Chaos in preinflationary Friedmann-Robertson-Walker universes

G. A. Monerat* and H. P. de Oliveira†
Universidade do Estado do Rio de Janeiro, Instituto de Física–Departamento de Física Teórica, CEP 20550-013 Rio de Janeiro, RJ, Brazil

I. Damião Soares‡
Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud, 150 CEP 22290-180, Rio de Janeiro–RJ, Brazil

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The dynamics of a preinflationary phase of the universe, and its exit to inflation, is discussed. This phase is modeled by a closed Friedmann-Robertson-Walker geometry, the matter content of which is radiation plus a scalar field minimally coupled to the gravitational field. The energy-momentum tensor of the scalar field is split into a cosmological constant type term, corresponding to the vacuum energy of the scalar field plus the energy-momentum tensor of the spatially homogeneous expectation value of the scalar field. This simple configuration, with two effective degrees of freedom only, presents a very complicated dynamics connected with the existence of critical points of saddle-center-type and saddle-type in the phase space of the system. Each of these critical points is associated with an extremum of the scalar field potential. The topology of the phase space about the saddle centers is characterized by homoclinic cylinders emanating from unstable periodic orbits, and the transversal crossing of the cylinders, due to the nonintegrability of the system, results in a chaotic dynamics. The topology of the homoclinic cylinders provides an invariant characterization of chaos. The model exhibits one or more exits to inflation, associated with one or more strong asymptotic de Sitter attractors present in phase space, but the way out from the initial singularity into any of the inflationary exits is chaotic. We discuss possible mechanisms, connected with the spectrum of inhomogeneous fluctuations in the models, which would allow us to distinguish physically the several exits to inflation. [S0556-2821(98)03116-6]

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I. INTRODUCTION

The existence of an inflationary phase in the early stages of our Universe has become one of the paradigms of modern cosmology [1], and is now being subject to experimental verification through the crucial measurements of small scale anisotropies in the cosmic background radiation. The basic physical ingredient for this inflationary phase is the existence of a scalar field—the inflaton field, the vacuum energy of which plays the role of a cosmological constant, engendering via the gravitational dynamics an exponential expansion in the comoving scales of the universe. This model may be thought to have evolved from a preinflationary phase just exiting the Planck era. Our attempt here will be to discuss a model for this pre-inflationary phase and its exit to inflation, by using a minimal set of ingredients: in its simplest version the model can be described as a Friedmann-Robertson-Walker (FRW) universe, the matter content of which is a perfect fluid (which we take as radiation) plus a scalar field minimally coupled to the gravitational field. The energy momentum tensor of the scalar field may be split into a cosmological constant-type term (corresponding to the vacuum energy of the scalar field) plus the energy momentum tensor of the spatially homogeneous expectation value of the scalar field. We assume from the start a closed FRW universe. This apparently simple configuration, with two effective degrees of freedom only, will present a very complicated dynamics. In fact, we will show that (i) the model exhibits one or more exits with inflation, associated to one or more asymptotic de Sitter attractors in phase space, depending on the structure of the scalar field potential, (ii) the dynamics will imply that in any of the exits to inflation the scalar field will be frozen in one of the vacuum states (with its corresponding vacuum energy playing the role of the cosmological constant for that exit) associated with an extremum of the scalar field potential, and (iii) the exit to inflation is chaotic.

In the literature of inflation, the exponential expansion of the scales of the model, whenever a positive cosmological constant is present, has been extensively discussed and constitutes the basis of the so-called cosmic no-hair conjecture. Wald [2], in the realm of spatially homogeneous cosmologies, and Starobinskii [3], in the more general case of inhomogeneous models, showed that all initially expanding models evolve towards the de Sitter configuration if a positive cosmological constant is present. Also, the introduction of dynamical degrees of freedom associated with the scalar fields, in addition to a cosmological constant, was shown to produce nontrivial dynamics in the early stages of inflation. In particular, Calzetta and El Hasi [4], and Cornish and Levin [5] exhibited chaotic behavior in the dynamics of FRW models with a cosmological constant term and scalar fields conformally and/or minimally coupled with gravitation, implying that small fluctuations in initial conditions of the model preclude or induce the Universe to inflate. In both cases, the presence of the cosmological constant induces the existence of separatrices in the unperturbed phase space of the models, connecting critical points of the saddle-type.

*Email address: germano@symbcomp.uerj.br
†Email address: oliveira@symbcomp.uerj.br
‡Email address: ivano@lca1.drp.cbpf.br.

These connections are known to be highly unstable, and their breaking and transversal crossing, due to perturbations originated from the coupling of the gravitational variable with the scalar field, are basically responsible for the chaotic dynamics. This chaotic behavior constitutes the so-called Poincare's homoclinic phenomena [6].

The new features in the preinflationary dynamics introduced by our model arise from the presence of a positive cosmological constant and a perfect fluid (matter in radiation form) in a closed FRW universe, whenever a scalar field is also present even in the form of small perturbations, and have not been considered in the literature yet. The degrees of freedom are the scale factor of the FRW geometry and one scalar field. As a consequence, the phase space of the system present one or more critical points, depending on the number of extrema of the potential of the scalar field. If the extremum is a minimum, the corresponding critical point is identified as a saddle center. As a consequence, we have a very complex dynamics based on the so-called homoclinic cylinders which emanate from unstable periodic orbits that exist in a neighborhood of a saddle center. Analogous to the breaking and crossing of heteroclinic curves in above referred homoclinic phenomena, these cylinders will cross each other in the nonintegrable cases producing a chaotic dynamics. These topological structures allow an invariant characterization of chaos in the model [7], as we will discuss, and will have a deep implication for the occurrence or not of inflation in its several exits, as well as for the physics in the early stages of inflation. Chaotic exit to inflation, as a consequence of the existence of a saddle center and its associated structure of homoclinic cylinders, has been examined for the first time in the literature by de Oliveira, Soares, and Stuchi [7] (subsequently referred to as OSS), and contains several technical results which we will often refer to here. The paper is organized as follows. In Sec. II, we describe the Hamiltonian dynamics of the model and the critical points in the phase space. We discuss the nature of these critical points, and describe the topology of homoclinic cylinders about each saddle-center critical point. The numerical evidence for a physically relevant chaotic behavior is shown in Sec. III, where two cases concerning the choices of the potential will be discussed. Finally, in Sec. IV, we conclude and trace some perspectives of the present work.

II. THE DYNAMICS OF THE MODEL

We consider closed FRW cosmological models characterized by the scale function $a(t)$, with line element

$$ds^2=dt^2-a^2(t)[(w^1)^2+(w^2)^2+(w^3)^2]. \tag{2.1}$$

Here t is the cosmological time and (w^1,w^2,w^3) are invariant Bianchi type-IX 1-forms [8]. The matter content of the model is represented by a perfect fluid with four-velocity $U^\mu=\delta^\mu_0$ in the comoving coordinate system used, plus a homogeneous minimally coupled scalar field with potential $V(\varphi)$. The total energy momentum tensor is described by

$$T_{\mu\nu}=(\rho+p)U_\mu U_\nu-pg_{\mu\nu}+\partial_\mu\varphi\partial_\nu\varphi \\ -\frac{1}{2}(\partial^\alpha\varphi\partial_\alpha\varphi-2V(\varphi))g_{\mu\nu}, \tag{2.2}$$

where ρ and p are the energy density and pressure of the fluid, respectively. In the present model we assume the equation of state for the fluid, $p=1/3\rho$. This assumption is justified since the models are supposed to describe a phase of the Universe emerging from the Planck era [9]. Einstein's equations for the metric (2.1) and the energy momentum tensor (2.2) are equivalent to Hamiltonian's equations generated by the Hamiltonian constraint

$$H=-\frac{p_\varphi^2}{2a^3}+\frac{p_a^2}{12a}+3a-a^3V(\varphi)-\frac{E_0}{a}=0, \tag{2.3}$$

where p_a and p_φ are the momenta canonically conjugated to a and φ, respectively, E_0 is a constant arising from the first integral of Bianchi identities, and obviously proportional to the total matter energy of the models. The complete dynamics is governed by Hamilton's equations

