Lorentz symmetry and the internal structure of the nucleon

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To investigate the internal structure of the nucleon, it is useful to introduce quantities that do not transform properly under Lorentz symmetry, such as the four-momentum of the quarks in the nucleon, the amount of the nucleon spin contributed by quark spin, etc. In this paper, we discuss to what extent these quantities provide Lorentz-invariant descriptions of the nucleon structure.
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In field theory, one often encounters various *densities* consisting of elementary fields, space-time coordinates, and their derivatives: baryon current, momentum density, angular momentum density, etc. These densities are Lorentz covariant, i.e., under Lorentz tranformations, they transform properly as four-vectors or four-tensors. In many cases, we are interested also in the *charges* defined from these densities. Considering a generic density $j^{\mu\alpha\cdots}$, one can define a charge according to

$$Q^{\alpha\cdots}=\int d^3x j^{0\alpha\cdots}. \tag{1}$$

Generally speaking, $Q^{\alpha\cdots}$ no longer transforms properly under Lorentz transformations. The condition for $Q^{\alpha\cdots}$ to be Lorentz covariant is well known [1]: The density $j^{\mu\alpha\cdots}$ must be conserved relative to the index μ,

$$\partial_\mu j^{\mu\alpha\cdots}=0. \tag{2}$$

Indeed in most of the applications, one considers charges from conserved densities.

Nevertheless, it is useful to consider charges defined from nonconserved densities. For instance, in quantum chromodynamics (QCD), the energy-momentum density consists of the sum of quark and gluon contributions

$$T^{\mu\nu}=T_q^{\mu\nu}+T_g^{\mu\nu}. \tag{3}$$

The total density is conserved because of translational invariance $\partial_\mu T^{\mu\nu}=0$. Therefore, the total momentum operator P^μ,

$$P^\mu=\int d^3x T^{0\mu}, \tag{4}$$

tranforms in a manner similar to a four vector under Lorentz symmetry. Meanwhile, one can also introduce the notion of the four momenta carried separately by quarks and gluons,

$$P^\mu_{q,g}(\mu)=\int d^3x T^{0\mu}_{q,g}, \tag{5}$$

where the nonconservation of $T^{0\mu}_{q,g}$ calls for a renormalization scale μ. It is quite obvious that $P^\mu_{q,g}$ does not transform similar to four-vectors, and therefore the significance of such quantities appears doubtful. However, the exact transformation property of the expectation values of $P^\mu_{q,g}$ is simple to derive.

The forward matrix element of the total energy-momentum density in a nucleon state is

$$\langle p|T^{\mu\nu}|p\rangle=2p^\mu p^\nu. \tag{6}$$

Here the covariant normalization of the nucleon state is used. From the above equation, one can easily obtain the usual matrix element of the momentum operator. The matrix elements of the quark and gluon parts of the density involve two Lorentz structures

$$\langle p|T^{\mu\nu}_{q,g}|p\rangle=2A_{q,g}(\mu)p^\mu p^\nu+2B_{q,g}(\mu)g^{\mu\nu}, \tag{7}$$

where $A_{q,g}$ and $B_{q,g}$ are scalar constants. In comparison with Eq. (6), one has the following constraints:

$$A_q(\mu)+A_g(\mu)=1,$$
$$B_q(\mu)+B_g(\mu)=0. \tag{8}$$

Moreover, the quark and gluon contributions to the nucleon four momentum are

$$\langle p|P^\mu_{q,g}|p\rangle=A_{q,g}(\mu)p^\mu+B_{q,g}(\mu)g^{\mu 0}/(2p^0). \tag{9}$$

The above equation defines transformation properties of the expectation values of $P^\mu_{q,g}$. The presence of the second term denies them a proper Lorentz transformation.

On the other hand, if one is interested in the *three momentum* of the nucleon only, the second term in Eq. (9) drops out and three components of the matrix elements transform just like those of a four-vector. Because $A_{q,g}(\mu)$ are Lorentz scalars, one concludes that the fractions of the nucleon three momentum carried by quarks and gluons are invariant under Lorentz tranformations. Such a statement, although drawn for non-Lorentz-covariant quantities, does carry important physical significance. Phenomenologically, $A_{q,g}(\mu)$ have been extracted from the parton distributions, which have simple interpretations only in the infinite momentum frame. According to the above discussion, the fractions of the nucleon mometum carried by quarks and gluons are also the same in ordinary frames. In particular, if a nucleon has a momentum of 1 GeV/c, then according to the recent analysis in Ref. [2], roughly 420 MeV/c is carried by gluons in the form of the Poynting vector $\int d^3x \vec{E}\times\vec{B}$ in the $\overline{\rm MS}$ scheme and at $\mu=1.6$ GeV.

