Easy way to solve two-loop vertex integrals

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We consider four two-loop three-point vertex diagrams with arbitrary exponents of propagators and dimension, three of them with two legs on-shell and one with all legs off-shell. As far as we know there is no similar calculation in the literature, and our original results are neatly expressed in terms of products of gamma functions. [S0556-2821(98)00816-9]

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I. INTRODUCTION

The comprehension and achievements of quantum field theory are mostly because of the perturbative approach, where the calculation of Feynman diagrams is an inevitable task. There are several techniques to solve the associated integrals, wherein the simpler the method to solve them the better. We can just mention a few: the Mellin-Barnes representation of massive propagators [1], the Gegenbauer polynomial approach in configuration space [2], and some others [3].

But none of them uses the principle of analytic continuation in such an interesting and beautiful way as the technique [4] known as the negative dimensional integration method (NDIM). Moreover, NDIM gives *simultaneously* several results and shares all the concepts of dimensional regularization [5]. We do not know of another approach which has this amazing property.

Our aim here is to present original results for four scalar integrals pertaining to three-point vertex diagrams at two-loop level. They are original in the sense that all the exponents of propagators and the space-time dimension are left arbitrary, and as far as we know there is no similar calculation in the literature. These results can be used to study two-loop on-shell form factors in massless QCD, see for instance [6].

II. FEYNMAN GRAPHS WITH FIVE AND FOUR MASSLESS PROPAGATORS

To calculate the scalar integral that contributes to the Feynman diagram of Fig. 1, we will follow the procedure and notation of [7,8]. This integral, in negative D, reads

$$\mathcal{A}(i_1,j_1,l_1,m_1,n_1;D)\equiv\int d^Dr\; d^Dq\,(q^2)^{i_1}(r^2)^{j_1}(p-r-q)^{2l_1}\times(q+k)^{2m_1}(q+k+r)^{2n_1}, \tag{1}$$

where one notes that its integrand is formed by the squares of the internal momenta flowing in the diagram raised to arbitrary positive integer powers. Were these powers negative, one would have the usual Feynman integral for the diagram with propagators in the denominator.

The result for the referred integral, in our real physical world, is given by a simple product of gamma functions

$$\begin{aligned}\mathcal{A}^{AC}(i_1,j_1,l_1,m_1,n_1;D) &= \pi^D(p^2)^{\sigma_1}(-i_1|\sigma_1)(-l_1|\sigma_1)(\sigma+\tfrac{1}{2}D|-2\sigma_1-\tfrac{1}{2}D)\\ &\quad\times(-j_1|i_1+j_1+n_1+\tfrac{1}{2}D)(j_1+l_1+m_1+D|-l_1-m_1\\ &\quad-\tfrac{1}{2}D)(i_1+j_1+n_1+D|-i_1-j_1+l_1+m_1-n_1-\tfrac{1}{2}D),\end{aligned} \tag{2}$$

where $\sigma_a=i_a+j_a+l_a+m_a+n_a+D$.

Consider now the graph of Fig. 2 and its scalar integral

$$\mathcal{B}(i_2,j_2,l_2,m_2,n_2;D)\equiv\int d^Dr\quad d^Dq\quad(r^2)^{i_2}(p-q)^{2j_2}\times(t+q)^{2l_2}(r^2)^{m_2}(t+q-r)^{2n_2}, \tag{3}$$

which, after the convenient analytic continuation to positive D, gives

$$\begin{aligned}\mathcal{B}^{AC}(i_2,j_2,l_2,m_2,n_2;D) &= \pi^D(p^2)^{\sigma_2}(-m_2|m_2+n_2+\tfrac{1}{2}D)(-n_2|m_2+n_2+\tfrac{1}{2}D)\\ &\quad\times(-i_2|\sigma_2)(-j_2|\sigma_2)(m_2+n_2+D|-2m_2-2n_2-\tfrac{3}{2}D)\\ &\quad\times(\sigma_2+\tfrac{1}{2}D|-2\sigma_2-\tfrac{1}{2}D).\end{aligned} \tag{4}$$