$$\begin{aligned}\dot{a}&=\frac{p_a}{6a},\\ \dot{\varphi}&=-\frac{p_\varphi}{a^3},\\ \dot{P}_a&=-\frac{p_\varphi^2}{a^4}-6+4a^2V(\varphi),\\ \dot{P}_\varphi&=a^3V'(\varphi),\end{aligned} \tag{2.4}$$

where $'$ denotes derivative with respect to φ. We have used the Hamiltonian constraint (2.3) to simplify the third of Eqs. (2.4). The dynamical system (2.4) has critical points E_{φ_0} in the finite region of the phase space whose coordinates are

$$E_{\varphi_0}:\varphi=\varphi_0, a=a_0=\sqrt{\frac{3}{2V(\varphi_0)}}, p_a=0, p_\varphi=0, \tag{2.5}$$

where φ_0 is solution of the equation $V'(\varphi_0)=0$. The number of critical points E_{φ_0} is therefore equal to the number of extrema φ_0 of the potential $V(\varphi)$. The energy associated with each critical point is given by

$$E_0=E_{cr}=\frac{3a_0^2}{2}. \tag{2.6}$$

Also, the existence of the critical points demands that $V(\varphi_0)>0$; these critical points correspond to the configuration of the Einstein universe, with the respective cosmological constant given by each $V(\varphi_0)$. The stability analysis is obtained by linearizing the system (2.4) about the critical points E_{φ_0}, resulting in the constant matrix determining the linear system about each E_{φ_0}, with the four eigenvalues

$$\lambda_{1,2}=\pm\sqrt{\frac{4V(\varphi_0)}{3}},\quad \lambda_{3,4}=\pm\sqrt{-V''(\varphi_0)}. \qquad (2.7)$$

From the above we can see that if the particular extremum of $V(\varphi)$ is a minimum, the corresponding critical point is a saddle center [10]. On the contrary, if $V''(\varphi_0)<0$, the critical point is a pure saddle, with four real eigenvalues.

The dynamical system (2.4) admits invariant planes $\mathcal{M}_{\varphi_0}$ defined generically by

$$\mathcal{M}_{\varphi_0}:\varphi=\varphi_0,\quad p_\varphi=0. \qquad (2.8)$$

On $\mathcal{M}_{\varphi_0}$ the dynamics is governed by the two-dimensional system

$$\dot{a}=\frac{p_a}{6a},$$

$$\dot{p}_a=-6+4a^2V(\varphi_0). \qquad (2.9)$$

The system (2.9) is integrable, with Hamiltonian constraint $H=p_a^2/12a+3a-a^3V(\varphi_0)-E_0/a=0$. Introducing the conformal time η by $d\eta=dt/a$, the dynamical system (2.9) and its associated Hamiltonian constraint become regular at $a=0$, so that the dynamics can be continuously extended to the region $a<0$[1] as shown in Fig. 1. The integral curves represent closed FRW models, with radiation plus an effective cosmological constant given by the value $V(\varphi_0)$. We remark that each critical point E_{φ_0} belongs to the respective invariant plane $\mathcal{M}_{\varphi_0}$. A straightforward analysis of the infinity of the phase space shows the presence of pairs of critical points, corresponding to the de Sitter solution, one acting as an attractor (stable de Sitter configuration) and the other as a repeller (unstable de Sitter configuration). Each pair of de Sitter attractors is associated with an invariant plane $\mathcal{M}_{\varphi_0}$, with its corresponding effective cosmological constant $V(\varphi_0)$. The scale factor $a(t)$ approaches the stable de Sitter attractors as $a(t)\sim e^{\sqrt{V(\varphi_0)/3}t}$. The stable de Sitter attractors define exits to inflation, and one of the questions to be examined in this paper is the characterization of sets of initial conditions for which one of them is attained. The existence of critical points of saddle-saddle or saddle-center character, and of distinct exits to inflation associated with the de Sitter attractors, is a striking novelty in the dynamics of our models. Moreover, we will show that—due to topology of the phase space of the models—the dynamics of the allowable exits to inflation is highly complex. For example, sets of initial conditions exist for which the exit to inflation is chaotic: arbitrarily small fluctuations of initial conditions in these sets may change the final state of the universe, not only from collapse to escape in the neighborhood of the same invariant plane, but also from collapse or escape about the neighborhood of one invariant manifold into collapse or escape about the neighborhood of another invariant plane.

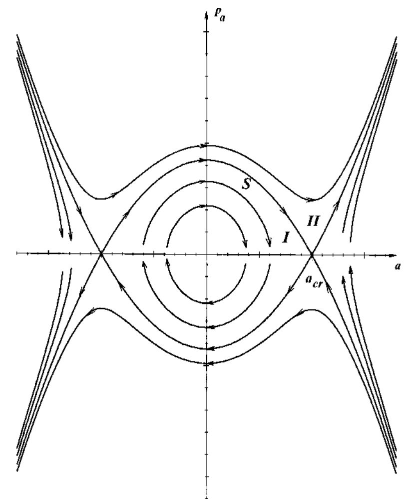

FIG. 1. Phase portrait of the invariant manifold $\varphi=\varphi_0$, $p_\varphi=0$. The orbits represent homogeneous and isotropic universes with radiation and cosmological constant.

This complex dynamics and some of its physical applications will be discussed in the following sections, together with various numerical experiments showing the above mentioned effects, for the case of two standard scalar field potentials: the first with one extremum (minimum) only, and the second having three extrema (two minima and one maximum).

III. THE TOPOLOGY OF PHASE SPACE ABOUT THE CRITICAL POINTS

Our starting point here is to linearize the Hamiltonian (2.3) about its critical points. As we have seen already these critical points are of two types, a saddle center if $V''(\varphi_0)>0$ or a pure saddle if $V''(\varphi_0)<0$. About the critical point, whose coordinates are given by Eq. (2.5), the Hamiltonian may be expressed

$$H=-\frac{p_\varphi^2}{2a_0^3}-\frac{a_0^3}{2}V''(\varphi_0)(\varphi-\varphi_0)^2+\frac{p_a^2}{12a_0}$$
$$-\frac{6}{a_0}(a-a_0)^2+\frac{1}{a_0}(E_{cr}-E_0)+\mathcal{O}(3)=0, \qquad (3.1)$$

where $\mathcal{O}(3)$ denotes higher order terms in the expansion. In a small neighborhood of the critical point, these higher order terms can be neglected and the motion is separable, with the partial energies

[1]We note however that $a=0$ corresponds to the singularity of the curvature tensor and we will obviously restrict our analysis to the physical region $a>0$.

$$E_1=\frac{p_\varphi^2}{2a_0^2}+\frac{a_0^4}{2}V''(\varphi_0)(\varphi-\varphi_0)^2, \quad E_2=\frac{1}{12}p_a^2-6(a-a_0)^2 \tag{3.2}$$

approximately conserved. We have

$$-E_1+E_2+E_{cr}-E_0\sim 0, \tag{3.3}$$

where the quantity $E_{cr}-E_0$ is small. We consider first the case of saddle centers. We note that E_1 is always positive, due to $V''(\varphi_0)>0$, and is associated with rotational motion, while E_2 has no fixed sign and corresponds to hyperbolic motion always. This is in accordance with a theorem by Moser [11] stating that there always exists a set of canonical variables such that, in a small neighborhood of a saddle center, the Hamiltonian is separable into rotational motion and hyperbolic motion pieces. In this approximation, we note from Eq. (3.2) that the scale factor $a(t)$ has pure hyperbolic motion, and is completely decoupled from the scalar field pure rotational motion. The general oscillatory behavior of the orbits are connected with the existence of a manifold of unstable periodic orbits, associated with the saddle center.