A more intriguing example concerns the spin structure of the nucleon, which depends on the QCD angular momentum density. The total density $M^{\mu\alpha\beta}$ is a mixed Lorentz tensor, expressible in terms of the energy-momentum density $T^{\mu\nu}$ in the Belinfante form [3]

$$M^{\mu\alpha\beta}=T^{\mu\beta}x^\alpha-T^{\mu\alpha}x^\beta. \tag{10}$$

The relevant charges $J^{\alpha\beta}=\int d^3xM^{0\alpha\beta}$ are the usual Lorentz generators (including the angular momentum operator $\vec{J}$), which transform as the Lorentz tensor (1,0)+(0,1).

According to Eq. (10), the angular momentum density has both quark and gluon contributions $M^{\mu\alpha\beta}=M_q^{\mu\alpha\beta}+M_g^{\mu\alpha\beta}$ where

$$M_{q,g}^{\mu\alpha\beta}=T_{q,g}^{\mu\beta}x^\alpha-T_{q,g}^{\mu\alpha}x^\beta. \tag{11}$$

Furthermore, the quark part contains the spin and orbital contributions $M_q^{\mu\alpha\beta}=M_{qL}^{\mu\alpha\beta}+M_{qS}^{\mu\alpha\beta}$ where

$$M_{qS}^{\mu\alpha\beta}=\frac{i}{4}\bar{\psi}\gamma^\mu[\gamma^\alpha,\gamma^\beta]\psi,$$

$$M_{qL}^{\mu\alpha\beta}=\bar{\psi}\gamma^\mu(x^\alpha iD^\beta-x^\beta iD^\alpha)\psi. \tag{12}$$

Accordingly, the QCD angular momentum operator can be written as a sum of three gauge-invariant contributions,

$$\vec{J}=\int d^3x\psi^\dagger\frac{\vec{\Sigma}}{2}\psi+\int d^3x\psi^\dagger\vec{x}\times(-i\vec{D})\psi+\int d^3x\vec{E}\times\vec{B}$$

$$\equiv\vec{S}_q+\vec{L}_q+\vec{J}_g. \tag{13}$$

An individual term in the above expression does not transform as a part of (1,0)+(0,1) and does not satisfy the angular momentum algebra by itself. Nonetheless, as we will argue, the decomposition is useful in studying the spin structure of the nucleon.

The angular momentum operator in Eq. (13) not only generates the spin of a composite particle, but also describes the orbital motion of its center of mass. To separate the two, Pauli and Lubanski introduced a spin vector W^μ [4],

$$W^\mu=-\epsilon^{\mu\alpha\beta\gamma}J_{\alpha\beta}P_\gamma/(2\sqrt{P^2}), \tag{14}$$

which reduces to the angular momentum operator only in the rest frame of the particle. The spin quantum number s characterizes the eigenvalue $s(s+1)$ of the scalar Casimir $W^\mu W_\mu$ in a representation of Poincare algebra. The separation of internal and external motions in a general Lorentz frame comes with a price: W^μ contains not only the angular momentum operators but also the boost operators. Consequently, the scalar character (or invariant notion) of the spin,

$$\langle p|W^\mu W_\mu|p\rangle/\langle p|p\rangle=\frac{1}{2}\left(\frac{1}{2}+1\right), \tag{15}$$

is maintained in an elaborated way in different reference frames. The above equation does not offer a convenient starting point to investigate the spin structure of the nucleon.

