Perturbations of solutions of the Einstein-Weyl equations: An example

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The coupled gravitational and neutrino field perturbations of a type D solution of the Einstein-Weyl equations are studied, reducing the problem to a system of four first-order ordinary differential equations. It is explicitly shown that there exist purely gravitational linear perturbations of the background solution considered here, such that the perturbed fields form an exact solution of the Einstein-Weyl equations.
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I. INTRODUCTION

The behavior of a test massless neutrino field, governed by the Weyl equation, on the Schwarzschild and the Kerr solution has been studied by various authors (e.g., Refs. [1–5]). Since these two metrics are of type D, the components of the neutrino field along the two principal spinors of the conformal curvature satisfy decoupled second-order partial differential equations, which are solvable by the separation of variables (see also Refs. [6,7]). As shown in Ref. [8], the solution of the Weyl neutrino equation in any algebraically special space-time that admits a shearfree congruence of null geodesics is given by a *single* scalar potential, ψ, that obeys a second-order partial differential equation which, in the Newman-Penrose notation, takes the form

$$[(\Delta-\bar{\gamma}+\bar{\mu})(D+\varepsilon)-(\bar{\delta}+\bar{\beta}-\bar{\tau})(\delta+\beta)]\psi=0, \tag{1}$$

assuming that the tetrad vector $D=l^{\mu}(\partial/\partial x^{\mu})$ is geodetic and shearfree and is a double principal null direction of the conformal curvature (i.e., $\kappa=\sigma=\Psi_0=\Psi_1=0$) (see also Ref. [9]). The components of the neutrino field are then given by

$$\bar{\eta}_{0'}=(D+\varepsilon)\psi, \quad \bar{\eta}_{1'}=(\delta+\beta)\psi. \tag{2}$$

(This result holds without any explicit restriction on the Ricci tensor.)

When there is a nonvanishing background neutrino field, the neutrino field perturbations are coupled to the gravitational perturbations; therefore, the study of the perturbations of an exact solution of the Einstein-Weyl (EW) equations involves simultaneously the gravitational and the neutrino field perturbations (see, however, Sec. III below). As in the case of the perturbations of a solution of the Einstein-Maxwell equations, where there is a mutual conversion of gravitational and electromagnetic waves, in a solution of the EW equations with a nonvanishing background neutrino field, there is a mutual conversion of gravitational and neutrino waves. Whereas the perturbations of some solutions of the Einstein-Maxwell equations (especially the Reissner-Nordström solution) have been studied, so far there are no results on perturbations of solutions of the EW equations.

In this paper, we consider an explicit example of the perturbations of a solution of the EW equations. Since we do not have an exact solution of the EW equations with a physical significance similar to that possessed by the Kerr-Newman or the Reissner-Nordström solution in the case of the Einstein-Maxwell equations, we shall consider the perturbations of a relatively simple type D solution of the EW equations such that the neutrino flux vector is geodetic and shearfree, making use of the fact that the complete perturbations of any solution of this class are determined by a set of four first-order partial differential equations for four scalar potentials [10] and taking into account that the type D solutions of the Einstein vacuum field equations are known to possess remarkable separability properties. We find that this system of equations admit separable solutions and that it reduces to a set of four coupled ordinary differential equations.

II. THE COMPLETE PERTURBATIONS

The EW equations are given by

$$\Phi_{ABA'B'}=2ik(\eta_{(A}\nabla_{B)A'}\bar{\eta}_{B'}-\bar{\eta}_{(A'}\nabla_{B')A}\eta_{B}),$$

$$\nabla^{AB'}\eta_A=0, \tag{3}$$

where $\Phi_{ABC'D'}$ denotes the spinor components of the trace-free part of the Ricci tensor, k is a real constant, η_A denotes the components of the Weyl neutrino field, $\bar{\eta}_{A'}\equiv\overline{\eta_A}$ and the parentheses denote symmetrization on the indices enclosed. As shown in Ref. [10] (see also Ref. [11]), the metric and neutrino field perturbations of a given exact solution of the EW equations such that the flux vector of the background neutrino field is tangent to a shearfree congruence of null geodesics, in a frame such that $\eta_0=0$ (which amounts to assume that l^{μ} is parallel to the flux vector of the neutrino field), are given by