To see this, let us briefly describe the topology of homoclinic cylinders in the phase space about a saddle center. A detailed description is done in OSS (cf. also references therein). Let us consider the possible motions in a small neighborhood N of the saddle centers. In the case $E_2=0$ and $p_a=0=a-a_0$, the motions are unstable periodic orbits τ_{E_0} in the plane (φ,p_φ). Such orbits depend continuously on the parameter E_0. For $E_2=0$, there is still the possibility $p_a=\pm(a-a_0)$, which defines the linear stable V_s and unstable V_u one-dimensional manifolds, which are tangent, at the critical point, to the separatrices S of the invariant plane associated with the saddle center (cf. Fig. 1). The separatrices are actually the nonlinear extension of V_s and V_u. The direct product of the periodic orbit τ_{E_0} with V_s and V_u generates, in the linear neighborhood N of the saddle center, the structure of pairs of stable $(\tau_{E_0}\times V_s)$ and unstable $(\tau_{E_0}\times V_u)$ cylinders. Orbits on the cylinders coalesce into the periodic orbit τ_{E_0} for times going to $+\infty$ and $-\infty$, respectively, the energy of the orbits being the same as that of the periodic orbit. The nonlinear extension of the plane of rotational motion, where the linear unstable periodic orbits reside, is a two-dimensional manifold, the *center manifold* [6], of unstable periodic orbits of the system, parametrized with the energy E_0. The intersection of the center manifold with the energy surface E_0 is a periodic orbit of energy E_0, from which two pairs of cylinders emanate, as in the linear case previously described. From Eq. (3.3) we can see that the intersection of the center manifold with the energy surface $E_0=E_{cr}$ is just the critical point; for $E_{cr}-E_0<0$, the energy surface does not intersect the center manifold. It follows that the structure of homoclinic cylinders is present only in the energy surfaces for which $E_{cr}-E_0>0$. In the case $E_2\neq 0$ and $E_{cr}-E_0>0$ the orbits are restricted to infinite cylindrical surfaces which, in a linear neighborhood N of the saddle center, are the product of periodic orbits of the central manifold with small hyperbolas in the plane (a,p_a), which are obviously solutions of Eq. (3.2). We remark that a picture of these hyperbolas is given by the curves of Fig. 1 contained in the small neighborhood N.

Now due to the nonintegrability of the system, the extension of the cylinders away from the periodic orbit is distorted and twisted, with eventual transversal crossings of the unstable cylinder with the stable one. These intersections will produce chaotic sets in the phase space [7,12,13], analogous to the case of breaking and crossing of homoclinic or heteroclinic connections in Poincare's homoclinic phenomena [14]. This provides an invariant characterization of chaos in the general relativistic dynamics of the models.

From the general chaotic behavior of the system, we will single out the following aspect which is of physical interest for inflation. A general orbit which visits the neighborhood N is characterized by $E_1\neq 0$, $E_2\neq 0$. In this region the orbit has an oscillatory approach to the cylinders, the closer as $E_2\to 0$. The *partition* of the energy $|E_{cr}-E_0|$, into the energies E_1 and E_2 of motion about the critical point, will determine the outcome of the oscillatory regime into collapse or escape to inflation (de Sitter attractor). For instance, initially expanding models with energy E_0 will go out the oscillatory regime into collapse or escape if the *partition* of $E_{cr}-E_0$ in N is such that $E_2<0$ or $E_2>0$, respectively. However, the non-integrability of the system (2.4), with the consequent twisting and crossing of homoclinic cylinders, will cause this *partition* of energy to be chaotic in general, and will characterize a chaotic exit to inflation towards the de Sitter attractor of the invariant plane associated with the saddle center. In other words, given a general initial condition of energy E_0, we are no longer able to foretell in what of regions I or II about the saddle center (cf. Fig. 1) the orbit will stay when it approaches the saddle center. Small fluctuations in E_0 or in the initial conditions in these sets will change the outcomes of the orbits from collapse into escape and vice versa, characterizing a chaotic exit to inflation. We note that this chaotic behavior will also set up associated with cylinders emanating from unstable periodic orbits of the center manifold which are not in a linear neighborhood of the saddle center. This will be illustrated thoroughly in the numerical experiments in the following section.

But this is not the whole story of the chaotic exit to inflation in the present model, as we will see if a critical point of the saddle-type is also present in phase space. In the case of a pure saddle, both E_1 and E_2 do not have a fixed sign [note that $V''(\varphi_0)<0$] and correspond obviously to hyperbolic motion only. Instead of homoclinic cylinders in this neighborhood, we have two sets of linear stable V_s and unstable V_u manifolds emanating from the pure saddle, associated with the pair of real eigenvalues (2.7). It is straightforward to see that one of the sets, associated with the eigenvalues $\lambda_{1,2}$, is the linearization (around the saddle point) of the separatrices of the invariant plane containing this critical point. The nonlinear extension of the second set (corresponding to the eigenvalues $\lambda_{3,4}$) constitutes a homoclinic curve that visits the nearest saddle center, approaching a periodic orbit in this neighborhood (cf. Fig. 6). The above-mentioned crossing of homoclinic cylinders will reach the neighborhood of the invariant plane associated with the pure saddle, producing also chaotic sets in this neighborhood.

The chaotic behavior of the system, associated with the exits to inflation, will be the main object of the next section.

IV. CHAOTIC EXITS TO INFLATION

In order to proceed in the numerical examination of the dynamics, we will restrict ourselves to the two choices of the scalar field potential $V(\varphi)$,

$$V_1(\varphi)=\Lambda+\frac{1}{2}m^2\varphi^2 \tag{4.1}$$

and

$$V_2(\varphi)=\Lambda+\frac{\lambda}{4}(\varphi^2-\sigma^2)^2, \tag{4.2}$$

where Λ is included as the vacuum energy of the scalar field (inflaton field), m is the mass of its expectation value, and λ and σ are positive constants. The constant Λ plays the role of a cosmological constant. The usual matter content will be represented by radiation. In the literature of inflation, extensive use has been made of both potentials [1]. In the numerical experiments performed here all calculations were made using the package *Poincarè* [15], where we enforce that the error of the Hamiltonian never exceeds a given threshold of 10^{-10}.

A. Case $V_1(\varphi)=\Lambda+\frac{1}{2}m^2\varphi^2$

The potential has only one extremum (a minimum) at $\varphi_0=0$, and the phase space has only one critical point P of saddle-center type, with coordinates

$$a_0=\sqrt{\frac{3}{2\Lambda}};\quad \varphi=\varphi_0=0;\quad p_a=p_\varphi=0, \tag{4.3}$$

and energy $E_{cr}=9/4\Lambda$. The eigenvalues (2.7) are given, respectively by $\pm\sqrt{4\Lambda/3}$ and $\pm im$, and they are associated with the hyperbolic motion in the plane (a,p_a) and rotational motion in the plane (φ,p_φ), in a neighborhood of P. Note that the eigenvalues corresponding to the rotational motion depend on the mass of the scalar field. There is also one invariant manifold $\mathcal{M}$ defined by $\varphi=p_\varphi=0$.

The phase space under consideration is not compact, and we will actually identify a chaotic behavior associated with the possible asymptotic outcomes of the orbits in this phase space, namely, escape to a de Sitter attractor at infinity representing the inflationary regime, or collapse after a burst of initial expansion.

