Boundary fields and renormalization group flow in the two-matrix model

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We analyze the Ising model on a random surface with a boundary magnetic field using matrix model techniques. We are able to exactly calculate the disk amplitude, boundary magnetization and bulk magnetization in the presence of a boundary field. The results of these calculations can be interpreted in terms of renormalization group flow induced by the boundary operator. In the continuum limit this RG flow corresponds to the flow from non-conformal to conformal boundary conditions which has recently been studied in flat space theories. [S0556-2821(98)04516-0]

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I. INTRODUCTION

Simple statistical mechanical lattice models, such as the Ising model, have been used for many years to gain insight into the behavior of a wide range of physical systems. The great utility of such simple models arises from the fact that in many cases they are exactly solvable. It is a remarkable fact that some questions which seem analytically intractable for models on a regular lattice yield exact solutions when the underlying lattice is itself taken to be a random element of a larger statistical ensemble. An example of such a situation is the Ising model with a bulk magnetic field. Although this model has not been analytically solved on a fixed lattice, it is possible to exactly compute the partition function and magnetization of the model on a random lattice [1].

In this paper, we consider the Ising model on a random lattice in the presence of a magnetic field on the boundary, rather than in the bulk. Again, this corresponds to a problem which does not seem to be analytically solvable on a fixed lattice. Summing over random lattices, however, we find that the partition function and magnetizations can be calculated exactly.

The Ising model on a random two-dimensional lattice can be described in terms of a matrix model. A great deal of technology has been developed to deal with matrix models, primarily as a tool for studying string theory. The techniques we use here were derived in an earlier pair of papers [2,3], and are related to methods described in [4,5]. Some of the results which we describe here appeared in a previous letter [6].

The Ising model on a regular lattice with a boundary magnetic field was studied many years ago [7]. This model has been of renewed interest recently because in the continuum limit, where the lattice spacing is taken to zero, it gives a simple example of a two-dimensional field theory which is conformal in the bulk but which has boundary conditions breaking conformal invariance [8,9,10]. The only boundary conditions for the Ising model which preserve conformal invariance [11] are free boundary conditions (where the boundary field h vanishes), and fixed spin boundary conditions (where $h=\pm\infty$). Putting an arbitrary field h on the boundary generates a renormalization group (RG) flow from the free boundary condition to the fixed boundary condition [12].

For a matrix model corresponding to fields on a random surface, a continuum limit can also be taken. In this limit, the theory describes conformal matter fields coupled to 2D quantum gravity. In this paper we take the continuum limit of the disk partition function and magnetizations and consider the implications of our results for the resulting theory of $c=1/2$ matter coupled to gravity. In particular, we find that the results are in accord with the hypothesis that the RG flow which has been understood in flat space is present in an appropriate form in the theory with gravity.

One provocative feature of our results is that as the boundary magnetic field is increased, except for a jump discontinuity when the field becomes nonzero, the expectation value of the magnetization of a randomly chosen spin in the bulk *decreases*. This counterintuitive result may be explained in terms of the effects of the matter fields on the geometry—roughly speaking, the increase in magnetic field produces a long ''throat'' which separates the boundary from the bulk and which increases the average distance of a bulk spin from the boundary, effectively decreasing the bulk magnetization. However, this result is also found to be a finite volume effect which may depend upon the precise choice of how the random lattice ensemble is defined.

Another context in which this work may be relevant is the current discussion of D-branes in string theory (see for example [13]). Just as there are two conformally invariant boundary conditions for the Ising theory, a conformal field theory of a single bosonic field can have two conformally

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invariant boundary conditions: Neumann and Dirichlet. Considering the continuum limit of the Ising model as a single free fermion, free and fixed Ising boundary conditions are related through supersymmetry to Neumann and Dirichlet boundary conditions on a bosonic field. The fact that RG flow behaves similarly in flat space and in the presence of a fluctuating metric suggests that perhaps Dirichlet boundary conditions in superstring theory naturally arise as an RG limit of a non-conformal boundary term.

In Sec. II of this paper we describe the discrete model we will use for the Ising model on a random surface. In Sec. III we apply the methods of [2] to this model, explicitly deriving a quartic equation satisfied by the disk amplitude and locating the associated critical point. In Sec. IV we take the continuum limit of the disk amplitude. In Sec. V we discuss the general formalism we will use for calculating magnetizations, and we apply this in Secs. VI and VII to compute the boundary and bulk magnetizations on the disk. In Sec. VIII we discuss implications of our results for renormalization group flow, duality, and the effects of gravity on the behavior of the matter theory.

II. THE MODEL

A discretized theory of $c=1/2$ matter coupled to 2D quantum gravity is described by the Ising model on a randomly triangulated surface. At the center of each (equilateral) triangle on the surface lives a single Ising spin, coupled to its nearest neighbors. This theory can be described by a matrix model [1] with partition function

$$\mathcal{Z}(g,c)=\int DUDV\ \exp(-NS(U,V)), \tag{1}$$

where the action is given by

$$S=\frac{1}{2}\ \mathrm{Tr}(U^2+V^2)-c\ \mathrm{Tr}\ UV-\frac{g}{3}\ \mathrm{Tr}(U^3+V^3). \tag{2}$$

In these expressions, U and V are $N\times N$ Hermitian matrices representing up and down spins respectively, g is a coupling constant corresponding to a Boltzmann weight for each triangle on the surface, and c describes the coupling between Ising spins. (This c is unrelated to the central charge.) The partition function can be expanded in a power series in g and $1/N$, and the coefficient of g^kN^{2-2h} is then given by a summation of the Ising partition function over all triangulations of a genus h Riemann surface by k triangles.

We are interested in amplitudes corresponding to various boundary conditions on spins living on a genus zero surface with boundary. For example, the disk amplitude for a configuration of plus and minus spins on the boundary, represented by an ordered string $w(U,V)$ of the matrices U and V, is given by the large-N limit of the matrix model expectation value,

$$p_w=\lim_{N\to\infty}\frac{1}{N}\langle \mathrm{Tr}\ w(U,V)\rangle. \tag{3}$$

(Since we will always be interested in the large-N limit, henceforth we will suppress the explicit $1/N$ in expectation values.) Again, p_w can be expanded in a power series in g, with the coefficient of g^k giving the sum over all disk triangulations by k triangles with a boundary having a fixed length and spin configuration $w(+,-)$.

There are two types of conformally invariant boundary condition in the Ising theory: ''fixed'' (corresponding to all boundary spins aligned) and ''free'' (corresponding to an equally weighted sum of all possible boundary spin configurations). For each of these we can define a generating function which encodes the amplitudes. Fixed boundary conditions are described by

$$\tilde{\phi}_{\mathrm{fixed}}(u)=\sum_{k=0}^{\infty}\ \langle \mathrm{Tr}\ U^k\rangle u^k, \tag{4}$$

while free boundary conditions are described by

$$\tilde{\phi}_{\mathrm{free}}(x)=\sum_{k=0}^{\infty}\ \langle \mathrm{Tr}(U+V)^k\rangle x^k. \tag{5}$$

The Ising model in this set of variables is symmetric under interchange of U and V (corresponding to the symmetry under spin reversal). The generating function for fixed boundary conditions can therefore also be described by summing over amplitudes with all V's on the boundary.