$$\begin{aligned}h_{\mu\nu}=2\{&l_{\mu}l_{\nu}[(\delta+3\beta+\bar{\alpha}-\tau)M_{1'}-\bar{\lambda}M_{0'}]\\&+m_{\mu}m_{\nu}(D+3\varepsilon-\bar{\varepsilon}-\rho)M_{0'}-l_{(\mu}m_{\nu)}\\&\times[(D+3\varepsilon+\bar{\varepsilon}-\rho+\bar{\rho})M_{1'}\\&+(\delta+3\beta-\bar{\alpha}-\tau-\bar{\pi})M_{0'}]\}+\text{c.c.}\end{aligned} \tag{4}$$

and

$$\overline{\eta}^{(1)}_{0'} = \frac{1}{4ik}(D+\varepsilon)\psi_{\rm N}, \quad \overline{\eta}^{(1)}_{1'} = \frac{1}{4ik}(\delta+\beta)\psi_{\rm N}, \tag{5}$$

respectively, with the complex scalar potentials $M_{0'}$, $M_{1'}$, $\psi_{\rm N}$ being governed by the equations

$$(\overline{\delta}+3\alpha+\overline{\beta}-\overline{\tau})M_{1'}-(\Delta+3\gamma-\overline{\gamma}+\overline{\mu})M_{0'}=\eta_1\psi_{\rm G},$$

$$(D+3\varepsilon+\overline{\varepsilon}+3\rho-\overline{\rho})M_{1'}-(\delta+3\beta-\overline{\alpha}+3\tau+\overline{\pi})M_{0'} = \eta_1\psi_{\rm N},$$

$$\eta_1[(\Delta+\gamma+\mu)\psi_{\rm N}-(\delta+3\beta+\tau)\psi_{\rm G}] = (3\Psi_2-2\Phi_{11})M_{1'}+2\Phi_{12}M_{0'}-ik\eta_1\overline{\eta}_{1'} \times[(D+3\varepsilon+\overline{\varepsilon}-\rho+\overline{\rho})M_{1'} +(\delta+3\beta-\overline{\alpha}-\tau-\overline{\pi})M_{0'}],$$

$$\eta_1[(\overline{\delta}+\alpha+\pi)\psi_{\rm N}-(D+3\varepsilon+\rho)\psi_{\rm G}] =(3\Psi_2+2\Phi_{11})M_{0'}-2ik\eta_1\overline{\eta}_{1'}(D+3\varepsilon-\overline{\varepsilon}-\rho)M_{0'}, \tag{6}$$

where $\psi_{\rm G}$ is an auxiliary potential. [Note that Eqs. (5) look like Eqs. (2); however the scalar potentials $\psi_{\rm N}$ and ψ do not obey the same differential equations. The neutrino field given by Eqs. (1) and (2) satisfies the Weyl equation and need not be a test field (see Eq. (13) below). On the other hand, Eqs. (5) yield a neutrino field perturbation which, together with the metric perturbation (4), satisfies the *linearized* EW equations.]

The perturbations of the components of the neutrino field given by Eqs. (5) correspond to the first-order difference between the components of the perturbed neutrino field with respect to the perturbed tetrad

$$\partial_{AB'}+\partial^{(1)}_{AB'}=\partial_{AB'}-\tfrac{1}{2}h_{ACB'D'}\partial^{CD'} \tag{7}$$

and the components of the background neutrino field with respect to the original tetrad $\partial_{AB'}$ (recall that $\partial_{00'}=D$, $\partial_{01'}=\delta$, $\partial_{10'}=\overline{\delta}$, $\partial_{11'}=\Delta$).