Following the procedure of OSS, our objective here is first to analyze numerically the behavior of the orbits, the initial conditions of which are taken in a small neighborhood of a point on the separatrix in the invariant manifold $\mathcal{M}$ $(\varphi=p_\varphi=0)$. We assume that $\Lambda=1.5$ and $m=4.0$ such that $a_0=1.0$ and $E_{cr}=1.5$. We select a point S_0 belonging to the separatrix $(E_0=E_{cr}=1.5)$ with coordinates $a=0.4$, $p_a=3.563818177$; around S_0 we construct a four dimensional sphere in phase space with arbitrary small radius $R=10^{-3},10^{-4}$, as the measure of the uncertainty in the initial conditions. The initial conditions $(a,p_a,\varphi,p_\varphi)$ are then taken in energy surfaces which have a nonempty intersection with this sphere, as evaluated from the Hamiltonian constraint. It is easy to see that such energy surfaces are those for which the range of the energy E_0 is in the interval $\mathcal{D}=(1.5-\Delta E_0,1.5-\Delta E_0)$ about the critical energy $E_0=E_{cr}=1.5$, with ΔE_0 of the order of, or smaller than, R. Actually, these initial conditions represent initially expanding models just after the singularity exhibiting small perturbations in the scalar field. The numerical experiments revealed, as expected, two possible outcomes: collapse or escape to the inflationary regime, depending on E_0 that varies from $1.5-\Delta E_0$ to $1.5+\Delta E_0$, with ΔE_0 of order R. Since the energy E_0 was chosen to be very close to the energy of the separatrix, it is not difficult to prove [7] that all orbits visit a small (of order of R) neighborhood N of the critical point before collapsing or escaping into inflation. The final state depends crucially on the *partition* of the energy $|E_0-E_{cr}|$ into the rotational motion mode and the hyperbolic motion mode, in the small neighborhood N, so that, if $E_2>0$ the orbits escape, whereas collapse is characterized by $E_2<0$. In Fig. 2 we illustrate the collapse and the escape of 400 orbits initially in a sphere of radius $R=10^{-4}$. Nevertheless, for each radius R, there exists a non-null interval of energy $\delta E^*=|E_{max}-E_{min}|\subset\mathcal{D}$ for which orbits have an indeterminate outcome; that is, fluctuations in initial conditions of the order of or smaller than $R=10^{-3},10^{-4},\ldots$ will change the long time behavior of an orbit from collapse to escape into inflation, and vice versa. Here E_{min} and E_{max} denote the values of the energy E_0 above or below which all orbits escape or collapse, respectively. In this sense we say that the exit to inflation is chaotic. In Fig. 3 this behavior is shown for 300 orbits corresponding to a sphere of initial conditions with radius $R=10^{-4}$. This result of fundamental importance is an evidence of the chaotic *partition* of the energy E_0-E_{cr} into the energy modes E_1 and E_2 when the orbits are in the small region around P. It is worth mentioning that the indeterminate outcome due to $\delta E^*\neq 0$ occurs only for $E_0-E_{cr}<0$, as expected. This is the energy condition for the presence of homoclinic cylinders. In the case $E_0-E_{cr}\geqslant 0$, we have $E_2>0$ and all orbits escape. By determining numerically the values of E_{min} and E_{max} for several values of R, we obtain the scaling law, $\delta E^*=kR^2$, where k is a constant depending on m. We will discuss this dependence on m elsewhere, but for $m=4$, we have $k\approx 2.576$. An analogous scaling relation was obtained in OSS, in the realm of anisotropic universes. We recall that the above relation is a manifestation of chaos resulting from the crossing of unstable and stable cylinders emanating from the periodic orbits of the center manifold.

The above chaotic behavior is not restricted to sets of initial conditions infinitesimally close to the invariant manifold and whose orbits visit a small neighborhood of P. In Fig. 4 we show the chaotic exit to inflation occurring in a

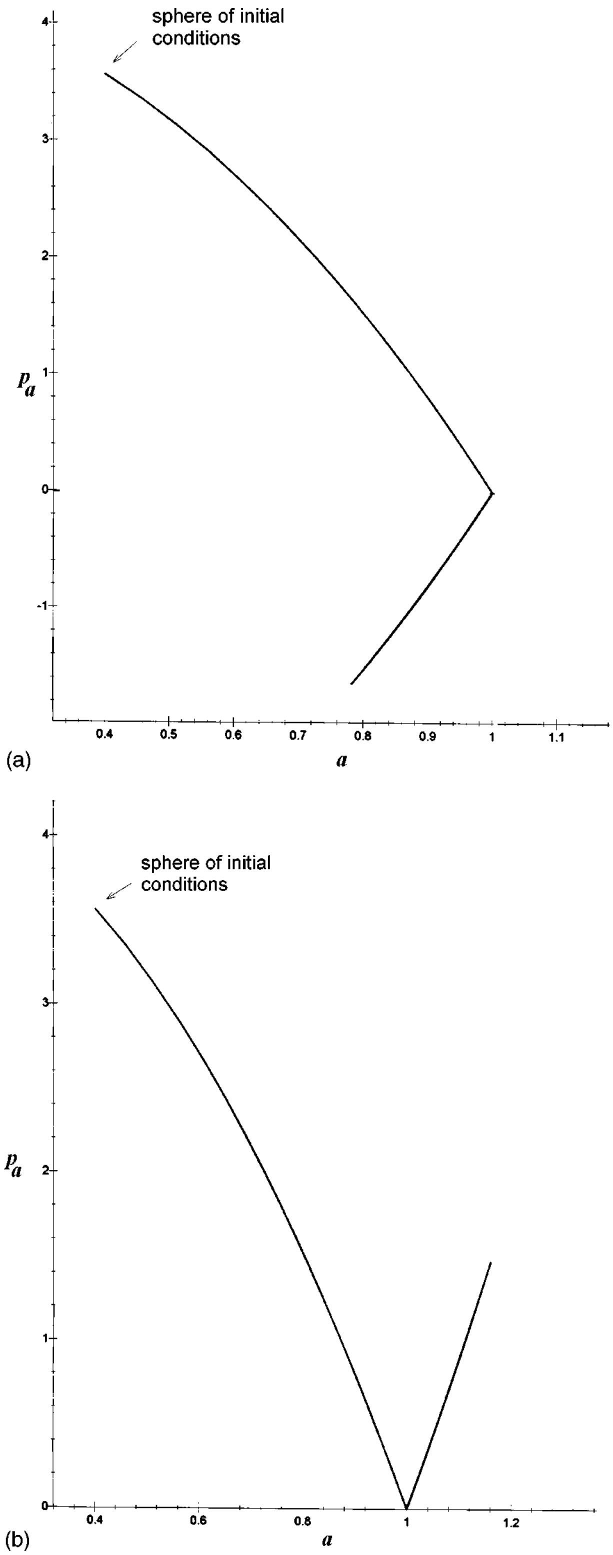

FIG. 2. (a) Collapse of 100 orbits with energy $E_0 = 1.49999994$ initially in a sphere of radius $R = 10^{-4}$ about the point $a = 0.4, p_a = 3.563818177$ on the separatrix of the invariant manifold associated with the saddle center. (b) Escape to inflation of 100 orbits with energy $E_0 = 1.499999999$ initially in a sphere of radius $R = 10^{-4}$ about the same point.

nonlinear neighborhood of the saddle center in which $a(t)$ and $p_a(t)$ oscillate, implying the breakdown of the linear approximation, since, according to Sec. III, the infinitesimal neighborhood of P displays only hyperbolic motions in the plane (a, p_a). In this nonlinear regime about P the scalar field degree of freedom pumps rotational energy into the degree of freedom associated with the gravitational scale factor. The initial conditions were obtained by constructing a small sphere about a point outside the invariant manifold. All orbits oscillate very close to a typical unstable periodic orbit of the center manifold before collapsing or escaping to inflation. The set of unstable periodic orbits are very special solutions that cannot be exactly attained by a physically relevant solution. This numerical experiment also indicates the generality of the chaotic exit to inflation due to the *partition* of the total energy into the rotational energy mode of the periodic orbit and the hyperbolic mode. The latter fixes along which cylinder (of collapse or escape) the motion will flow.