An alternative coupling of discretized $c=1/2$ matter to 2D quantum gravity is obtained by considering the dual Ising model on a random surface [14]. The partition function of the dual model is defined once again as a sum over surfaces with Ising spins at the face of each plaquette, where now the plaquettes may be polygons with arbitrary numbers of sides, but the coordination number at each vertex is constrained to be equal to three. Such a configuration is equivalent through duality to a triangulation in the original theory, but with spins located at vertices rather than on faces. The dual model may also be described as a theory of two matrices X and Y, with action

$$S=\frac{(1-c)}{2}\ \mathrm{Tr}\ X^2+\frac{(1+c)}{2}\ \mathrm{Tr}\ Y^2-\frac{\hat{g}}{3}\ \mathrm{Tr}(X^3+3XY^2). \tag{6}$$

There are once again two types of conformally invariant boundary conditions, fixed and free. In the dual model the matrix variables X and Y do not refer to the state of individual spins, but rather to the relative state of two spins across an edge; X denotes an edge separating two equal spins, while Y denotes a boundary between two opposite spins. The generating function for fixed boundary conditions is therefore

$$\hat{\phi}_{\mathrm{fixed}}(x)=\sum_{k=0}^{\infty}\ \langle \mathrm{Tr}\ X^k\rangle x^k, \tag{7}$$

while free boundary conditions are described by

$$\hat{\phi}_{\rm free}(u)=\sum_{k=0}^{\infty}\langle {\rm Tr}(X+Y)^k\rangle u^k. \tag{8}$$

Note that the dual model is symmetric under $Y\to -Y$; it is therefore the generating function for free boundary conditions which has an alternative description in these variables, as a sum over $\langle {\rm Tr}(X-Y)^k\rangle$. (Amplitudes with an odd number of boundary Y's vanish identically.)

Although the original and dual formulations of the Ising model represent different couplings to 2D gravity, there is a simple transformation that relates the original action (2) to the dual action (6):

$$X\to \frac{1}{\sqrt{2}}(U+V)$$

$$Y\to \frac{1}{\sqrt{2}}(U-V) \tag{9}$$

$$\hat{g}\to g/\sqrt{2}.$$

As a consequence, the partition functions for the two models on surfaces with no boundaries are identical. This does not, however, guarantee that all correlation functions in the two theories agree, since the transformation (9) has a nontrivial action on the states and operators of the theory. For example, the disorder operator Y in the dual theory is taken into the spin operator $U-V$ in the original theory. Similarly, the role of free and fixed boundary conditions is interchanged; we have

$$\tilde{\phi}_{\rm fixed}(u)=\hat{\phi}_{\rm free}(u/\sqrt{2}),$$
$$\tilde{\phi}_{\rm free}(x)=\hat{\phi}_{\rm fixed}(\sqrt{2}x). \tag{10}$$

The (Kramers-Wannier, or T-) duality of the model relates a specified state (fixed, free, or with additional operator insertions) in the original form of the model to the same state in the dual version. Although in the discrete version of the theory, this duality symmetry is explicitly violated by finite volume effects, it seems likely that the duality symmetry is restored for all correlation functions in the continuum limit. Evidence for duality in the continuum limit at genus zero was presented in [15,3], and the issue of higher genus was explored in [16,17,18].

Our interest in this paper is in non-conformally-invariant boundary conditions, which may be thought of as arising from the introduction of boundary fields or couplings. We therefore consider the original model in a new set of matrix variables Q, R, defined by

$$Q=e^hU+e^{-h}V,$$
$$R=e^hU-e^{-h}V. \tag{11}$$

Substituting these into the original action (2), we obtain

$$S=\tfrac{1}{4}\ {\rm Tr}[(\cosh(2h)-c)Q^2+(\cosh(2h)+c)R^2$$
$$-2\sinh(2h)QR]-\frac{g}{12}\ {\rm Tr}[\cosh(3h)(Q^3+3QR^2)$$
$$-\sinh(3h)(3Q^2R+R^3)]. \tag{12}$$

In this set of variables we can calculate the generating function for disk amplitudes with all Q's on the boundary,

$$\phi(q,h)=\sum_{k=0}^{\infty}\langle {\rm Tr}(e^hU+e^{-h}V)^k\rangle q^k=\sum_{k=0}^{\infty}\langle {\rm Tr}\ Q^k\rangle q^k. \tag{13}$$

This generating function describes boundary conditions which interpolate between fixed and free. The parameter h can be thought of as a boundary magnetic field applied to otherwise free boundary conditions; for $h=0$ we recover $\tilde{\phi}_{\rm free}$ in the original model, while for $h=\pm\infty$ the boundary spins are all driven to one value and we recover $\tilde{\phi}_{\rm fixed}$. Note that although the $h\to\pm\infty$ limit appears to be singular, this is just an issue of normalization, which can be absorbed by a constant rescaling of q. Indeed, we shall see that although the critical value of q goes to zero as $h\to\pm\infty$ with the chosen normalization, all quantities of interest (such as ϕ for example) are well behaved in these limits.

III. LOOP EQUATIONS FOR CORRELATION FUNCTIONS

The process of calculating the generating function $\phi(q)$ was described and essentially carried out in [2], without the explicit solution being written down. Here we will review the basics of that procedure, and examine the solution in detail. (The discussion in [2] was framed in terms of noncommuting variables; for our purposes here we may skip directly to functions of a single variable.)

We begin by defining an additional set of generating functions $\phi_{w(q,r)}(q)$. These functions describe disks whose boundaries include a fixed string of matrices $w(Q,R)$, plus any number of additional Q's:

$$\phi_{w(q,r)}(q)=\sum_{k=0}^{\infty}\langle {\rm Tr}\ w(Q,R)Q^k\rangle q^k. \tag{14}$$

For example, we have

$$\phi_{rqr}(q)=\sum_{k=0}^{\infty}\langle {\rm Tr}\ RQRQ^k\rangle q^k \tag{15}$$

$$=p_{rqr}+qp_{rqrq}+q^2p_{rqrqq}+q^3p_{rqrqqq}+\cdots\ . \tag{16}$$

Recall that $p_{w(q,r)}$ is the amplitude for a disk with boundary specified by the string $w(q,r)$; thus p_q corresponds to a single boundary edge labelled q, p_r corresponds to a single boundary edge labelled r, and $p_0=1$ corresponds to no boundaries.

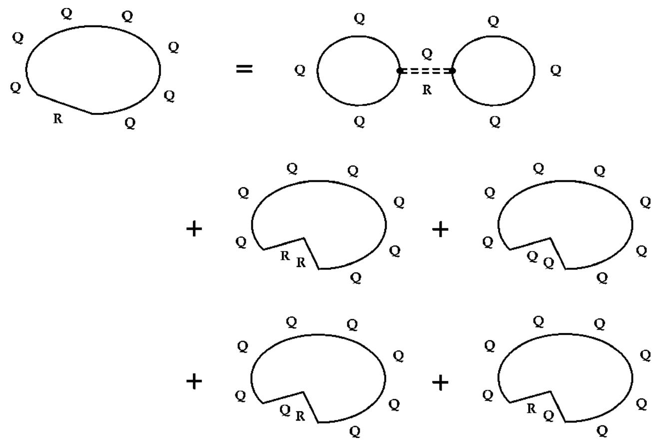

FIG. 1. Decomposition of ϕ_r. Removal of the external edge marked R on the left hand side leads to one of the possibilities shown on the right hand side, depending on whether that edge was connected to another exterior edge or an interior triangle.

We will also introduce a derivative operator D_q which acts on power series in q. Its effect is to annihilate terms independent of q, and to remove one power of q from all other terms:

$$\mathrm{D}_q q^k = \begin{cases} 0 & \text{for } k=0, \\ q^{k-1} & \text{for } k \geqslant 1. \end{cases} \tag{17}$$

The action of D_q on a generating function is to remove any constant term and divide the remainder by q. For example,

$$\begin{aligned} \mathrm{D}_q \phi &= q^{-1}(\phi - 1), \\ \mathrm{D}_q^2 \phi &= q^{-2}(\phi - 1 - p_q q), \\ \mathrm{D}_q \phi_r &= q^{-1}(\phi_r - p_r), \end{aligned} \tag{18}$$

and so on.