For the next graph, Fig. 3 — the simplest one of this work — we write

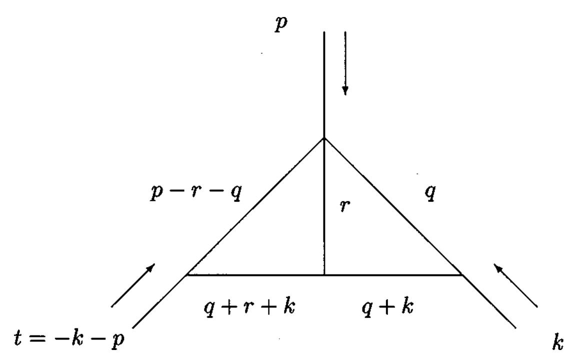

FIG. 1. Two-loop three-point vertex calculated with NDIM. We consider that the two lower external legs are on-shell, i.e., $k^2=t^2=0$.

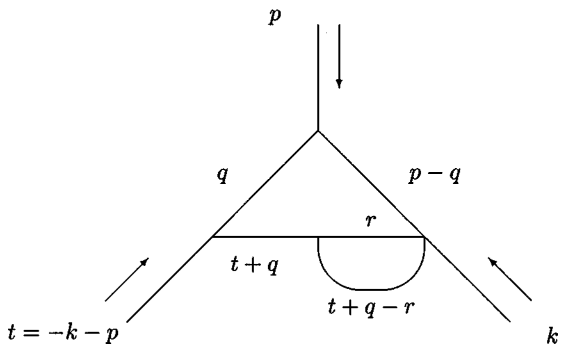

FIG. 2. Another two-loop three-point graph calculated in the NDIM approach.

$$\mathcal{C}(i_3,j_3,l_3,m_3,n_3;D)$$
$$\equiv \int d^Dr \;\; d^Dq \;\; (q^2)^{i_3}(r^2)^{j_3}(p-q)^{2l_3}(p-r)^{2m_3}$$
$$\times (k+r)^{2n_3}$$
$$=\pi^D(p^2)^{\sigma_3}(i_3+l_3+D|-i_3-l_3+m_3+n_3-\tfrac{1}{2}D)$$
$$\times(-i_3|-l_3-\tfrac{1}{2}D)(-l_3|i_3+l_3+\tfrac{1}{2}D)$$
$$\times(-j_3|-m_3-n_3-\tfrac{1}{2}D)(-m_3|l_3+m_3+\tfrac{1}{2}D)$$
$$\times(j_3+m_3+n_3+D|-m_3-\tfrac{1}{2}D). \qquad (5)$$

Now one could rightfully ask: this calculation was greatly simplified because two of the external legs were on-shell, is it so simple to perform these integrals when all the external legs are put off-shell? We will answer this question with such a calculation.

Consider the diagram of Fig. 4,

$$\mathcal{D}(i_4,j_4,l_4,m_4;D)=\int\int d^Dq \;\; d^Dr \;\; (q^2)^{i_4}[(q-p)^2]^{j_4}$$
$$\times(r^2)^{l_4}[(r-q+k)^2]^{m_4}, \qquad (6)$$

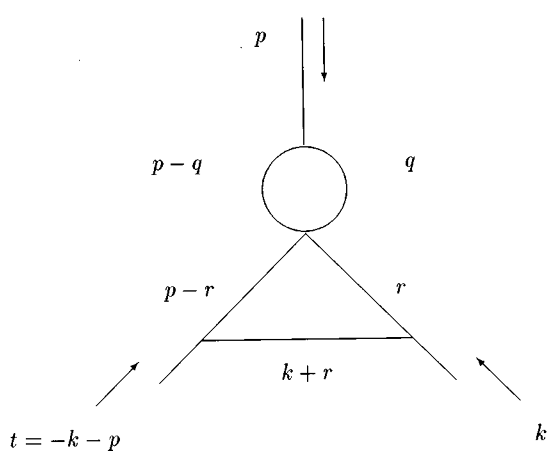

FIG. 3. The easiest two-loop three-point vertex diagram calculated in the NDIM approach.