B. Case $V_2(\varphi) = \Lambda + \lambda/4(\varphi^2 - \sigma^2)^2$

For the scalar field potential (4.2), three critical points are present on the phase space. Two of them are saddle centers whereas the third is a pure saddle denoted, respectively, by $P_\pm$ and P_0 whose coordinates are

$$P_\pm: \quad a_\pm = \sqrt{\frac{3}{2\Lambda}}; \varphi = \varphi_0 = \pm\sigma; p_a = p_\varphi = 0, \quad (4.4)$$

$$P_0: \quad a_0 = \sqrt{\frac{3}{2\Lambda_{ef}}}; \varphi = \varphi_0 = 0; p_a = p_\varphi = 0, \quad (4.5)$$

where $\Lambda_{ef} = \Lambda + (\lambda/4)\sigma^4$. The energies of $P_\pm$ and P_0 are given by $E_{cr} = 9/4\Lambda, 9/4\Lambda_{ef}$, respectively. Since the Hamiltonian (2.3) is invariant under the change $\varphi \rightarrow -\varphi$ and $p_\varphi \rightarrow -p_\varphi$, both critical points $P_\pm$ are physically identical. Due to the fact that $\Lambda_{ef} > \Lambda$, it follows $a_\pm > a_0$. The critical points are contained, respectively, in the three invariant manifolds $\mathcal{M}_\pm$ and $\mathcal{M}_0$, defined by $(\varphi = \pm\sigma,\ p_\varphi = 0)$, and $(\varphi = 0,\ p_\varphi = 0)$. The phase portrait of the invariant manifolds are schematically shown in Fig. 1. As we shall see in the sequence, the coexistence of three critical points of distinct topological nature produce very rich and complex dynamics resulting in several chaotic exits to inflation. In the numerical experiments performed in the sequence, we assume $\Lambda = 1.0$, $\lambda = 0.5$, $\sigma = \sqrt{3}$, which produces $a_0 = 0.840168050$, $E_{cr} = 1.058823529$ and $a_\pm = 1.224744871$, $E_{cr} = 2.25$ that are the values of the scale factor of the pure saddle and saddle centers together with the corresponding energies, respectively.

Analogous to the case A, we start the numerical study by examining the long time behavior of orbits generated from initial conditions inside a sphere (as before the radius R is of order $10^{-3}, \ldots,$ etc.) about a point on the separatrices of one of the invariant manifolds $\mathcal{M}_\pm$. These orbits represent universes with radiation and effective cosmological constant $V(\pm\sigma) = \Lambda$ with fluctuations of the scalar field about the symmetry-breaking scale $|\varphi| = \sigma$. The dynamics is similar to the one analyzed in the case A for initial conditions

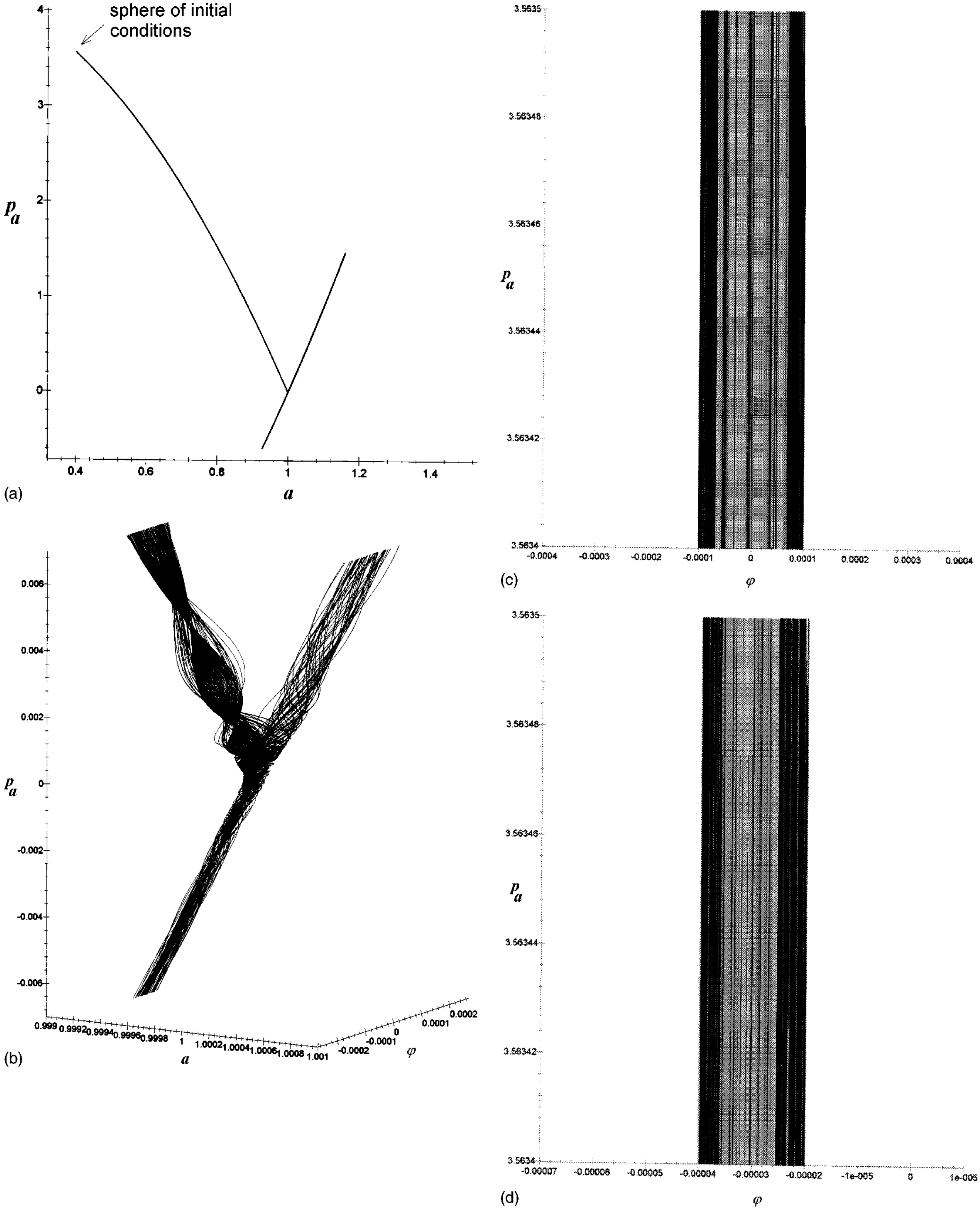

FIG. 3. (a) Chaotic exit to inflation of 300 orbits with energy $E_0 = 1.499999983$ initially in a sphere of $R = 10^{-4}$ about the same point of Fig. 2. (b) Three dimensional view of the region near the saddle center. Note the oscillations of the orbits in this region. (c) Projection of orbits near the sphere of intitial conditions in the plane (φ, p_φ). The strip in black indicates orbits that escape to de Sitter configuration, while those in gray correspond to orbits that collapse. (d) A small strip of (c) $(-0.00004 \leqslant \varphi \leqslant -0.00002)$ is magnified, and repeats the same pattern indicating a fractal structure.