We shall consider the background metric

$$ds^2=\frac{Q(y)}{y^2+a^2}(du-2axdv)^2-\frac{y^2+a^2}{Q(y)}dy^2 -(y^2+a^2)(dx^2+dv^2), \tag{8}$$

where x,y,u,v are real coordinates,

$$Q(y)=-2My+b \tag{9}$$

and a, b, and M are real constants. Then, the vector fields

$$D=\partial_y-\frac{y^2+a^2}{Q}\partial_u,$$

$$\Delta=-\frac{1}{2}\frac{Q}{y^2+a^2}\left(\partial_y+\frac{y^2+a^2}{Q}\partial_u\right),$$

$$\delta=\frac{1}{\sqrt{2}}\frac{1}{y-ia}(\partial_x+i(2ax\partial_u+\partial_v)),$$

$$\overline{\delta}=\frac{1}{\sqrt{2}}\frac{1}{y+ia}(\partial_x-i(2ax\partial_u+\partial_v)), \tag{10}$$

form a null tetrad, and the corresponding nonvanishing spin coefficients are

$$\rho=-\frac{1}{y+ia}, \quad \mu=-\frac{1}{2}\frac{Q}{y^2+a^2}\frac{1}{y+ia},$$

$$\gamma=-\frac{1}{2}\frac{M}{y^2+a^2}+\mu. \tag{11}$$

The only nonvanishing components of the curvature are given by

$$\Phi_{11}=\frac{b}{2}\frac{1}{(y^2+a^2)^2}, \quad \Psi_2=-\frac{M}{(y+ia)^3}+\frac{b}{(y^2+a^2)(y+ia)^2} \tag{12}$$

therefore, if M and b are not both zero, the metric is of type D.

Making use of Eqs. (10)–(12) one finds that the neutrino field

$$\eta_0=0, \quad \eta_1=\frac{A}{y+ia}, \tag{13}$$

where A is a complex constant, together with the metric (8) satisfy the EW equations (3) provided

$$b=4ka|A|^2. \tag{14}$$

[Note that the neutrino field (13) can be expressed in the form (2) with $\psi=\sqrt{2}\overline{A}x$.] This solution was found in Ref. [12] and, the case where $a=1$, in Ref. [13]. When $b=0$, the metric (8) is a solution of the Einstein vacuum field equations that coincides with one of the Newman-Unti-Tamburino metrics [14] and is a special case of the Carter [$\tilde{\rm B}(+)$] class of solutions [15]. When $a=0$ but $A\neq 0$, the energy-momentum tensor of the neutrino field vanishes and the metric is a vacuum solution that coincides with one of the solutions found in Ref. [16].

The metric (8), with $b>0$, can also be produced by an electromagnetic field. It can be readily seen that the metric (8) with the electromagnetic field given by $\varphi_0=0=\varphi_2$, $\varphi_1=\frac{1}{2}\sqrt{b}e^{i\phi}(y+ia)^{-2}$, where ϕ is a constant, with respect to the tetrad (10), satisfy the Einstein–Maxwell equations. The tetrad vectors l^μ and n^μ are principal null directions of this electromagnetic field.

Since $\eta_0=0$, the tetrad vector l^μ is parallel to the flux vector of the neutrino field, $\eta_A\overline{\eta}_{A'}$, and from the relations $\kappa=0=\sigma$, it follows that the flux vector is geodetic and shearfree, therefore, the complete perturbations of the solution given by Eqs. (8) and (13) are determined by the system of equations (6). Using the fact that u and v are ignorable coordinates, we look for separable solutions to Eqs. (6) of the form

$$M_{0'}=2AS_{-2}(x)R_1(y)e^{i(lu+mv)},$$
$$M_{1'}=\sqrt{2}AS_{-1}(x)R_2(y)e^{i(lu+mv)},$$
$$\psi_{\rm G}=S_{-2}(x)R_3(y)e^{i(lu+mv)},$$
$$\psi_{\rm N}=\sqrt{2}S_{-1}(x)R_4(y)e^{i(lu+mv)}, \tag{15}$$

