phi_{CD)EF}\\&\quad-\tfrac{2}{3}K_{(ABCD)}\phi-\Sigma E_{ABCD}\\&\quad+2i\Sigma_{(A}{}^{E}B_{BCD)E},\end{aligned} \tag{4.34}$$

$$\begin{aligned}\partial\phi_{AB}-\tfrac{2}{3}\partial_{AB}\phi+\partial^{CD}\phi_{ABCD}&=\tfrac{2}{3}K_{AB}\phi-\phi_{ABCD}K^{CD}\\&\quad+\tfrac{3}{2}K_{ABCD}\phi^{CD}-\tfrac{1}{2}K\phi_{AB}\\&\quad+E_{ABCD}\Sigma^{CD}-4\partial_{AB}\Lambda,\end{aligned} \tag{4.35}$$

$$\partial\phi+2\partial_{AB}\phi^{AB}=-2K_{AB}\phi^{AB}-2K^{ABCD}\phi_{ABCD}+\tfrac{4}{3}K\phi-2\partial\Lambda. \tag{4.36}$$

The evolution of the Weyl curvature

$$\partial E_{ABCD}-2i\partial_{(A}{}^{E}B_{BCD)E}=2iK_{(A}{}^{E}B_{BCD)E}-3K_{(AB}{}^{EF}E_{CD)EF}+2K_{ABCD} \tag{4.37}$$

$$\partial B_{ABCD}+2i\partial_{(A}{}^{E}E_{BCD)E} =-2iK_{(A}{}^{E}E_{BCD)E}-3K_{(AB}{}^{EF}B_{CD)EF} +2KB_{ABCD}. \quad (4.38)$$

The evolution of the conformal factor

$$\partial\Omega=\Sigma, \quad (4.39)$$

$$\partial\Sigma=-K_{AB}\Sigma^{AB}-\Omega\phi+2S, \quad (4.40)$$

$$\partial\Sigma_{AB}=\frac{1}{2}K_{AB}\Sigma-\Omega\phi_{AB}, \quad (4.41)$$

$$\partial S=\frac{1}{2}\phi\Sigma-\phi_{AB}\Sigma^{AB}+\Omega\partial\Lambda+2\Lambda\Sigma. \quad (4.42)$$

This completes the list of evolution equations. We need to make several remarks.

The gauge functions in this system are the ''harmonicity'' F, the shift functions T^i, the frame rotations F_{AB}, and the scalar curvature Λ. These eight functions can be chosen almost at will. From the form of the metric, we infer a condition which has to be satisfied by the shift functions. Since the vector ∂_t needs to be time-like, we find the inequality

$$D_k^{AB}D_{ABi}T^iT^k>2. \quad (4.43)$$

The full gauge freedom of eleven functions has been reduced to eight because of our fixing of the time-like leg of the tetrad.

Since the operator ∂ is the directional derivative along the vector field t^a when acting on functions, most of the equations are simple advection equations along that vector field. This is obvious from the explicit form of $\partial f=(1/N)\partial_t-T^i\partial_{x^i}$. There are three subsystems for which this is not the case. These are the systems describing the evolution of K_{AB} and K_{ABCD}, of the Ricci curvature and of the Weyl curvature. They will have considerable significance later when it comes to the numerical treatment of the equations at the boundary.

It is a useful property of the space spinor formalism that the equations automatically come out separated into constraints and evolution equations and that the evolution equations automatically come out as a symmetric hyperbolic system. This is the case for the above system. As written, it is in symmetric hyperbolic form. The symmetric hyperbolicity is the basic property which allows the proof of existence and uniqueness of solutions for various initial value problems see, e.g. [10].

In a similar way, the constraint equations can be grouped according to their geometric meaning starting with the frame components

$$0=\partial_{AB}N-\tfrac{1}{2}K_{AB}N, \quad (4.44)$$

$$0=\partial_{(A}{}^{C}C^i_{B)C}. \quad (4.45)$$

Next are the extrinsic curvature and the acceleration of the time-like unit normal to the surfaces

$$0=\partial_{(A}{}^{C}K_{B)C}, \quad (4.46)$$

$$0=\partial_{(A}{}^{E}K_{B)ECD}+i\Omega B_{ABCD}-\epsilon_{(A(C}\phi_{D)B)}, \quad (4.47)$$