We can now derive a set of loop equations which relate these functions to each other, by considering the possible outcomes of removing a marked edge on the boundary. For example, let us examine the effect of removing the edge marked R from the disks represented by $\phi_r(q)$. Since ϕ_r represents a sum over various triangulated geometries of the disk, we can consider the effect of removing R from each of the terms separately. For each term, there are two basic possibilities: the edge might be identified with another edge elsewhere on the boundary, or it might belong to a triangle. In the first case, removing the marked edge and the one it was connected to leaves two disconnected triangulations, both with all Q's on the boundary and therefore representing $\phi(q)$. In the second case, removing the triangle reveals two new boundary edges, which may be marked with two Q's, two R's, or one Q and one R; these alternatives relate the initial generating function to $\mathrm{D}_q^2\phi$, ϕ_{rr}, and $\mathrm{D}_q\phi_r$. This decomposition of ϕ_r is shown schematically in Fig. 1. The amount which each term contributes to ϕ_r can be derived from the action (12), as detailed in [2].

By this procedure we can derive a closed system of eight independent equations in the quantities $(\phi, \phi_r, \phi_{rr}, \phi_{rqr}, \phi_{rrr}, \phi_{rqqr}, \phi_{rqrr}, \phi_{rrrr})$:

$$\begin{aligned}
\phi &= 1 + \beta q^2 \phi^2 + a q \phi_{rr} + a \mathrm{D}_q \phi + 2 d \phi_r \\
\phi_r &= \gamma q \phi^2 + b \phi_{rr} + b \mathrm{D}_q^2 \phi + 2 e \mathrm{D}_q \phi_r \\
\mathrm{D}_q \phi_r &= \gamma \phi + \beta q \phi \phi_r + d \phi_{rqr} + a \phi_{rrr} \\
&\quad + a \mathrm{D}_q^2 \phi_r + d \mathrm{D}_q \phi_{rr} \\
\mathrm{D}_q^2 \phi_r &= \gamma p_q \phi + \beta \phi_r + \beta q \phi \mathrm{D}_q \phi_r + d \phi_{rqqr} \\
&\quad + a \phi_{rqrr} + a \mathrm{D}_q^3 \phi_r + d \mathrm{D}_q \phi_{rqr} \\
\mathrm{D}_q \phi_{rr} &= \gamma p_r \phi + \gamma \phi_r + \beta q \phi \phi_{rr} + d \phi_{rqrr} \\
&\quad + a \phi_{rrrr} + a \mathrm{D}_q^2 \phi_{rr} + d \mathrm{D}_q \phi_{rrr} \\
\phi_{rr} &= \alpha \phi + \gamma q \phi \phi_r + e \phi_{rqr} + b \phi_{rrr} \\
&\quad + b \mathrm{D}_q^2 \phi_r + e \mathrm{D}_q \phi_{rr} \\
\phi_{rqr} &= \alpha p_q \phi + \gamma \phi_r + \gamma q \phi \mathrm{D}_q \phi_r + e \phi_{rqqr} \\
&\quad + b \phi_{rqrr} + b \mathrm{D}_q^3 \phi_r + e \mathrm{D}_q \phi_{rqr} \\
\phi_{rrr} &= \alpha p_r \phi + \alpha \phi_r + \gamma q \phi \phi_{rr} + e \phi_{rqrr} \\
&\quad + b \phi_{rrrr} + b \mathrm{D}_q^2 \phi_{rr} + e \mathrm{D}_q \phi_{rrr}.
\end{aligned} \tag{19}$$

(In [2] we listed ten equations in these variables, but the last two were not linearly independent from the first eight.) Here we have defined the new variables

$$\begin{aligned}
\alpha &= \frac{2}{1-c^2}[\cosh(2h) - c] \\
\beta &= \frac{2}{1-c^2}[\cosh(2h) + c] \\
\gamma &= \frac{2}{1-c^2} \sinh(2h) \\
a &= \frac{g}{2(1-c^2)}[\cosh h + c \cosh(3h)] \\
b &= -\frac{g}{2(1-c^2)}[\sinh h - c \sinh(3h)] \\
d &= -\frac{g}{2(1-c^2)}[\sinh h + c \sinh(3h)] \\
e &= \frac{g}{2(1-c^2)}[\cosh h - c \cosh(3h)].
\end{aligned} \tag{20}$$

The set of equations (19) is completely algebraic, since we can replace the derivative operators with algebraic functions as in Eqs. (18).

Note that we can cut down somewhat on the number of undetermined variables by looking at Eqs. (19) order by order. These give the following relations:

$$p_q = ap_{rr} + ap_{qq} + 2dp_{qr} \tag{21}$$

$$p_{qq} = \beta + ap_{qrr} + ap_{qqq} + 2dp_{qqr}$$

$$p_r = bp_{rr} + bp_{qq} + 2ep_{qr}$$

$$p_{qr} = \gamma + bp_{qrr} + bp_{qqq} + 2ep_{qqr}$$

$$p_{qr} = \gamma + 2dp_{qrr} + ap_{rrr} + ap_{qqr}$$

$$p_{rr} = \alpha + 2ep_{qrr} + bp_{rrr} + bp_{qqr}.$$

Furthermore, each of these amplitudes may be expressed as a sum of amplitudes in the original Ising model; for example, $p_q = e^h p_u + e^{-h} p_v = 2(\cosh h) p_u$. We use these relations to eliminate everything but $p_1 = p_u$ and $p_3 = p_{uuu}$.

It is now a straightforward but tedious exercise to solve the system (19) by deriving a single quartic equation for ϕ as a function of q, h, c, g, p_1 and p_3. The quartic is given by