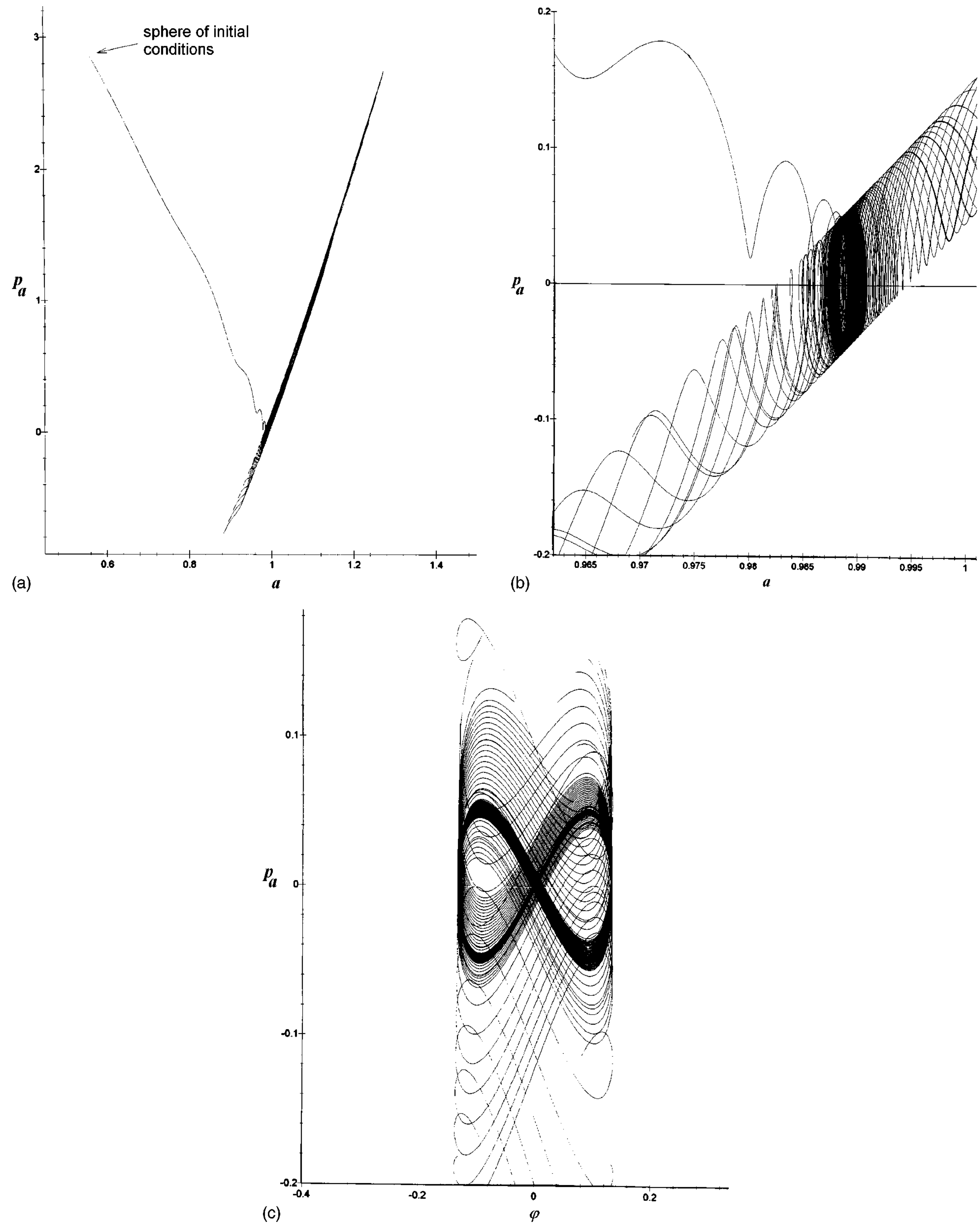

FIG. 4. (a) Chaotic exit to inflation of 30 orbits with energy $E_0 = 1.3660351$ initially inside a ball of radius $= 10^{-8}$ about the point with coordinates $a = 0.560834374$, $p_a = 2.857528660$, $\varphi = 0.106525696$, $p_\varphi = 0.251541127$, close to the invariant manifold of the saddle center. (b) These orbits approach an unstable periodic orbit of the center manifold in a small, but not infinitesimal, neighborhood of saddle center, in such a way that the scale factor $a(t)$ as well as $p_a(t)$ oscillate several times before collapsing or escaping. (c) Projection of the same orbits in the plane (φ, p_a) showing that the frequency of the motion of p_a is twice the frequency of the motion in φ.

sets taken in a neighborhood of the invariant manifold of the saddle center, with a chaotic exit to inflation of the same type as shown in Fig. 3. This study displays only the dynamics of case B occurring in a region of phase space close to the invariant manifold of the saddle center. The dynamical aspects due to the existence of another critical point of the pure saddle nature have not been evidenced, as well as the dynamics in the region between the two invariant manifolds. For instance, the extension of the center manifold and its associated structure of homoclinic cylinders will permeate the neighborhood of the pure saddle invariant plane $\mathcal{M}_0$, producing the complex dynamics to be discussed next.

Now we proceed by taking initial conditions near the invariant manifold $\mathcal{M}_0$ associated with the pure saddle. The idea is, again, to select a point on $\mathcal{M}_0$ and construct a small sphere of initial conditions with radius R that represents the uncertainty about the point under consideration due to fluctuations around the local maximum of the potential. Depending on the energy E_0 collapse and escape to inflation take place. Nevertheless, it can be shown numerically that there always exists an interval of energy $\delta E^* = |E_{max} - E_{min}|$ for each radius R, assumed sufficiently small as $R = 10^{-3}$, 10^{-4}, 10^{-5}, . . . , etc., in which the boundaries of collapse and escape to the de Sitter configuration are chaotically mixed. Again E_{min} and E_{max} denote the values of the energy E_0 for above or below which all orbits escape or collapse, respectively. The chaotic exits to inflation occur for the energy inside the interval δE^*, and are a direct consequence of the nonintegrability of the dynamics between the saddle centers and the pure saddle. We recall that the twisting and crossing of homoclinic cylinders emanating from periodic orbits of the center manifold extend to the region of phase space between the saddle centers and the pure saddle reaching the neighborhood of $\mathcal{M}_0$. The interplay of cylinders in a neighborhood of the pure saddle, and the consequent several chaotic exits to inflation, can be revealed more clearly by the following experiments. Consider now a point lying on the separatrix S of $\mathcal{M}_0$ and whose coordinates are $a = 0.4$, $p_a = 2.756570759$, $\varphi = p_\varphi = 0$. Choosing the radius R sufficiently small, all orbits remains close to $\mathcal{M}_0$ until they reach a region of the same order of R around the pure saddle. From this region, orbits will collapse or escape into the de Sitter attractor associated with the invariant plane of pure saddle (with $\Lambda_{ef} = \Lambda + \lambda\sigma^4/4$), depending on the energy E_0. However, there also exists a domain δE^* for which the outcome of orbits is chaotic. According to Fig. 5, for a given energy inside the chaotic domain δE^*, we note three types of orbits. Type I orbits approach the pure saddle from which some collapse and some escape to inflation. Indeed, in this linear region about the pure saddle, the *partition* of $|E_0 - E_{cr}|$ into the hyperbolic modes energies E_1 and E_2 is completely indeterminate so that we are not able to foretell which orbit will collapse or escape once the initial conditions are generated. Type IIa orbits visit the neighborhood of the saddle center, oscillate to follow with collapse or escape, whereas type IIb return to the neighborhood of the pure saddle to proceed with collapse or escape. In these situations the *partition* of $|E_0 - E_{cr}|$ into the hyperbolic and rotational energies modes around the saddle center, and in the hyperbolic

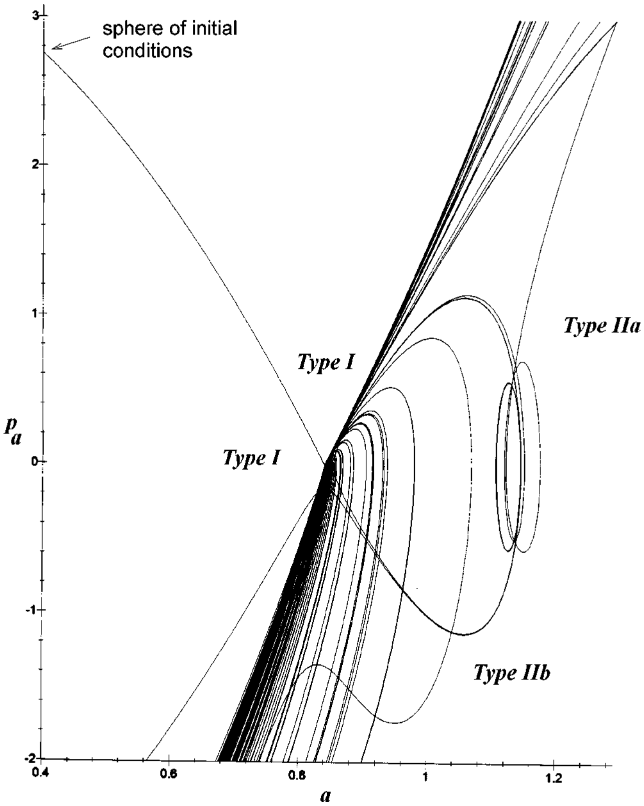

FIG. 5. Chaotic exit to inflation (case of symmetry breaking potential) of 100 orbits with energy $E_0 = 1.058823529$ evolving from a sphere of $R = 10^{-7}$ about a point with $a = 0.4$, $p_a = 2.756570760$ on the separatrix of the invariant manifold associated to the pure saddle point. The orbits remain close to the invariant manifold until they arrive at the small neighborhood of the pure saddle. Type I orbits collapse or escape to inflation after the approach to the pure saddle. Type II orbits are directed towards one of the saddle centers and, after some oscillations, either escape/collapse (type IIa), or return to a neighborhood of the pure saddle (type IIb) to collapse/escape.

energy modes associated with the pure saddle, are chaotic. In Fig. 6, we refine the numerical experiment in such a way to select only type IIb orbits. By projecting them in the plane φ, p_φ, the approach to the homoclinic trajectory appears. Therefore, in the same sense that orbits shown in Fig. 4 approached a given unstable periodic orbit of the center manifold, the orbits of Fig. 6 approach the *homoclinic orbit*.