where l and m are constants and some constant factors have been introduced for later convenience. Substituting Eqs. (10)–(13) and (15) into Eqs. (6) one obtains the ordinary differential equations

$$C\phi R_2+\phi^4\overline{\phi}Q^2\mathcal{D}^\dagger Q^{-1}\phi^{-3}R_1=\phi R_3,$$
$$C\overline{\phi}R_1+\phi^{-3}\overline{\phi}\mathcal{D}\phi^3\overline{\phi}^{-1}R_2=\phi R_4,$$
$$C\phi\overline{\phi}R_3-\phi^4\overline{\phi}Q^{3/2}\mathcal{D}^\dagger Q^{-1/2}\phi^{-2}R_4$$
$$=2(3\Psi_2-2\Phi_{11})R_2-2ik|A|^2$$
$$\times(\phi^2\mathcal{D}\phi^{-1}\overline{\phi}R_2-C\phi\overline{\phi}^2R_1),$$
$$C\phi^2R_4-\mathcal{D}\phi R_3=2(3\Psi_2+2\Phi_{11})R_1-4ik|A|^2\phi^2\overline{\phi}\mathcal{D}\phi^{-1}R_1 \tag{16}$$

and

$$\mathcal{L}S_{-1}=CS_{-2},\quad \mathcal{L}^\dagger S_{-2}=-CS_{-1} \tag{17}$$

where C is another separation constant, $\phi\equiv 1/(y+ia)$, and

$$\mathcal{D}\equiv\partial_y-\frac{il(y^2+a^2)}{Q},\quad \mathcal{D}^\dagger\equiv\partial_y+\frac{il(y^2+a^2)}{Q},$$
$$\mathcal{L}\equiv\partial_x+2alx+m,\quad \mathcal{L}^\dagger\equiv\partial_x-2alx-m. \tag{18}$$

Assuming that al is different from zero, by means of the change of variable $z\equiv(2alx+m)/\sqrt{|2al|}$, the operators $\mathcal{L}$ and $\mathcal{L}^\dagger$ take the form

$$\mathcal{L}=\sqrt{|2al|}(\epsilon\partial_z+z),\quad \mathcal{L}^\dagger=\sqrt{|2al|}(\epsilon\partial_z-z), \tag{19}$$

where $\epsilon\equiv\mathrm{sgn}(al)=(al)/|al|$. Therefore, except for a constant factor, $\mathcal{L}$ and $\mathcal{L}^\dagger$ correspond to the well-known ladder operators of the one-dimensional harmonic oscillator in quantum mechanics, and from Eqs. (17) one finds that the functions S_s satisfy the equations

$$\frac{1}{2}(-\partial_z^2+z^2)S_{-1}=\left(\frac{C^2}{4|al|}-\frac{\epsilon}{2}\right)S_{-1},$$
$$\frac{1}{2}(-\partial_z^2+z^2)S_{-2}=\left(\frac{C^2}{4|al|}+\frac{\epsilon}{2}\right)S_{-2}, \tag{20}$$

whose solutions are the parabolic cylinder functions. The solutions of Eqs. (20) that are bounded as $x\to\pm\infty$ are proportional to $e^{-z^2/2}H_{n-1}(z)$ and $e^{-z^2/2}H_n(z)$, respectively, if $al>0$ or to $e^{-z^2/2}H_n(z)$ and $e^{-z^2/2}H_{n-1}(z)$, respectively, if $al<0$, where H_n are the Hermite polynomials and $n\equiv C^2/|4al|$ is an integer greater than or equal to zero [we take $H_{-1}\equiv 0$, as required by Eqs. (17)]. When $al=0$, Eqs. (17) and (18) yield $\partial_x^2S_s=(m^2-C^2)S_s$, and the solutions are linear combinations of $\exp\pm\sqrt{m^2-C^2}z$.