$$f_4\phi^4 + f_3\phi^3 + f_2\phi^2 + f_1\phi + f_0 = 0, \tag{22}$$

with

$$\begin{aligned}
f_4 &= 2g^2[1-\cosh(6h)]q^8,\\
f_3 &= -4g^3\cosh(3h)q^5 + 4g^2[\cosh(4h)+\cosh(2h)+c(3-\cosh(6h))]q^6\\
&\quad + 2g[c^2\cosh(9h)+c\cosh(7h)-(1+3c^2)\cosh(3h)-(1+5c)\cosh(h)]q^7,\\
f_2 &= -g^4q^2 + 4g^3[\cosh(h)-2c\cosh(3h)]q^3\\
&\quad + g^2[4c^2\cosh(6h)+12c\cosh(4h)-2(1-5c)\cosh(2h)-(3-23c^2)]q^4\\
&\quad + 2g[-2c^2\cosh(7h)-2c(1+2c)\cosh(5h)-(5c+13c^3)\cosh(3h)+(1-4c-19c^2)\cosh(h)]q^5\\
&\quad + 2[c^3\cosh(8h)+c(2c+2c^3+g^2)\cosh(6h)+(c+2c^2+5c^3-g^2+2p_1g^3)\cosh(4h)\\
&\quad + (2c+4c^2+6c^3+g^2-2p_1g^3)\cosh(2h)+c(1+4c+2c^3-g^2)]q^6,\\
f_1 &= -2cg^4 + 2cg^3[5\cosh(h)+c\cosh(3h)]q - 2cg^2[2c\cosh(4h)+4(1+c)\cosh(2h)+(5+3c^2)]q^2\\
&\quad + 2g[c^2(1+c)\cosh(5h)+c(1+6c-5c^3+3g^2)\cosh(3h)+(6c+2c^2-4c^3-g^2+2p_1g^3)\cosh(h)]q^3\\
&\quad + 2[-c^2(c-c^3+g^2)\cosh(6h)+2c(-c+c^3-2g^2+p_1g^3)\cosh(4h)\\
&\quad + (-c-2c^2+c^3+2c^4+g^2-2p_1g^3-6cp_1g^3)\cosh(2h)+(-c+c^5+g^2-5c^2g^2-2p_1g^3)]q^4\\
&\quad + 2g[c^2(1-p_1g)\cosh(7h)+c(1-p_1g+2cp_1g)\cosh(5h)\\
&\quad + 3c(c^2+p_1g)\cosh(3h)+(-1+4c^2+2p_1g+2cp_1g+c^2p_1g)\cosh(h)]q^5,\\
f_0 &= 2cg^4 + 2cg^3[-c\cosh(3h)+(-5+2p_1g)\cosh(h)]q\\
&\quad + g^2[2c^2(2-p_1g)\cosh(4h)+2c(4+4c-3p_1g-3cp_1g)\cosh(2h)+(10c-6c^3+4p_1g-14cp_1g-3g^2-4p_3g^3)]q^2\\
&\quad + 2g[c^2(-1-c+p_1g+cp_1g)\cosh(5h)+c(-1-6c+5c^3-3p_1g+13cp_1g+3g^2+4p_3g^3)\cosh(3h)\\
&\quad + (-6c-2c^2+4c^3-4p_1g+13cp_1g+2c^2p_1g+c^3p_1g+3g^2+4p_3g^3)\cosh(h)]q^3\\
&\quad + [2c^2(c-c^3+p_1g-3cp_1g-g^2-p_3g^3)\cosh(6h)+2c(2c-2c^3+2p_1g-6cp_1g-g^2-p_1g^3-2p_3g^3)\cosh(4h)\\
&\quad + 2(c+2c^2-c^3-2c^4+p_1g-cp_1g-6c^2p_1g-g^2-2cg^2+p_1g^3+cp_1g^3-p_3g^3-2cp_3g^3-p_1^2g^4)\cosh(2h)\\
&\quad + (2c-2c^5+2p_1g-6cp_1g+2c^2p_1g-6c^3p_1g-g^2-c^2g^2-2p_1g^3-2p_3g^3-2c^2p_3g^3+2p_1^2g^4)]q^4.
\end{aligned} \tag{23}$$

The analytic solutions to such an expression are of course rather unwieldy, but fortunately they are also of little interest to us. Instead, we are interested in the expansion of ϕ around the critical point of the model, which encodes the continuum limit of the theory. (See [3] for a discussion of the extraction of the continuum limit.) The critical values of the quantities c, g, p_1 and p_3 are well known [1,3]:

$$c_c = \frac{1}{27}(-1+2\sqrt{7}), \tag{24}$$

$$g_c = 3^{-9/2}\sqrt{10}(-1+2\sqrt{7})^{3/2}, \tag{25}$$

$$g_c p_{1c} = \frac{1}{5}(3-\sqrt{7}), \tag{26}$$

and

$$g_c p_{3c} = \frac{1}{100}(-699+40\times 7^{3/2}). \tag{27}$$

We would now like to find the critical value of q for any given boundary field h. The critical point is defined as the radius of convergence of the power series expansion for the generating function, interpreted physically as the point where boundaries with an infinite number of segments begin to dominate. The generating functions for fixed boundary conditions, $\tilde{\phi}_{\rm fixed}(u)$ in the original model and $\hat{\phi}_{\rm fixed}(x)$ in the dual model, obey cubic equations, and the critical point is simply the value of x or u for which the cubic has repeated roots. For the quartic this story is slightly more complicated.

Figure 2 shows a numerically generated plot of the real part of the solutions to the quartic, as a function of q, for $\exp(h)=5/3$ and the other parameters at their critical values. Only three distinct curves are visible, as two of the solutions have identical real parts. At three points on the graph the curves intersect, representing multiple roots; as q increases, there is first a triple root, then a double root, and another triple root. To decide which of these represents the actual critical point, we follow the behavior of the physical solution as q is increased from zero along the real axis, and look for a branch point (which will indicate the radius of convergence). From the definition of the generating function we know that the physical solution is the one which equals one at $q=0$. As q is increased, the first triple root does not represent a branch point for this solution, and is therefore not the critical point. A similar situation holds for the double root; even though the physical solution is one of the double roots, one can verify numerically that circling the double root in the complex q plane does not take you to a distinct Riemann sheet, and hence this value is not a branch point. The critical value of q is therefore at the second of the two triple roots, as indicated by the point (c) in the figure.

To discover an analytic expression for the critical value q_c, we note that at this point the quartic can be factored into the form

$$f_4(\phi-\phi^{(1)})(\phi-\phi^{(3)})^3=0, \tag{28}$$

where $\phi^{(3)}$ is the triple root and $\phi^{(1)}$ is the single root. Setting the coefficients of Eq. (28) equal to those of Eq. (22) yields a set of equations from which we can eliminate $\phi^{(1)}$ and $\phi^{(3)}$ to obtain a single octic equation for q_c as a function of h. Happily, this octic factors into the product of two quadratics (one of which is cubed). It is most conveniently written in terms of q_c/g_c:

$$\begin{aligned} 0 = {} & [729+(z_1\cosh 3h+z_2\cosh h)(q_c/g_c) \\ & +(z_3\cosh 4h+2z_3\cosh 2h+z_4)(q_c/g_c)^2]^3 \\ & \times[729+(z_1\cosh 3h+z_2\cosh h)(q_c/g_c) \\ & +(z_5\cosh 4h+2z_5\cosh 2h+z_6)(q_c/g_c)^2], \end{aligned} \tag{29}$$

where

$$\begin{aligned} z_1 &= 54(1-2\sqrt{7}) \\ z_2 &= -1458 \\ z_3 &= 6(10+7\sqrt{7}) \\ z_4 &= 2(262-11\sqrt{7}) \\ z_5 &= 18(-16+5\sqrt{7}) \\ z_6 &= 2(784-83\sqrt{7}). \end{aligned} \tag{30}$$

The critical point is the solution obtained by setting the cubed quadratic to zero; one root of the quadratic will be the critical point for $h\geqslant 0$, and the second for $h\leqslant 0$. For $h\geqslant 0$ the critical value is

$$q_c(h) = \frac{g_c(1+2\sqrt{7})e^{3h}}{1+(-1+\sqrt{7})e^{2h}+(2+\sqrt{7})e^{4h}}, \tag{31}$$

whereas, for $h\leqslant 0$,

$$q_c(h) = \frac{g_c(1+2\sqrt{7})e^{-3h}}{1+(-1+\sqrt{7})e^{-2h}+(2+\sqrt{7})e^{-4h}}, \tag{32}$$

so that although $q_c(h)$ is continuous at $h=0$, it is non-analytic there.

The corresponding critical value of ϕ is

$$\phi_c(h) = \frac{(3e^{2h}-1)(1+(-1+\sqrt{7})e^{2h}+(2+\sqrt{7})e^{4h})}{10e^{6h}}. \tag{33}$$

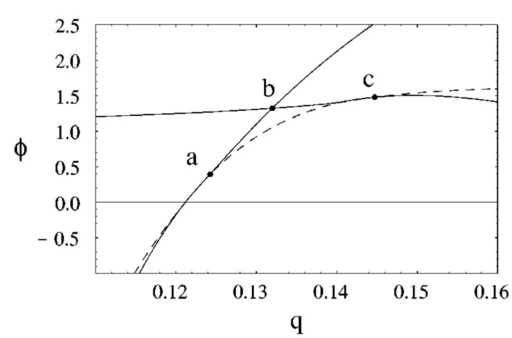

FIG. 2. The four roots of the quartic, as a function of q. The dashed line represents two roots which are complex conjugates of each other. Moving from left to right, we find the triple root ''a,'' the double root ''b,'' and the triple root ''c'' which is the critical point. The physical branch is the one that goes to 1 as q goes to zero.