the intrinsic connection

$$0=2\partial_{(A}{}^{E}\Gamma_{B)ECD}+\tfrac{1}{2}KK_{ABCD}-\tfrac{1}{2}K_{AB}{}^{EF}K_{CDEF} -\tfrac{1}{4}\epsilon_{C(A}\epsilon_{B)D}(K_{EFGH}K^{EFGH}-K^2)+2\Gamma^{E}{}_{(A}{}^{F}{}_{|C|}\Gamma_{B)EDF} -\tfrac{1}{3}\epsilon_{C(A}\epsilon_{B)D}(\phi+6\Lambda)+\Omega E_{ABCD}+\phi_{ABCD}, \quad (4.48)$$

the Ricci curvature components

$$0=\partial_{(A}{}^{E}\phi_{BCD)E}-\tfrac{1}{2}K_{(ABC}{}^{E}\phi_{D)E}-\frac{i}{2}B_{ABCD}-\Sigma_{(A}{}^{E}E_{BCD)E}, \quad (4.49)$$

$$0=\partial^{CD}\phi_{ABCD}+\tfrac{1}{3}\partial_{AB}\phi+\tfrac{1}{2}K_{ABCD}\phi^{CD} -\tfrac{1}{2}K\phi_{AB}+\Sigma^{CD}E_{ABCD}+2\partial_{AB}\Lambda, \quad (4.50)$$

$$0=\partial_{(A}{}^{E}\phi_{B)E}+K_{(A}{}^{CDE}\phi_{B)CDE}-i\Sigma^{CD}B_{ABCD}, \quad (4.51)$$

and the constraints for the Weyl spinor

$$0=\partial^{CD}E_{ABCD}-iK_{(A}{}^{CDE}B_{B)CDE}, \quad (4.52)$$

$$0=\partial^{CD}B_{ABCD}+iK_{(A}{}^{CDE}E_{B)CDE}. \quad (4.53)$$

The rest of the constraints is concerned with the conformal factor and its derivatives

$$0=\partial_{AB}\Omega-\Sigma_{AB}, \quad (4.54)$$

$$0=\partial_{AB}\Sigma+K_{ABCD}\Sigma^{CD}+\Omega\phi_{AB}, \quad (4.55)$$

$$0=\partial_{AB}\Sigma_{CD}-\tfrac{1}{2}K_{ABCD}\Sigma+\Omega\phi_{ABCD} +\tfrac{1}{6}\epsilon_{C(A}\epsilon_{B)D}(\Omega\phi+6S), \quad (4.56)$$

$$0=\partial_{AB}S+\phi_{ABCD}\Sigma^{CD}+\tfrac{1}{2}\phi_{AB}\Sigma -\tfrac{1}{6}\phi\Sigma_{AB}-\Omega\partial_{AB}\Lambda+2\Lambda\Sigma_{AB}. \quad (4.57)$$

The final constraint is the algebraic condition mentioned earlier

$$0=2\Omega S-2\Omega^2\Lambda-\tfrac{1}{2}\Sigma^2-\Sigma_{AB}\Sigma^{AB}. \quad (4.58)$$

Finally, a note on the name conventions of the various spinor components. In the above equations, all the spinor fields are irreducible except for the extrinsic curvature and the intrinsic three connection which have the decompositions

$$K_{ABCD}=K_{4ABCD}-\tfrac{1}{3}\epsilon_{C(A}\epsilon_{B)D}K, \quad (4.59)$$

$$\Gamma_{ABCD} = \gamma_{4ABCD} + \tfrac{1}{2}\epsilon_{A(C}\gamma_{2D)B} + \tfrac{1}{2}\epsilon_{B(C}\gamma_{2D)A} - \tfrac{1}{3}\epsilon_{C(A}\epsilon_{B)D}\gamma \tag{4.60}$$

with K_{4ABCD} being totally symmetric. The components of the irreducible parts with respect (o_A, ι_A) are defined as follows for a four index spinor α_{ABCD}

$$\alpha_{ABCD} = \alpha_4 o_A o_B o_C o_D - 4\alpha_3 o_{(A} o_B o_C \iota_{D)} + 6\alpha_2 o_{(A} o_B \iota_C \iota_{D)} - 4\alpha_1 o_{(A} \iota_B \iota_C \iota_{D)} + \alpha_0 \iota_A \iota_B \iota_C \iota_D, \tag{4.61}$$

and for a two index spinor

$$\alpha_{AB} = \alpha_2 o_A o_B - 2\alpha_1 o_{(A} \iota_{B)} + \alpha_0 \iota_A \iota_B. \tag{4.62}$$

When we have spinors with the same kernel symbol, but different numbers of indices like ϕ_{AB} and ϕ_{ABCD} we specify the number of indices in the name of the components, thus obtaining, e.g., ϕ_{40} as a component of ϕ_{ABCD} and ϕ_{22} as a component of ϕ_{AB}.