The system of equations (16) is analogous to the system of radial equations obtained in the study of the perturbations of the Reissner-Nordström solution by means of scalar potentials [17] and, in the latter case, the four equations can be transformed into two independent pairs of first-order differential equations. We have not found such a partial decoupling for Eqs. (16); however, in the particular case where the separation constant C vanishes, Eqs. (16) constitute two decoupled pairs of equations, one for R_1 and R_3 and another for R_2 and R_4. It may be noticed that when $C=0$ and $al>0$ the potential $\psi_{\rm N}$ vanishes, as well as the neutrino field perturbation. As we shall show in the next section, the solution of the EW equations given by Eqs. (8) and (13) admits linear perturbations such that only the metric tensor is perturbed and the resulting fields yield an *exact* solution of the EW equations.

A remarkable feature of the neutrino perturbations is that $\eta_A^{(1)}$ depends on $\overline{\psi_{\rm N}}$ but not on its complex conjugate [see Eq. (5)], which means that in the case of a separable solution of Eqs. (6) of the form (15), the perturbations of the neutrino components $\eta_A^{(1)}$ will contain only a factor $e^{-i(lu+mv)}$ (assuming l and m real), instead of a combination of $e^{i(lu+mv)}$ and $e^{-i(lu+mv)}$. This behavior contrasts with that of the curvature perturbations [since the metric perturbation depends on $M_{0'}$, $M_{1'}$ and their conjugates; see Eq. (4)], for which the simultaneous presence of the factors $e^{i(lu+mv)}$ and $e^{-i(lu+mv)}$ means that the polarization of the gravitational waves changes upon its interaction with the background fields. (This effect is clearly seen in the case of the electromagnetic and gravitational perturbations of the Reissner-Nordström solution [18].)

III. EXACT SOLUTIONS FROM LINEAR PERTURBATIONS

As shown in Ref. [19], given an exact solution $(g_{\mu\nu},\eta_A)$ of the EW equations such that the neutrino flux vector is geodetic, if l_μ is parallel to the neutrino flux vector, the perturbation $h_{\mu\nu}=2Hl_\mu l_\nu, \eta_A^{(1)}=0$ satisfies the linearized EW equations if and only if $(g_{\mu\nu}+2Hl_\mu l_\nu,\eta_A)$ is an exact solution of the EW equations. Hence, if there exists a solution of the linearized EW equations such that the metric perturbation $h_{\mu\nu}$ is proportional to $l_\mu l_\nu$, and the neutrino field is left unchanged, the sum of the background solution and the perturbation is an exact solution of the EW equations. Moreover, since an analogous result holds for the Einstein-Maxwell equations (with l_μ being a principal null direction of the background electromagnetic field [19,20]) and, when

$b>0$, the metric (8) is part of an exact solution of the Einstein-Maxwell equations, the perturbed metric also corresponds to an *exact* solution of the Einstein-Maxwell equations, with the electromagnetic field unchanged.

In order to find metric perturbations proportional to $l_\mu l_\nu$, instead of Eqs. (4), it is more convenient to use the expressions given in Ref. [19], which in the present case reduce to

$$\tfrac{1}{2}(D-\rho+\overline{\rho})(D+\rho-\overline{\rho})H-\rho^2 H=0$$

$$\delta(D+\rho-\overline{\rho})H=0$$

$$\delta\overline{\delta}H-(\Delta+\gamma+\overline{\gamma}+\mu)\rho H-\mu(D+\rho)H+HD\mu=0$$

$$\tfrac{1}{2}(D+\rho-\overline{\rho})(D+\rho-\overline{\rho})H-2(D-\overline{\rho})\rho H+\rho^2 H=0. \quad (21)$$