Note that as expected, $q_c \to 0$ as $h \to \infty$, but ϕ_c is finite and non-zero for all h.

IV. CONTINUUM LIMIT AND DISK AMPLITUDE

To extract information about the theory in the continuum limit, we analyze the behavior of the discrete model in the vicinity of the critical point. The coupling constant c is set to the critical value c_c given by Eq. (24); deviations from this value would correspond to introducing a mass for the Majorana fermion in the continuum limit of the Ising model, which we do not examine in this paper. We then trade the independent variables g and q for new variables t and z, defined by

$$g = g_c e^{-\epsilon^2 t}, \qquad q = q_c e^{-\epsilon z} \tag{34}$$

where ϵ is a small parameter indicating the distance from the critical point. The expansions of p_1 and p_3 in powers of ϵ are then given by

$$gp_1 = \frac{3-\sqrt{7}}{5}\left[1 - \frac{22+10\sqrt{7}}{9}\,\epsilon^2 t + \frac{5^{1/3}(55+25\sqrt{7})}{36}\,\epsilon^{8/3}t^{4/3} + \frac{5^{5/3}(11+5\sqrt{7})}{216}\,\epsilon^{10/3}t^{5/3}\right] + \mathcal{O}(\epsilon^4 t^2) \tag{35}$$

and

$$gp_3 = \frac{-699+40\times 7^{3/2}}{100} - \frac{4(121-5\times 7^{3/2})}{25}\,\epsilon^2 t + \frac{9\times 5^{1/3}}{2}\,\epsilon^{8/3}t^{4/3} + \frac{3\times 5^{2/3}}{4}\,\epsilon^{10/3}t^{5/3} + \mathcal{O}(\epsilon^4 t^2). \tag{36}$$

These expansions may be substituted into the quartic (22), which may then be solved for ϕ as a sum of increasing powers of ϵ. We obtain

$$\phi = \phi_c(h) - \frac{3+2(-2+\sqrt{7})e^{2h}+(1+2\sqrt{7})e^{4h}}{10e^{4h}}\,\epsilon Z + \frac{1}{5\times 2^{7/3}\alpha(h)}\,\epsilon^{4/3}\Phi + \frac{2^{4/3}(1+4e^{2h}+e^{4h})}{15\alpha(h)(1-e^{4h})} \times\left[\frac{\epsilon^{5/3}Z(2T+Z^2)\Phi}{\Phi^2-(4T)^{4/3}}\right] + \mathcal{O}(\epsilon^2), \tag{37}$$

where for convenience we have rescaled the variables to

$$T = 5t \tag{38}$$

and

$$Z = \frac{z}{\alpha(h)}, \tag{39}$$

and the function $\Phi(Z,T)$ is given by

$$\Phi(Z,T) \equiv (Z+\sqrt{Z^2-4T})^{4/3} + (Z-\sqrt{Z^2-4T})^{4/3}. \tag{40}$$

For $h>0$,

$$\alpha(h) = \frac{e^{2h}(1+e^{2h})}{1+(-1+\sqrt{7})e^{2h}+(2+\sqrt{7})e^{4h}}, \tag{41}$$

but this function is discontinuous at $h=0$. If we take the continuum limit in Eq. (22) after h is set to zero (cf. the calculation in [3]), we find that ϕ is given by Eq. (37) with

$$\alpha(0) = \frac{1}{\sqrt{2}(1+\sqrt{7})}. \tag{42}$$

The universal part of ϕ is the first non-analytic term, and appears at order $\epsilon^{4/3}$. By virtue of the discontinuity in $\alpha(h)$, both the numerical coefficient of the universal term, and the amplitude of ϕ as a function of z, are discontinuous.

The universal part can be converted into the asymptotic form of the disk amplitude $\tilde{\phi}(l,a)$ for fixed boundary length l and disk area a. These forms of the amplitude are related through a Laplace transform

$$\frac{1}{5\times 2^{7/3}\alpha(h)}\,\epsilon^{4/3}\Phi(z/\alpha(h),5t) = \int \mathrm{d}l \int \mathrm{d}a\, e^{-zl-ta}\tilde{\phi}(l,a). \tag{43}$$

Inverting the Laplace transform, we have

$$\tilde{\phi}(l,a) = \frac{1}{25\sqrt{3}\pi}(\alpha(h)l)^{1/3}(a/5)^{-7/3}e^{-5(\alpha(h)l)^2/a} = \frac{1}{25\sqrt{3}\pi}L^{1/3}A^{-7/3}e^{-L^2/A}, \tag{44}$$

with the rescalings

$$L = \alpha(h)l, \quad A = \frac{a}{5}. \tag{45}$$

Up to an irrelevant multiplicative constant, this is precisely the form of the disk amplitude when the boundary conditions are conformal [19,4,20,3] (i.e., with $h=0$ or $h=\pm\infty$); however, the boundary length l is rescaled by the factor $\alpha(h)$ which depends discontinuously on the boundary magnetic field. Note that this amplitude includes an extra factor of l corresponding to a marked point on the boundary.

V. EXPECTATION VALUES

Now that we have defined the continuum limit of the model and computed the disk amplitude, we would like to calculate a number of correlation functions related to the boundary and bulk magnetizations in the theory. In this section we describe the formalism necessary to perform such calculations efficiently.

An example of the type of calculation we need to perform is the limiting value of the boundary magnetization on a disk with a large number of triangles and boundary segments. The boundary magnetization for a spin on the boundary of a disk with k boundary edges and n triangles is given by

$$\langle m\rangle_{n,k}=\frac{\langle \mathrm{Tr}(e^hU-e^{-h}V)(e^hU+e^{-h}V)^{k-1}\rangle_n}{\langle \mathrm{Tr}(e^hU+e^{-h}V)^k\rangle_n}, \quad (46)$$

where by $\langle\ \rangle_n$ we indicate a sum over triangulations restricted to geometries with n spins (the coefficient of g^n in an expansion in g). The quantity $\langle m\rangle$ is defined to be the large n and k limit of Eq. (46).

More generally, we define the expectation value of an operator in a specified state ψ as

$$\langle \mathcal{A}\rangle=\lim_{k\to\infty,\ n\to\infty}\frac{\langle \mathcal{A}\psi\rangle_{n,k}}{\langle\psi\rangle_{n,k}}, \quad (47)$$

where $\langle\psi\rangle_{n,k}$ is taken to mean the sum over all triangulations, with appropriate weights, with n triangles and k boundary edges. $\langle\mathcal{A}\psi\rangle_{n,k}$ is the same quantity, but with the weights adjusted by the operator $\mathcal{A}$.

For the cases we are considering in this paper, ψ is a sum over all triangulations, with weights determined by the boundary magnetic field. Thus, for zero field, ψ is simply a sum which is equally weighted for all boundary configurations (free boundary conditions), whereas for infinite (positive) h, the only boundary configurations with non-zero weight are those with all spins pointing up (fixed boundary conditions). When $\mathcal{A}$ represents a boundary spin operator, for example, then for each configuration the boundary spin is evaluated at a particular site, and the weight acquires a ± 1 depending on whether that spin is up or down. When $\mathcal{A}$ is a bulk spin operator, the spin is evaluated at a site in the bulk.