V. THE SYMMETRY REDUCTION

Finally we discuss a simplification which has been used to reduce the resource requirements. Since the main interests in this project are the development of procedures to locate $\mathcal{J}$, to extract the radiation information from there and to study several various gauge choices, it is legitimate to assume the existence of a continuous symmetry. This is because to locate $\mathcal{J}$ as the zero-set of the conformal factor is not much more difficult in three dimensions than it is in two. On the other hand, one needs to have at least two nontrivial spatial dimensions because otherwise the Weyl curvature vanishes identically so that there will be no radiation present. Hence, in the sequel we will assume the existence of a space-like Killing field ξ^a in the physical space-time, which in addition is required to be hypersurface orthogonal. In order to exclude numerical problems with coordinate singularities, we follow Schmidt [24], who suggested to look at situations, where the Killing vector has no singular points. This excludes the physically intuitive axisymmetry which has fixed points on the axis and leaves us with a completely nonphysical toy problem. It also has the implication that $\mathcal{J}$ no longer has spherical cross sections, but toroidal ones. These global questions are not relevant to local considerations such as the influence of different choices of gauge functions on the solution or even the question of stability of the numerical method. They do, however, forcefully come to the fore when it comes to defining and interpreting global quantities such as the Bondi mass or the radiation flux. These are issues that have not yet been discussed. We will consider them in somewhat more detail in [20].

We adapt the gauge to the symmetry. First of all, the Killing vector in the physical space-time becomes a conformal Killing vector in the unphysical space-time. Choosing an appropriate conformal gauge, we may achieve that it becomes a Killing vector. Then the conformal factor is invariant under the symmetry and the conformal gauge freedom is reduced to multiplication with functions which are also invariant. We assume that the foliation of M into space-like hypersurfaces is compatible with the symmetry, i.e., that ξ^a is everywhere tangent to the hypersurfaces (i.e., orthogonal to t^a). Next we choose the frame in such a way that one of the space-like legs points along the Killing vector and that it is invariant under the action of the symmetry group. Note, that this restricts the available frame rotation from the orthogonal group to rotations around the Killing vector. Finally, we take one of the coordinates, say x^3 to be the coordinate along the integral curves of the Killing vector. Then all components of the geometric quantities with respect to the adapted frame are independent of that coordinate.

These choices can be made irrespective of whether the Killing vector is hypersurface orthogonal or not. They do not entail much simplification in the equations except for the fact that some frame components and connection coefficients vanish. In particular, the Weyl curvature still has all of its ten components. However, assuming hypersurface orthogonality simplifies things considerably. This is because it is equivalent to the existence of a discrete symmetry $\xi^a \mapsto -\xi^a$. To explain the simplification, it is best to consider an example. Let C_{abcd} be the Weyl tensor on M. The electric part with respect to the Killing vector ξ^a is proportional to $\xi^a \xi^c C_{abcd}$ which is symmetric under the discrete symmetry. However, the magnetic part with respect to ξ^a which is proportional to $\xi^a \xi^c C^{\star}_{abcd}$ changes sign under the symmetry hence it has to vanish. This reduces the Weyl curvature down to five independent components. Similar consideration can be made for the geometric quantities showing that the number of independent variables reduces from fifty-three down to thirty-three.

In the space spinor formalism, we take the Killing vector to be of the form $\xi^{AB} = \xi^{BA} \propto o^{(A}\iota^{B)}$. Then the discrete symmetry implies that the components of almost all of the spinor fields vanish if they are obtained by contraction with an odd number of o^A and ι^A. Only the fields F_{AB}, Γ_{ABCD} and B_{ABCD} have a different behavior. For them it is the even components which have to vanish. So, e.g., the Weyl curvature is described by the five nonvanishing components E_0, E_2, E_4, B_1 and B_3 in agreement with the argument given above.

VI. CONCLUSION

The purpose of the present article was the presentation of the conformal field equations in the space spinor formalism. This paper is intended to serve as a reference for future articles which are intended to discuss the numerical implementation of this set of equations for solving the hyperboloidal initial value problem. We have discussed the simplification obtained from assuming the existence of a hypersurface orthogonal Killing vector field.

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