Another chaotic exit to inflation is obtained if we consider a point of coordinates $a = 0.4$, $p_a = 4.427039907$, $\varphi = p_\varphi = 0$ ($E_0 = 2.058823529$) on a trajectory (not the separatrix) lying entirely on $\mathcal{M}_0$. For a given energy inside the chaotic domain the initial conditions generated about this point evolve to one of the saddle centers, perform some oscillations in its neighborhood (the scale factor a and its canonical momentum p_a also oscillate) to collapse or to escape to inflation, as shown in Fig. 7. In this case, the orbits have approached a periodic orbit of the center manifold analogously as shown in Fig. 4.

Finally, the chaotic behavior of the system described above, associated with the several exits to inflation, can be summarized as follows:

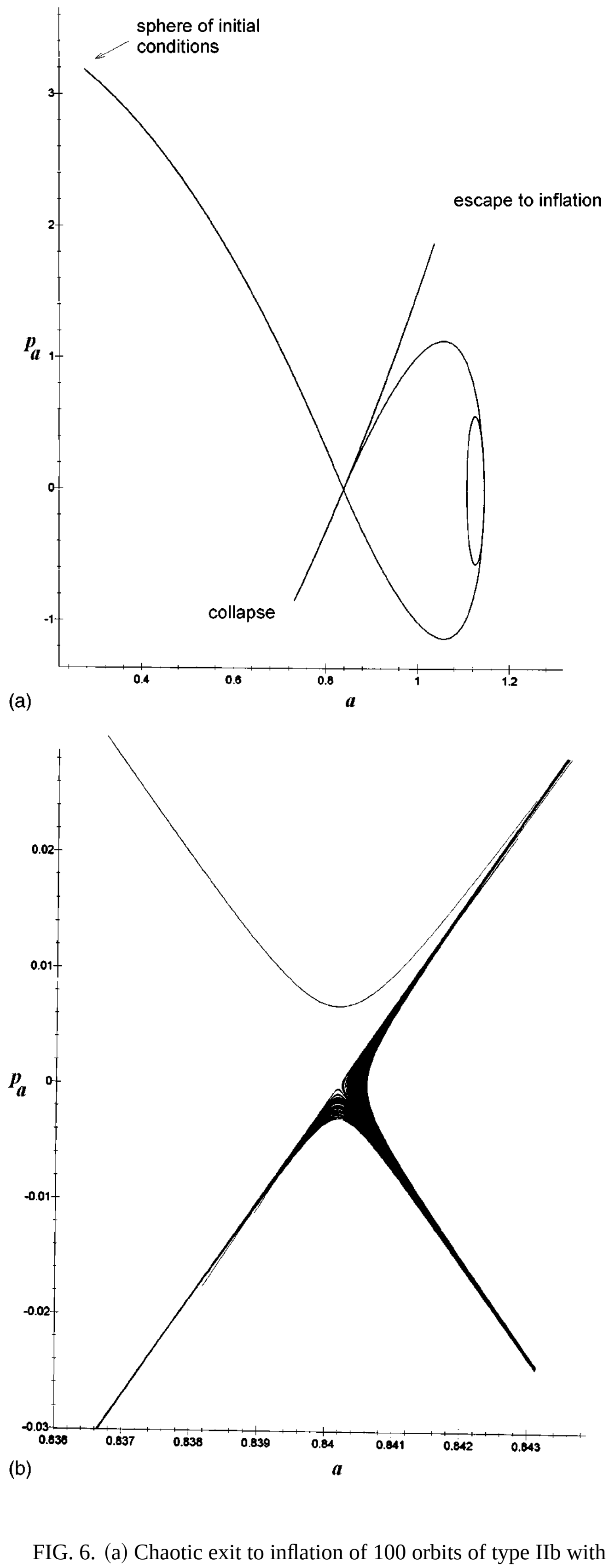

FIG. 6. (a) Chaotic exit to inflation of 100 orbits of type IIb with $E_0 = 1.058825026$. Note the approach of these orbits to the homoclinic orbit extending from the pure saddle to the saddle center. (b) Zoom of the region near the pure saddle showing the chaotic exit to inflation.

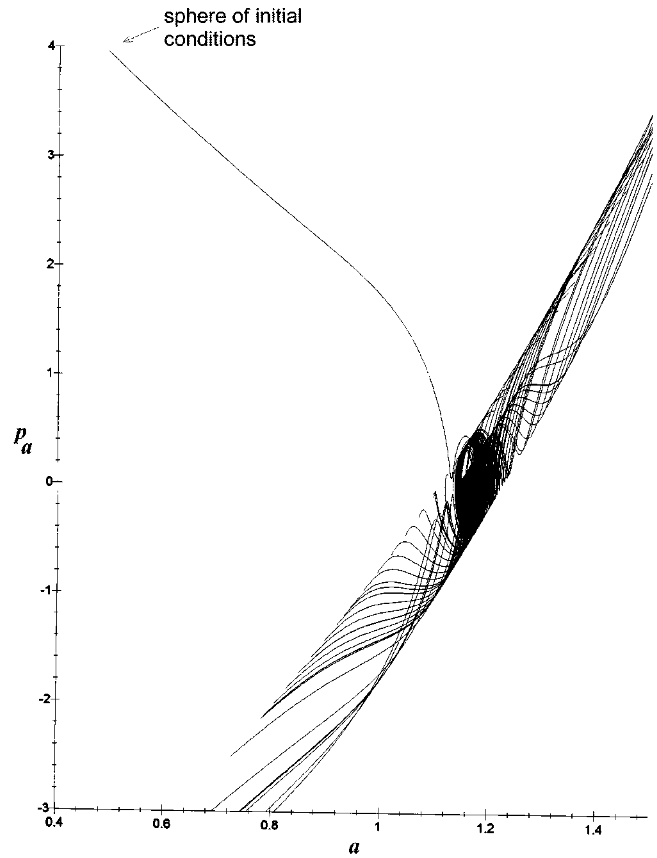

FIG. 7. Chaotic exit to inflation of 60 orbits with energy $E_0 = 1.623311538$, and initial conditions taken about the point, $a = 0.4$, $p_a = 4.427039909$, $\varphi = 0$, $p_\varphi = 0$. This point belongs to the invariant manifold of the pure saddle point, but not on the separatrix. The orbits visit the neighborhood of one of the saddle centers and perform some oscillations before the collapse/escape.

(1) Small fluctuations of initial conditions taken on chaotic sets in a neighborhood of the invariant plane associated with a saddle center will change one of the following asymptotic outcomes into another of the remaining ones: visit the neighborhood of the saddle center and escape to inflation, towards the de Sitter attractor associated with the invariant plane of the saddle center; visit the neighborhood of the saddle center, then visit a neighborhood of the pure saddle, and escape to inflation, towards the de Sitter attractor associated with the invariant plane of the pure saddle; visit the neighborhood of the saddle center and collapse; visit the neighborhood of the saddle center, then visit the neighborhood of the pure saddle, and collapse.

(2) Analogously, small fluctuations of initial conditions taken on chaotic sets in a neighborhood of the invariant plane associated with a pure saddle will change one of the following asymptotic outcomes into another of the remaining ones: visit the neighborhood of the pure saddle and escape to inflation, towards the de Sitter attractor associated with the invariant plane of the pure saddle; visit the neighborhood of the pure saddle, then visit a neighborhood of the saddle center, and escape to inflation, towards the de Sitter attractor associated with the invariant plane of the saddle center; visit the neighborhood of the pure saddle and collapse; visit the

neighborhood of the pure saddle, then visit the neighborhood of the saddle center, and collapse.