The limits in Eq. (47) can be understood in terms of the asymptotic behavior of $\langle\mathcal{A}\psi\rangle_{n,k}$ and $\langle\psi\rangle_{n,k}$. For large n,k these functions scale asymptotically as

$$\langle\psi\rangle_{n,k}\sim g_c^{-n}q_c^{-k}f(n,k) \quad (48)$$

and

$$\langle\mathcal{A}\psi\rangle_{n,k}\sim g_c^{-n}q_c^{-k}g(n,k). \quad (49)$$

Thus

$$\langle\mathcal{A}\rangle\sim\lim_{k\to\infty,\ n\to\infty}\frac{g(n,k)}{f(n,k)}. \quad (50)$$

For large n and k, it is appropriate to replace the number of triangles and boundary edges by the area and length variables $a=\epsilon^2 n$ and $l=\epsilon k$, so that $\langle\mathcal{A}\rangle$ will appear as a function of a,l.

The continuum limit of an operator expression of this type can be easily determined by taking the continuum limits of the quantities $\langle\psi\rangle$ and $\langle\mathcal{A}\psi\rangle$. Consider the continuum limits of the sums

$$\sum_{n,k=0}^{\infty}\langle\psi\rangle_{n,k}g^nq^k \quad (51)$$

and

$$\sum_{n,k=0}^{\infty}\langle\mathcal{A}\psi\rangle_{n,k}g^nq^k. \quad (52)$$

The universal behaviors of these two sums give the Laplace transforms of the two functions $f(n,k)$ and $g(n,k)$ with respect to z and t. For example

$$\sum_{n,k=0}^{\infty}\langle\psi\rangle_{n,k}q^kg^n\sim\int dk\int dne^{-\epsilon^2nt}e^{-\epsilon kz}f(n,k)=F_u(t,z), \quad (53)$$

where the subscript ''u'' indicates the universal part. In order to recover the functions f and g, it suffices to perform an inverse Laplace transform on the universal parts $F_u(t,z)$ and $G_u(t,z)$ of the sums to obtain the functions $\tilde{f}(a,l)$, for which

$$F_u(t,z)=\int dl\int dae^{-zl-ta}\tilde{f}(a,l) \quad (54)$$

and similarly $\tilde{g}(a,l)$. Thus

$$\langle\mathcal{A}\rangle=\frac{\tilde{g}(a,l)}{\tilde{f}(a,l)}. \quad (55)$$

We shall see this explicitly in the following sections.

VI. BOUNDARY MAGNETIZATION

In this section we will apply the discussion above to compute the one- and two-point boundary magnetizations in the presence of a boundary field.

A. One-point boundary magnetization

The boundary magnetization $\langle m\rangle$ is given by the large n,k limit of

$$\langle m\rangle_{n,k}=\frac{\langle \mathrm{Tr}(e^hU-e^{-h}V)(e^hU+e^{-h}V)^{k-1}\rangle_n}{\langle \mathrm{Tr}(e^hU+e^{-h}V)^k\rangle_n}. \quad (56)$$

We may follow the route described in the previous section to compute $\langle m\rangle$. The first step, the critical expansion of ϕ in the continuum limit, was given in Eq. (37). The other quantity we need to expand is

$$\psi_r\equiv q\phi_r=\sum_{k=0}^{\infty}\langle \mathrm{Tr}\ RQ^k\rangle q^{k+1}. \quad (57)$$

When $h=0$, ϕ_r vanishes by symmetry. When $h\neq 0$, we can compute $\phi_r(h)$ by solving a linear combination of the first two loop equations of (19). From these it follows that

$$\phi_r = \frac{(1-e^{2h})[(c-e^{2h}+ce^{2h}+ce^{4h})(1-\phi)+2p_1e^{2h}g-2\phi^2q^2(1+e^{2h}+e^{4h})]}{cq+e^{2h}q+e^{4h}q+ce^{6h}q-2e^{3h}g}, \tag{58}$$

and hence, expanding and multiplying by $q_c e^{-\epsilon z}$, we obtain

$$\psi_r = \frac{(e^{2h}-1)(-1+(3-\sqrt{7})e^{2h}-(4-3\sqrt{7})e^{4h})}{10e^{6h}} - \frac{(e^{2h}-1)(3+(1+2\sqrt{7})e^{2h})}{10e^{6h}}\epsilon Z + \frac{(e^{2h}-1)(3+(2+\sqrt{7})e^{2h})}{5\times 2^{7/3}e^{2h}(1+e^{2h})}\epsilon^{4/3}\Phi$$
$$- \frac{2^{4/3}(3+(2+\sqrt{7})e^{2h})(1+4e^{2h}+e^{4h})}{15e^{2h}(1+e^{2h})^2}\left[\frac{\epsilon^{5/3}Z(2T+Z^2)\Phi}{\Phi^2-(4T)^{4/3}}\right] + \mathcal{O}(\epsilon^2). \tag{59}$$

The universal part of ψ_r is equal to that of ϕ, up to an h-dependent constant. There is therefore no need to explicitly compute $\tilde{\psi}_r(a,l)$, the inverse Laplace transform of $\psi_r(Z,T)$, in order to determine the boundary magnetization. We need only compute the ratio of the universal parts to get the l and A independent result (for $h>0$)

$$\langle m \rangle = \frac{\tilde{\psi}_r}{\tilde{\phi}} = \frac{(e^{2h}-1)(3+(2+\sqrt{7})e^{2h})}{(1+(-1+\sqrt{7})e^{2h}+(2+\sqrt{7})e^{4h})}. \tag{60}$$

Note that $\langle m \rangle$ is independent of l and A, and is continuous at $h=0$. (Note also that the expression given in [6] contains a typographical error in the numerator.)

In this particular case, there happens to be a simple argument that gives $\langle m \rangle$ more directly than the computation outlined above. In the large k limit,

$$\langle Q^k \rangle_n \sim q_c(h)^{-k} g_c^{-n} f(n,k). \tag{61}$$

Differentiating both sides with respect to h, we obtain

$$k\langle Q^{k-1}R\rangle_n \sim k[-q_c(h)^{-(k+1)}q_c'(h)f(n,k)+\mathcal{O}(1/k)]g_c^{-n}, \tag{62}$$

from which it follows directly that, in the large k limit,

$$\langle m \rangle = -\frac{q_c'(h)}{q_c(h)} = \frac{(e^{2h}-1)(3+2e^{2h}+\sqrt{7}e^{2h})}{1+(-1+\sqrt{7})e^{2h}+(2+\sqrt{7})e^{4h}}, \tag{63}$$

where we have used Eq. (31) for $q_c(h)$. This confirms the result obtained in Eq. (60).

