Extension of the Nielsen-Ninomiya theorem

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The index theorem is employed to extend the no-go theorem for lattice chiral Dirac fermions to translation noninvariant and nonlocal formulations. [S0556-2821(98)03817-X]

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Lattice regularization of functional integrals is the basis of the most powerful methods of nonperturbative treatment of non-Abelian gauge theories. As any regularization, the lattice breaks some properties of the underlying continuous theory. The Nielsen-Ninomiya theorem [1] imposes nontrivial limitations on those properties of the fermion action that can be maintained on the lattice. In particular, this theorem states that if a lattice Dirac operator provides a correct free fermion spectrum in the continuum limit, it cannot simultaneously be (i) lattice translation invariant, (ii) local, and (iii) chirally invariant (for a more detailed formulation, see [2]).

Among the properties (i)–(iii) chiral symmetry is of particular interest due to the role it plays both in the low energy dynamics of QCD and in the construction of chiral gauge theories. Various attempts have been made to hold chiral invariance on the lattice at the cost of losing translation invariance [3], locality [4], or the correct free fermion spectrum [5].

At the same time, one can hardly consider a lattice definition of the Dirac operator as satisfactory if it does not reproduce, exactly or in some limit, all known exact properties of the continuum Dirac operator. One of such nontrivial properties is a consequence of the Atiyah-Singer index theorem [6]. This is a relation between the number of chiral zero modes of the Dirac operator in a smooth external gauge field on a compact manifold and the topological number Q of this external field, so that on the manifolds of index zero (like a sphere and a torus) one has [7]

$$Q=n_{-}-n_{+}, \tag{1}$$

where n_{+} (n_{-}) is the number of zero modes of positive (negative) chirality. Note that on a finite torus relation (1) is nontrivial also in the Abelian case (see, for example, [8]).

Inability of some particular lattice Dirac operator to reproduce this property were discussed in [9]. The crucial role of the index theorem in defining both vector and chiral gauge theories on a lattice was emphasized in [10].

The aim of this paper is to demonstrate that by requiring relation (1) to hold, at least approximately, on a finite lattice one can extend the no-go theorem to generic chirally invariant lattice Dirac operators including translation noninvariant and nonlocal ones, provided they satisfy a mild spectral condition.

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We consider a gauge theory defined on a *finite lattice* with the fermion action

$$S_f=\sum_{m,n} \bar{\psi}_m D_{mn}(A)\psi_n, \tag{2}$$

where A stands for gauge variables. The lattice may be nonregular with arbitrary boundary conditions. The Dirac operator D may be translation noninvariant or nonlocal; it also may not be gauge invariant.

The generic form of the Dirac operator in the chiral representation of γ matrices, $\gamma_5=\mathrm{diag}(I,-I)$, $\gamma_\mu^\dagger=\gamma_\mu$, reads as

$$D=\begin{pmatrix} M_+ & D_- \\ D_+ & M_- \end{pmatrix}. \tag{3}$$

Matrices D_+ and D_- are lattice transcriptions of the covariant Weyl operators $\sigma_\mu(\partial_\mu+iA_\mu)$ and $\sigma_\mu^\dagger(\partial_\mu+iA_\mu)$, respectively, where A_μ is the gauge field, $\sigma_\mu=(1,i)$ in two dimensions and $\sigma_\mu=(I,i\sigma_i)$ in four dimensions. Matrix $M=\mathrm{diag}(M_+,M_-)\neq\mathrm{const}\times I$ is the measure of breaking of chiral symmetry: $D\gamma_5+\gamma_5 D=2M\gamma_5$.

We now assume that the chirally invariant part of D is normal, i.e. $(D-M)(D-M)^\dagger=(D-M)^\dagger(D-M)$. This implies that

$$D_+^\dagger D_+=D_-D_-^\dagger,$$
$$D_+D_+^\dagger=D_-^\dagger D_-, \tag{4}$$

and that operator $D-M$ has a complete orthonormal set of eigenvectors. This is *the only condition* we impose on the form of the Dirac operator. Note that Eq. (4) is automatically satisfied, if $D_\pm$ obey the same relation as the corresponding continuum operators: $D_-=-D_+^\dagger$.

Now all is fixed to prove the following

Theorem. A necessary condition for the lattice Dirac operator (3), (4) to reproduce a nonzero index (1) is $M\neq 0$.

We give two proofs: a ''mathematical,'' that uses only properties of linear operators in finite-dimensional Hilbert spaces, and ''physical,'' that uses customary technique of quantum field theory. Both are very simple.

Proof 1 (''mathematical''). If $M=0$, by definition $n_+-n_-=\dim\ker D_+-\dim\ker D_-$, where $\dim\ker D_\pm$ mean the dimensions of the kernels of the operators $D_\pm$. By virtue of condition (4) and finiteness of the lattice, one has $\dim\ker\ D_+ = \dim\ker\ D_+^\dagger D_+ = \dim\ker\ D_- D_-^\dagger = \dim\ker\ D_-^\dagger = \dim\ker\ D_-$. Hence, $n_+ - n_- = 0$.