It is easy to see that all the (real) solutions of these linear equations are given by

$$H=c\frac{y}{y^2+a^2}, \quad (22)$$

where c is an arbitrary real constant. Taking into account that

$$l_\mu dx^\mu=-\frac{y^2+a^2}{Q}dy-(du-2ax\,dv), \quad (23)$$

from Eqs. (8), (22) and (23) we obtain the perturbed metric

$$(g_{\mu\nu}+2Hl_\mu l_\nu)dx^\mu dx^\nu$$
$$=\frac{Q+2cy}{y^2+a^2}(du-2ax\,dv)^2-\frac{y^2+a^2}{Q^2}(Q-2cy)dy^2$$
$$+\frac{4cy}{Q}dy(du-2ax\,dv)-(y^2+a^2)(dx^2+dv^2) \quad (24)$$

which, together with the neutrino field given by Eqs. (13) with respect to the perturbed tetrad (7), is not only an approximate solution of the EW equations but an exact solution of the EW equations. In fact, replacing the coordinate u by u' defined by

$$du'=du+\left[\frac{1}{Q}-\frac{1}{Q+2cy}\right](y^2+a^2)dy, \quad (25)$$

one finds that Eq. (24) reads

$$(g_{\mu\nu}+2Hl_\mu l_\nu)dx^\mu dx^\nu$$
$$=\frac{Q+2cy}{y^2+a^2}(du'-2ax\,dv)^2$$
$$-\frac{y^2+a^2}{Q+2cy}dy^2-(y^2+a^2)(dx^2+dv^2), \quad (26)$$

which is of the form (8), with Q replaced by $Q+2cy$ or, equivalently, with the parameter M replaced by $M-c$. Thus, the metric (24) is not essentially different from the unperturbed metric (8).

IV. CONCLUDING REMARKS

Making use of the general results of Ref. [10], we have shown that the coupled gravitational and neutrino field perturbations of the exact solution of the EW equations given by Eqs. (8) and (13) are determined by a set of four first-order ordinary differential equations [Eqs. (16)] which, among other things, determines the conversion factors between the gravitational and neutrino waves. Despite the fact that the background solution considered here is relatively simple, the system of equations (16) looks somewhat complicated; however, this system is the only condition that remains to be solved from the linearized EW equations, which is a set of ten real second-order partial differential equations and two complex first-order partial differential equations for ten real and two complex unknowns.

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[1] D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. **29**, 465 (1957).
[2] S. A. Teukolsky, Astrophys. J. **185**, 635 (1973).
[3] W. Unruh, Phys. Rev. Lett. **31**, 1265 (1973).
[4] W. Unruh, Phys. Rev. D **10**, 3194 (1974).
[5] S. Chandrasekhar and S. Detweiler, Proc. R. Soc. London **A352**, 325 (1977).
[6] N. Kamran and R. G. McLenaghan, in *Gravitation and Geometry: A volume in Honor of I. Robinson*, edited by W. Rindler and A. Trautman (Bibliopolis, Naples, 1987).
[7] G. F. Torres del Castillo, J. Math. Phys. **29**, 2078 (1988).
[8] L. S. Kegeles and J. M. Cohen, Phys. Rev. D **19**, 1641 (1979).
[9] G. F. Torres del Castillo, Gen. Relativ. Gravit. **22**, 1085 (1990).
[10] G. F. Torres del Castillo, Class. Quantum Grav. **5**, 649 (1988).
[11] G. F. Torres del Castillo and J. C. Flores-Urbina (unpublished).
[12] G. F. Torres del Castillo and J. F. Plebański, J. Math. Phys. **26**, 477 (1985).
[13] C. A. Kolassis, J. Phys. A **16**, 749 (1983).
[14] E. Newman, L. Tamburino, and T. Unti, J. Math. Phys. **4**, 915 (1963).
[15] B. Carter, Commun. Math. Phys. **10**, 280 (1968).
[16] C. D. Collinson and P. B. Morris, J. Phys. A **6**, 915 (1973).
[17] G. F. Torres del Castillo, Class. Quantum Grav. **4**, 1133 (1987).
[18] G. F. Torres del Castillo and R. Cartas-Fuentevilla, Phys. Rev. D **54**, 4886 (1996).
[19] G. F. Torres del Castillo, J. Math. Phys. **37**, 4584 (1996); **39**, 3511(E) (1998).
[20] G. F. Torres del Castillo and J. A. Mondragón-Sánchez, Gen. Relativ. Gravit. **30**, 379 (1998).