A graph of the boundary magnetization is shown in Fig. 3 (bold curve). As expected, with no field the magnetization is zero, and for an infinite field the magnetization is 1. This result is compared with the boundary magnetization on a half-plane in flat space, computed by McCoy and Wu [7] (dashed curve). Whereas in flat space the magnetization scales as $h \ln h$ for small h, leading to a divergence in the magnetic susceptibility at the critical temperature, on a random surface we find that the magnetization is linear at $h=0$, with a finite susceptibility

$$\chi = \partial_h \langle m \rangle|_{h=0} = \frac{1+2\sqrt{7}}{3}. \tag{64}$$

B. Two-point boundary magnetization

Having computed the magnetization at a single point on the boundary of the disk in the presence of a boundary magnetic field, we would now like to compute the correlation between two spins on the boundary of the disk, which are separated by k and l edges in the two directions around the boundary. To compute the two point magnetization,

$$\langle m^2 \rangle = \frac{\langle \mathrm{Tr}(e^hU-e^{-h}V)(e^hU+e^{-h}V)^k(e^hU-e^{-h}V)(e^hU+e^{-h}V)^l\rangle_n}{\langle \mathrm{Tr}(e^hU+e^{-h}V)^{k+l+2}\rangle_n}, \tag{65}$$

we need expressions for

$$\Sigma(q_1,q_2) \equiv q_1q_2\sigma(q_1,q_2) \tag{66}$$

where

$$\sigma(q_1,q_2) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle \mathrm{Tr}\, RQ^kRQ^l\rangle q_1^k q_2^l, \tag{67}$$

and for

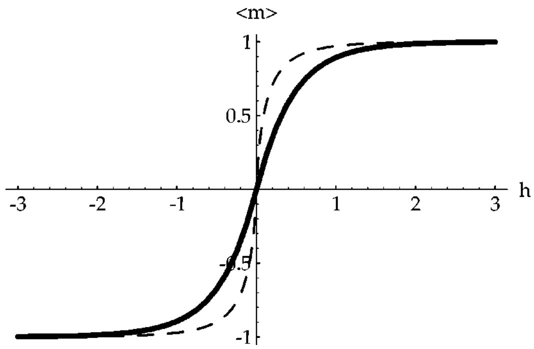

FIG. 3. Boundary magnetization $\langle m\rangle$ as a function of boundary field h in flat space (dotted line) and on a random surface (bold line).

$$\rho(q_1,q_2)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle \mathrm{Tr}\ Q^{k+l+2}\rangle q_1^{k+1}q_2^{l+1}. \tag{68}$$

Again, using the loop equation techniques of [2], one can derive the following set of equations for σ [here, a derivative D_{q_i} denotes a combinatorial derivative as in Eq. (17), where all variables other than q_i are held constant]:

$$\begin{aligned}
\sigma(q_1,q_2)&=b\mathrm{D}_{q_1}^2\overline{\phi}(q_1,q_2)+e(\mathrm{D}_{q_1}\sigma(q_1,q_2)\\
&\quad+\mathrm{D}_{q_2}\sigma(q_1,q_2))+\alpha\phi(q_1)\phi(q_2)\\
&\quad+\gamma\overline{\phi}(q_1,q_2)(\phi(q_1)q_1+\phi(q_2)q_2)\\
&\quad+b\sigma_r(q_1,q_2)\\
\overline{\phi}(q_1,q_2)&=b\mathrm{D}_{q_2}^2\phi(q_2)+2e\mathrm{D}_{q_2}\phi_r(q_2)\\
&\quad+b\phi_{rr}(q_2)+a\mathrm{D}_{q_1}^2\overline{\phi}(q_1,q_2)q_1\\
&\quad+d(\mathrm{D}_{q_1}\sigma(q_1,q_2)+\mathrm{D}_{q_2}\sigma(q_1,q_2))q_1\\
&\quad+\gamma\phi(q_1)\phi(q_2)q_1+\gamma\phi(q_2)^2q_2\\
&\quad+\beta\overline{\phi}(q_1,q_2)q_1(\phi(q_1)q_1\\
&\quad+\phi(q_2)q_2)+aq_1\sigma_r(q_1,q_2)\\
\phi(q_1)&=1+a\mathrm{D}_{q_1}^2\phi(q_1)q_1+2d\mathrm{D}_{q_1}\phi_r(q_1)q_1\\
&\quad+a\phi_{rr}(q_1)q_1+\beta\phi(q_1)^2{q_1}^2\\
\phi(q_2)&=1+a\mathrm{D}_{q_2}^2\phi(q_2)q_2+2d\mathrm{D}_{q_2}\phi_r(q_2)q_2\\
&\quad+a\phi_{rr}(q_2)q_2+\beta\phi(q_2)^2{q_2}^2\\
\phi_r(q_1)&=b\mathrm{D}_{q_1}^2\phi(q_1)+2e\mathrm{D}_{q_1}\phi_r(q_1)\\
&\quad+b\phi_{rr}(q_1)+\gamma\phi(q_1)^2q_1\\
\phi_r(q_2)&=b\mathrm{D}_{q_2}^2\phi(q_2)+2e\mathrm{D}_{q_2}\phi_r(q_2)\\
&\quad+b\phi_{rr}(q_2)+\gamma\phi(q_2)^2q_2,
\end{aligned}$$

where

$$\sigma_r(q_1,q_2)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle \mathrm{Tr}\ RQ^kRQ^lR\rangle q_1^kq_2^l \tag{69}$$

and

$$\overline{\phi}(q_1,q_2)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle \mathrm{Tr}\ Q^kRQ^l\rangle q_1^kq_2^l. \tag{70}$$

It is easily verified that $\overline{\phi}(q_1,q_2)$ is given by

$$\overline{\phi}(q_1,q_2)=\frac{q_1\mathrm{D}_{q_1}\phi_r(q_1)-q_2\mathrm{D}_{q_2}\phi_r(q_2)}{q_1-q_2}. \tag{71}$$

An expression for $\sigma(q_1,q_2)$ in terms of $\phi(q_1)$, $\phi(q_2)$ and p_1 and p_3 can be obtained by solving these equations. The expression is too long to be included here. On the other hand $\rho(q_1,q_2)$ can be directly expressed in terms of $\phi(q_1)$ and $\phi(q_2)$ in a very simple way as

$$\begin{aligned}
\rho(q_1,q_2)&=\frac{q_2q_1^2\mathrm{D}_{q_1}^2\phi(q_1)-q_1q_2^2\mathrm{D}_{q_2}^2\phi(q_2)}{q_1-q_2}\\
&=1+\frac{q_2\phi(q_1)-q_1\phi(q_2)}{q_1-q_2}.
\end{aligned} \tag{72}$$

Armed with expressions for $\sigma(q_1,q_2)$ and $\rho(q_1,q_2)$, it is then straightforward, if rather tedious, to obtain the critical expansions of Σ and ρ. They are given by

$$\begin{aligned}
\Sigma(Z_1,Z_2,T)&=\sigma_c+\frac{(e^{2h}-1)^2(3+(2+\sqrt{7})e^{2h})^2}{20\times 2^{1/3}e^{4h}(1+e^{2h})^2}\\
&\quad\times\left[\frac{\Phi(Z_1,T)-\Phi(Z_2,T)}{Z_1-Z_2}\right]\epsilon^{1/3}+\mathcal{O}(\epsilon^{2/3})
\end{aligned} \tag{73}$$

and

$$\begin{aligned}
\rho(Z_1,Z_2,T)&=\rho_c+\frac{1}{20\times 2^{1/3}\alpha(h)^2}\\
&\quad\times\left[\frac{\Phi(Z_1,T)-\Phi(Z_2,T)}{Z_1-Z_2}\right]\epsilon^{1/3}+\mathcal{O}(\epsilon^{2/3}).
\end{aligned} \tag{74}$$

As in the case of the one-point magnetization, the universal parts of Σ and ρ depend on Z_1, Z_2, and T in the same way. Consequently, the ratio of the Laplace transforms of these universal parts will simply be the h-dependent ratio of the universal parts themselves. It follows that the two-point boundary magnetization,

$$\langle m^2\rangle=\frac{\tilde{\Sigma}}{\tilde{\rho}}=\frac{(e^{2h}-1)^2(3+(2+\sqrt{7})e^{2h})^2}{(1+(-1+\sqrt{7})e^{2h}+(2+\sqrt{7})e^{4h})^2}, \tag{75}$$

is precisely the square of the one-point magnetization. We find an absence of polynomial corrections to this result (at least to the first subleading order), indicating that correlations between boundary spin operators decay exponentially, as one would expect in analogy with the flat space theory.