Proof 2 (''physical''). Define the partition function in an external gauge field

$$Z_f = \int \prod_n d\psi_n d\bar{\psi}_n \exp\left(-S_f - \epsilon \sum_n \bar{\psi}_n \psi_n\right) = \det(D+\epsilon), \tag{5}$$

and the fermion expectation values

$$\langle O(\psi,\bar{\psi})\rangle_f = Z_f^{-1} \int \prod_n d\psi_n d\bar{\psi}_n O(\psi,\bar{\psi}) \times \exp\left(-S_f - \epsilon \sum_n \bar{\psi}_n \psi_n\right), \tag{6}$$

where ϵ is an infinitesimal mass introduced to avoid possible singularities.

The Ward identity associated with the singlet chiral transformations $\psi_n = \exp(i\alpha_n\gamma_5)\psi'_n$, $\bar{\psi}_n = \bar{\psi}'_n \exp(i\alpha_n\gamma_5)$ [11] reads as

$$i\frac{\partial}{\partial\alpha_n} \ln Z_f|_{\alpha=0} = \left\langle \bar{\psi}_n \sum_m \gamma_5 D_{nm}\psi_m + \sum_m \bar{\psi}_m D_{mn}\gamma_5\psi_n \right\rangle_f + 2\epsilon\langle\bar{\psi}_n\gamma_5\psi_n\rangle_f = 0. \tag{7}$$

Taking the sum over n we obtain

$$\sum_{mn} \langle\bar{\psi}_m M_{mn}\gamma_5\psi_n\rangle_f + \epsilon\sum_n \langle\bar{\psi}_n\gamma_5\psi_n\rangle_f = 0. \tag{8}$$

Now let $M=0$. Then, by our condition D is normal and anticommutes with γ_5. Therefore it has complete orthonormal set of eigenfunctions such that if f_λ is an eigenfunction with the eigenvalue λ, the function $\gamma_5 f_\lambda$ is an eigenfunction with the eigenvalue $-\lambda$, and zero modes f_0 can be chosen to be chiral, i.e., such that $\gamma_5 f_0 = \pm f_0$. Then, one has

$$\epsilon\sum_n \langle\bar{\psi}_n\gamma_5\psi_n\rangle_f = -\epsilon\sum_{f_\lambda,n} \frac{f_\lambda^\dagger(n)\gamma_5 f_\lambda(n)}{\lambda+\epsilon} = n_- - n_+ = 0, \tag{9}$$

independently of the configuration of the external gauge fields. QED.

Let us make some comments. If on a finite lattice one has $n_+ - n_- = 0$, then a nonzero result cannot be obtained in any limit. If $M\neq 0$, the left-hand side in Eq. (9) generally does not equal $n_+ - n_-$ and may not be integer. However, in such a case relation (1) can be approached in some limit, as it happens in the case of Wilson fermions [8] (for more references see, for instance, [12]). Thus these considerations exclude all chirally invariant Dirac operators defined on a finite lattice and satisfying condition (4).

The index theorem (1) is closely related to the existence of the anomaly in the divergence of the singlet axial current [13]. On a lattice the current may be defined up to the terms vanishing in the naive continuum limit. Therefore chiral invariance of a lattice Dirac operator does not exclude *a priori* the appearance of the anomaly at some definition of the current. By employing the global relation (1) we completely avoid this ambiguity.

Although the considered theorem directly concerns only Dirac fermions, it has some bearing on the construction of chiral gauge theories. For instance, it demonstrates that at least one of the operators $D_\pm$ satisfying condition (4) always fails to reproduce all the properties of its continuum counterpart. In particular, this means that to define a Weyl fermion on a lattice it is not sufficient only to define an anti-Hermitian lattice transcription of the covariant derivative $\partial_\mu + iA_\mu$ (see also [10]).

The condition $M\neq 0$ does not necessarily lead to the complications typical of Wilson fermions. Very interesting examples are the operators constrained by the Ginsparg-Wilson condition [14–16]: $2M\gamma_5 = rD\gamma_5 D$, where r is a nonzero parameter. In fact, this condition ensures a nontrivial realization of exact chiral symmetry on a lattice [17,2], at least in some region of phase space [18]. It is an interesting question whether the properties of the Dirac operators satisfying the Ginsparg-Wilson relation can facilitate the construction of chiral gauge theories. Note that finite lattice versions of chirally invariant nonlocal fixed-point action [14,19] corresponding to $r=0$ should be rejected. An example of the Dirac operator that, despite $M\neq 0$, has gauge invariant modulus of the determinant when the gauge coupling is chiral is proposed in [20] (where this operator has been incorrectly called chirally invariant). This operator obeys the condition $M(D-M)=0$.

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