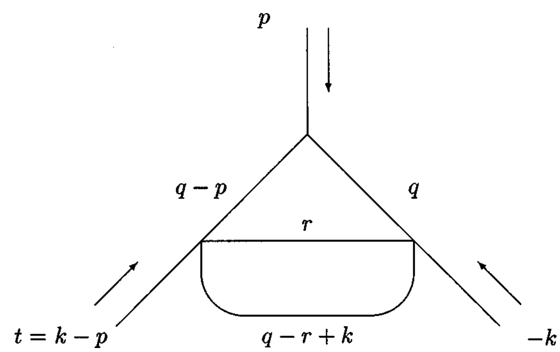

FIG. 4. Two-loop three-point vertex calculated with all external legs off-shell.

for this integral NDIM gives 21 solutions, each one valid in a given region and all of them related by analytic continuation. Two of them are of interest because we can study special cases. The first one,

$$\mathcal{D}_1^{AC}(i_4,j_4,l_4,m_4;D)=\pi^D(p^2)^{\rho}P_1^{AC}F_4(-\rho,-l_4-m_4$$
$$-\tfrac{1}{2}D;1+j_4-\rho,1+i_4-\rho|x,y), \qquad (7)$$

where F_4 is a hypergeometric function of two variables and

$$P_1^{AC}=(-i_4|\rho)(-j_4|\rho)(-l_4|-m_4-\tfrac{1}{2}D)$$
$$\times(-m_4|l_4+m_4+\tfrac{1}{2}D)(l_4+m_4+D|-l_4-\tfrac{1}{2}D)$$
$$\times(\rho+\tfrac{1}{2}D|-2\rho-\tfrac{1}{2}D), \qquad (8)$$

and $\rho=\sigma_4-n_4$, $x=k^2/p^2$, $y=t^2/p^2$. This one allows us to consider $k^2=t^2=0$ [7] or $k=0$, $p^{\nu}=t^{\nu}$ [8]:

$$\mathcal{D}_2^{AC}(i_4,j_4,l_4,m_4;D)$$
$$=f_2\sum_{a,b,c=0}^{\infty}\frac{(y^{-1})^{a+b}(z)^c}{a!b!c!}\,\frac{(m_4+\tfrac{1}{2}D|a)}{(1-i_4-j_4-\tfrac{1}{2}D|a+b)}$$
$$\times\frac{(l_4+\tfrac{1}{2}D|b)(-i_4|a+b+c)(-\rho|a+b+c)}{(l_4+m_4+D|a+b)(1-i_4-l_4-m_4-D|c)}, \qquad (9)$$

where

$$f_2=\pi^D(t^2)^{\rho}(-j_4|\rho)(-l_4|l_4+m_4+\tfrac{1}{2}D)$$
$$\times(-m_4|l_4+m_4+\tfrac{1}{2}D)$$
$$\times(l_4+m_4+D|i_4+j_4-l_4-m_4-\tfrac{1}{2}D)$$
$$\times(\rho+\tfrac{1}{2}D|-2\rho-\tfrac{1}{2}D),$$

and $z=k^2/t^2$. We can put p^{ν} and k^{ν} on-shell in this solution. The resulting graph contributes to a more complicated one [6]. It also admits $p=0$ and $k^{\nu}=t^{\nu}$.

III. CONCLUSIONS

NDIM's amazing feature of transfering the task of solving parametric integrals to solving systems of linear algebraic equations allows us to greatly simplify the effort to solve loop integrals. With two-loop vertex diagrams as examples we worked out rather cumbersome integrals with ease. The analytic continuation of space-time dimension to *negative* values has shown advantages: we interpret the analytic continuation similar to that in dimensional regularization but solve the Feynman integrals in a much simpler way because they are polynomial integrals. We presented original results for four two-loop three-point scalar integrals, (see Figs. 1–4) for arbitrary D and exponents of the propagators (2), (4), (5), (7), and (9).

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