To specify the spin states of a nucleon, a polarization vector s^μ is usually introduced [5]. In the rest frame of the nucleon, s^μ specifies the direction of the spin quantization axis; the convention in the literature is such that s^μ represents a state with positive spin projection 1/2. Thus, in any Lorentz frame,

$$\langle ps|W^\mu s_\mu|ps\rangle/\langle ps|ps\rangle=\frac{1}{2}. \tag{16}$$

The above equation is linear in angular momentum operators and is suitable for studying the spin structure if the boost operators are not present. This is the case for a special choice of $s^\mu=(|\vec{p}|/M,p^0\hat{p}/M)$, where $\hat{p}=\vec{p}/|\vec{p}|$. $W^\mu s_\mu$ then is just the well-known helicity operator $h=\vec{J}\cdot\hat{p}$ and the above equation reduces to

$$\langle p+|J\cdot\hat{p}|p+\rangle/\langle p+|p+\rangle=\frac{1}{2}, \tag{17}$$

where + denotes the positive helicity. Because of the special choice, Eq. (17) is invariant only under a special class of Lorentz transformations, i.e., the boosts along the momentum without reversing its direction and rotations around the momentum axis. While the conclusion is quite obvious, our analysis shows that the study of the spin structure can at best be done in a restricted class of Lorentz frames.

In the remainder of the paper, we are going to show that when the angular momentum operator is split into a sum as in Eq. (13), the individual contributions to the spin of the nucleon are invariant under the special Lorentz transformations that preserve the helicity.

Without loss of generality, let us assume that the nucleon momentum is in the positive z direction. Then the helicity operator is just $J^z=J^{12}=\int d^3xM^{012}$. Consider the matrix elements of $\int d^4xM^{\mu\alpha\beta}$ in the nucleon state,

$$\left\langle p\left|\int d^4xM^{\mu\alpha\beta}\right|p\right\rangle=\bar{U}(p)\left[\frac{i}{2}A(p^\mu[\gamma^\alpha,\gamma^\beta]+\gamma^\mu[\gamma^\beta p^\alpha-\gamma^\alpha p^\beta])+B(g^{\mu\alpha}\gamma^\beta-g^{\mu\beta}\gamma^\alpha)\right]U(p)\times(2\pi)^4\delta^4(0)+\cdots, \tag{18}$$

where the elipsis denotes terms with derivatives on the δ function which are related to the orbital motion. Taking $\mu=0$, $\alpha=1$, and $\beta=2$, we have

$$\langle p+|J^z|p+\rangle/\langle p+|p+\rangle=A. \tag{19}$$

Thus A must be 1/2. Now suppose $M^{\mu\alpha\beta}=\Sigma_iM_i^{\mu\alpha\beta}$. The Lorentz symmetry yields

$$\left\langle p\left|\int d^4x M_i^{\mu\alpha\beta}\right|p\right\rangle=\overline{U}(p)\bigg[\frac{i}{2}A_i(\mu)p^\mu[\gamma^\alpha,\gamma^\beta]+B_i(\mu)\gamma^\mu(\gamma^\beta p^\alpha-\gamma^\alpha p^\beta)+C_i(\mu)(g^{\mu\alpha}\gamma^\beta-g^{\mu\beta}\gamma^\alpha)\bigg]\times U(p)(2\pi)^4\delta^4(0)+\cdots, \quad (20)$$

where $A_i(\mu)$, $B_i(\mu)$, and $C_i(\mu)$ are scalar constants. Then the consistency between Eqs. (18) and (20) yields

$$\sum_i A_i(\mu)=A=\frac{1}{2}. \quad (21)$$

Equation (20) allows us to calculate the contributions to the nucleon spin from different terms in $\vec{J}=\Sigma_i\vec{J}_i$,

$$\langle p+|J_i^z|p+\rangle/\langle p+|p+\rangle=A_i(\mu), \quad (22)$$

where $J_i^z=\int d^3x M_i^{012}$. The result is *independent* of the magnitude of the nucleon momentum. Futhermore, Eq. (21) yields a nucleon spin sum rule invariant under the special Lorentz transformations. Denoting the scalar matrix elements A_i of the angular momentum densities in Eqs. (11),(12) as $A_{qS}\equiv\frac{1}{2}\Delta\Sigma$, $A_{qL}\equiv L_q$, and $A_g\equiv J_g$, Eq. (21) becomes [6]

$$\frac{1}{2}\Delta\Sigma(\mu)+L_q(\mu)+J_g(\mu)=\frac{1}{2}. \quad (23)$$

As the nucleon momentum goes to infinity, the result can be interpreted according to light-front quantization, in which the Lorentz generators are defined according to $\int d^3x M^{+\alpha\beta}$. As such, the quark spin and orbital and gluon angular momentum contributions to the nucleon are the same in both ordinary and light-front coordinates.

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