We recall that, in any of the exits to inflation, the scalar field will be frozen in one of the vacuum states (with its corresponding vacuum energy playing the role of the cosmological constant for that exit) associated with one of the extrema of the scalar field potential $V(\varphi)$.

V. FINAL REMARKS AND CONCLUSIONS

In this paper we have discussed the dynamics of closed Friedmann-Robertson-Walker models which may provide a description of preinflationary stages of the universe and its exit to inflation. The basic physical ingredients of the models are radiation plus a scalar field minimally coupled to the gravitational field. The energy momentum tensor of the scalar field is split into a cosmological constant-type term (corresponding to the vacuum energy of the scalar field), plus the energy momentum tensor of the spatially homogeneous expectation value of the scalar field. This simple configuration, with two effective degrees of freedom, presents a complex dynamics. The basic features of the dynamics result from the presence of saddle center and pure saddle critical points in the phase space of the system. In our model, the critical points are associated with extrema of the scalar field potential, a minimum and a maximum corresponding, respectively, to a saddle center and pure saddle. Each critical point is related to an invariant plane of the dynamics and to a de Sitter attractor. The scale factor approaches the de Sitter attractors exponentially, defining exits to inflation, one for each critical point. The region of phase space about a saddle center has the structure of homoclinic cylinders, emanating from the center manifold of unstable periodic orbits, resulting in a general orbit with an oscillatory behavior in the neighborhood of the saddle center. Due to the nonintegrability of the system, the extension of homoclinic cylinders away from the periodic orbit is distorted and twisted, with eventual transversal crossings of the unstable cylinders with the stable ones. These intersections produce chaotic sets in phase space, in a manner analogous to the breaking and crossing of homoclinic/heteroclinic curves in Poincaré's homoclinic phenomena, and provide a topological characterization of chaos in the general relativistic dynamics of the model. As we have shown, these phenomena extends to the neighborhood of the nearest pure saddle invariant plane, producing also chaotic sets of initial conditions in the region of phase space laying between the invariant planes. A physically relevant manifestation of chaos is the chaotic exit to inflation through one of the de Sitter attractors present in phase space. Small fluctuations of initial conditions taken on chaotic sets change drastically the long time behavior of orbits, with all possibilities listed at the end of Sec. IV. This is a fundamental result, illustrated extensively in Figs. 3, 4, 5, 6 and 7. For instance, a typical chaotic exit to inflation is realized when orbits, emanating from a small ball about a point on the separatrix of one of the invariant manifolds associated with the saddle centers, reach a linear neighborhood of the saddle center to proceed with collapse or escape to inflation. In this case, the chaotic domain is characterized by the gap of energy δE^* in which collapse and escape are possible, implied by small fluctuations in initial conditions. Nevertheless, the chaotic exit to inflation seems to be a general feature of the system. As made evident in the text, there is always a gap of energy for which orbits initially in a small sphere can visit a nonlinear region about the saddle center to evolve afterwards to collapse or escape. The new and important effect is the oscillation of the scale factor induced by the scalar field, or, equivalently, due to the approach to a given unstable orbit of the center manifold. Considering a model with symmetry breaking potential three critical points, two saddle centers and one pure saddle are present and the dynamics in the region between them produces several chaotic exits to inflation, which are related to the extension of the cylindrical structure to this region. Therefore, we have shown numerically that there always exists a gap of energy in which orbits initially close to the invariant manifold of the pure saddle, for instance, can visit a small neighborhood of one of the saddle centers to (i) collapse or escape, or (ii) return to the pure saddle to collapse or escape.

Finally, an interesting perspective of this work is the possibility of the physical distinction between the exits to inflation, namely, whether the exit occurred towards the saddle-center de Sitter attractor or towards the pure saddle de Sitter attractor. This possibility is based on the growing of a selected spectrum of Fourier components of inhomogeneous perturbations, due a resonance mechanism generated by the oscillations of the scale factor, as already pointed out previously [7]. Indeed, as we have seen, the scale factor $a(t)$ and the scalar field $\varphi(t)$ oscillate, as the orbit visits a neighborhood of the saddle center, with frequency determined by the unstable periodic orbit approached (see Fig. 4, for instance). We remark that initial conditions always exist such that the oscillations take an arbitrary fixed time before collapse or escape to a de Sitter phase. Therefore, inhomogeneous scalar field perturbations and/or matter perturbations in this gravitational background will have a selected spectrum of Fourier components amplified by a mechanism of resonance with the oscillations, the amplification occurring for the particular Fourier components having periods approximately equal to an integer times the period of the periodic orbit approached. Even if the universe inflates afterwards the relative rate of amplitudes produced after the resonance amplification would be maintained as an imprint in the ''initial spectrum'' of density fluctuations. This mechanism however is absent in the case of the exit towards the pure saddle de Sitter attractor, since no oscillations appear when the orbit visits the neighborhood of the pure saddle before escaping. The two cases can in principle be observationally distinguished, based on restrictions imposed by observations in the initial spectrum of density fluctuations. If the exit to inflation occurred via a saddle-center de Sitter attractor the resonance amplification mechanism referred to above will give rise to a nonflat [17] ''initial spectrum'' of density fluctuations [16]. The above analysis obviously excludes orbits of type IIb.

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[1] Edward W. Kolb and Michael S. Turner, *The Early Universe*, Frontiers of Physics (Addison-Wesley, New York, 1990).
[2] R. M. Wald, Phys. Rev. D **28**, 2118 (1983).
[3] A. A. Starobinski, JETP Lett. **37**, 66 (1983).
[4] E. Calzetta and C. El Hasi, Class. Quantum Grav. **10**, 1825 (1993); S. Blanco, G. Domenech, C. El Hasi, and O. Rosso, Gen. Relativ. Gravit. **26**, 1131 (1994); E. Calzetta and C. El Hasi, Phys. Rev. D **51**, 2713 (1995).
[5] N. J. Cornish and J. J. Levin, Phys. Rev. D **53**, 3022 (1996); see also Phys. Rev. Lett. **78**, 998 (1997); Phys. Rev. D **55**, 7489 (1997).
[6] J. Guckenheimer and P. Holmes, *Dynamical Systems and Bifurcations of Vector Fields*, Appl. Math. Sciences Vol. 42 (Springer-Verlag, New York, 1983); J. Koiller, J. R. T. M. Neto, and I. Damião Soares, Phys. Lett. **110A**, 260 (1985).
[7] H. P. de Oliveira, I. Damião Soares, and T. J. Stuchi, Phys. Rev. D **56**, 730 (1997); H. P. de Oliveira, I. Damião Soares, and T. J. Stuchi, ''Chaotic exit to inflation: the dynamics of preinflationary universes,'' gr-qc/9711014 and G. A. Monerat, H. P. de Oliveira, and I. Damião Soares, presented at the Proceedings of the VII Marcel Grossmann Meeting, gr-qc/9711023.
[8] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[9] J. Halliwell and S. Hawking, Phys. Rev. D **31**, 1777 (1985).
[10] S. Wiggins, *Global Bifurcations and Chaos* (Springer-Verlag, Berlin, Heidelberg, 1988).
[11] M. A. Moser, Commun. Pure Appl. Math. **11**, 257 (1958).
[12] A. M. Osório de Almeida, N. De Leon, M. A. Metha, and C. C. Marston, Physica D **46**, 265 (1990).
[13] W. M. Vieira and A. M. Osório de Almeida, Physica D **90**, 9 (1996).
[14] M. V. Berry, *Regular and Irregular Motion*, Proceedings of the Workshop on Topics in Nonlinear Dynamics, edited by S. Jorna, AIP Conf. Proc. 46 (AIP, New York, 1978).
[15] E.S. Cheb-Terrab and H. P. de Oliveira, Comput. Phys. Commun. **95**, 171 (1996).
[16] H. P. de Oliveira and I. Damião Soares, Mod. Phys. Lett. A **13**, 1881 (1998).
[17] A. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic Publishers, Chur, Switzerland, 1993).