VII. BULK MAGNETIZATION

The expectation value of the bulk magnetization in the presence of a boundary magnetic field, on a disk with boundary length k and area n, is given by

$$\langle M\rangle = \frac{\langle \mathrm{Tr}(e^{h}U+e^{-h}V)^{k}\cdot \mathrm{Tr}(U-V)\rangle_{n}}{\langle \mathrm{Tr}(e^{h}U+e^{-h}V)^{k}\cdot \mathrm{Tr}(U+V)\rangle_{n}}. \tag{76}$$

This can be evaluated by considering cylinder amplitudes with one boundary having a boundary magnetic field, and the other with a single boundary edge. The second boundary represents a marked point on the bulk. Although the second boundary corresponds to only a single edge rather than 3 edges as would be appropriate for a triangle corresponding to a single spin, this distinction should not be relevant in the continuum limit where the boundary becomes pointlike. Again, a quantity such as Eq. (76) can be computed by the method of loop equations [3].

To compute the magnetization, we require two punctured-disk amplitudes:

$$\tau(h) = \sum_{k=0}^{\infty} \langle \mathrm{Tr}(e^{h}U+e^{-h}V)^{k}\ \mathrm{Tr}(U-V)\rangle q^{k} \tag{77}$$

and

$$\lambda(h) = \sum_{k=0}^{\infty} \langle \mathrm{Tr}(e^{h}U+e^{-h}V)^{k}\ \mathrm{Tr}(U+V)\rangle q^{k}. \tag{78}$$

As in Eq. (14), we define functions related to τ but with additional words corresponding to sequences of spins on the outer boundary:

$$\tau_{w(q,r)}(q) = \sum_{k=0}^{\infty} \langle \mathrm{Tr}\ w(Q,R)Q^{k}\cdot \mathrm{Tr}(U-V)\rangle q^{k}. \tag{79}$$

The first step in computing the bulk magnetization is now to derive a set of eight independent equations which close on the quantities $(\tau,\tau_r,\tau_{rr},\tau_{rqr},\tau_{rrr},\tau_{rqqr},\tau_{rqrr},\tau_{rrrr})$:

$$\tau = 2\beta q^{2}\phi\tau + aq\tau_{rr} + a\mathrm{D}_q\tau + 2d\tau_r + c_q q\phi$$

$$\tau_r = 2\gamma q\phi\tau + b\tau_{rr} + b\mathrm{D}_q^{2}\tau + 2e\mathrm{D}_q\tau_r + c_r\phi$$

$$\tau_{rr} = \alpha\tau + \gamma q(\tau\phi_r + \phi\tau_r) + e\tau_{rqr} + b\tau_{rrr} + b\mathrm{D}_q^{2}\tau_r + e\mathrm{D}_q\tau_{rr} + c_r\phi_r$$

$$\mathrm{D}_q\tau_r = \gamma\tau + \beta q(\tau\phi_r + \phi\tau_r) + d\tau_{rqr} + a\tau_{rrr} + a\mathrm{D}_q^{2}\tau_r + d\mathrm{D}_q\tau_{rr} + c_q\phi_r$$

$$\mathrm{D}_q^{2}\tau_r = \gamma(p_q\tau + t_q\phi) + \beta\tau_r + \beta q(\tau\phi_r + \phi\tau_r) + d\tau_{rqqr} + a\tau_{rqrr} + a\mathrm{D}_q^{3}\tau_r + d\mathrm{D}_q\tau_{rqr} + c_q\mathrm{D}_q\phi_r \tag{80}$$

$$\mathrm{D}_q\tau_{rr} = \gamma(p_r\tau + t_r\phi) + \gamma\tau_r + \beta q(\tau\phi_{rr} + \phi\tau_{rr}) + d\tau_{rqrr} + a\tau_{rrrr} + a\mathrm{D}_q^{2}\tau_{rr} + d\mathrm{D}_q\tau_{rrr} + c_q\phi_{rr}$$

$$\tau_{rqr} = \alpha(p_q\tau + t_q\phi) + \gamma\tau_r + \gamma q(\tau\mathrm{D}_q\phi_r + \phi\mathrm{D}_q\tau_r) + e\tau_{rqqr} + b\tau_{rqrr} + b\mathrm{D}_q^{3}\tau_r + e\mathrm{D}_q\tau_{rqr} + c_r\mathrm{D}_q\phi_r$$

$$\tau_{rrr} = \alpha(p_r\tau + t_r\phi) + \alpha\tau_r + \gamma q(\tau\phi_{rr} + \phi\tau_{rr}) + e\tau_{rqrr} + b\tau_{rrrr} + b\mathrm{D}_q^{2}\tau_{rr} + e\mathrm{D}_q\tau_{rrr} + c_r\phi_{rr},$$

where

$$t_q \equiv \langle \mathrm{Tr}\ Q\cdot \mathrm{Tr}(U-V)\rangle, \quad t_r \equiv \langle \mathrm{Tr}\ R\cdot \mathrm{Tr}(U-V)\rangle. \tag{81}$$

As before, $p_q = \langle \mathrm{Tr}\ Q\rangle$, $p_r = \langle \mathrm{Tr}\ R\rangle$, and the various constants take the same values as in Eq. (19).

The equations (80) can now be used to express $\tau(h)$ as a polynomial function of $\phi(h)$. However, this equation will also contain a number of unknown correlation functions t_q, t_{qq}, t_{qqq}, t_r, t_{rr} and t_{rrr}, some of which appear explicitly in Eqs. (80) and some of which arise from the derivatives of τ, since

$$\mathrm{D}_q\tau = q^{-1}(\tau-1), \qquad \mathrm{D}_q^{2}\tau = q^{-2}(\tau - 1 - t_q q), \tag{82}$$

and so on. As in the computation of ϕ, these correlation functions can be reduced to a much smaller number of unknowns by expanding Eqs. (80) order-by-order. It turns out that after using all the relations in (80) [cf. Eq. (21)], two extra relations are required between

$$t_u \equiv \langle \mathrm{Tr}\ U\cdot \mathrm{Tr}\ U\rangle, \quad t_{uuu} \equiv \langle \mathrm{Tr}\ U^{3}\cdot \mathrm{Tr}\ U\rangle, \qquad t_v \equiv \langle \mathrm{Tr}\ V\cdot \mathrm{Tr}\ U\rangle, \quad t_{vvv} \equiv \langle \mathrm{Tr}\ V^{3}\cdot \mathrm{Tr}\ U\rangle. \tag{83}$$

The first extra relation comes from the calculation in [3] of the critical expansion of

$$w_0 \equiv \frac{1}{2}\langle \mathrm{Tr}(U-V)\cdot \mathrm{Tr}(U-V)\rangle = t_u - t_v, \tag{84}$$

which is given by

$$w_0 = \frac{1+2\sqrt{7}}{5}\,(1 + 5^{2/3}\epsilon^{4/3}t^{2} + \mathcal{O}(\epsilon^{2}t^{3})). \tag{85}$$

The second relation is obtained by differentiating the matrix integral expression for p_1 with respect to g:

$$\partial_g p_1 = \frac{t_{uuu} + t_{vvv}}{3}. \tag{86}$$

Then

$$\partial_g p_1 = \frac{\partial_t p_1}{-3g\epsilon^2 t^2} \tag{87}$$

can be used to obtain the critical expansion of $\partial_g p_1$.

Armed with these extra relations, we have all the information required to expand τ in ϵ. We obtain

$$\tau = \frac{2^{4/3} g_c (1+2\sqrt{7})^2 (1+(-1+\sqrt{7})e^{2h} + (2+\sqrt{7})e^{4h}) Z}{15 e^{2h}(1+e^{2h})(\Phi(Z,T)+(4T)^{2/3})} \epsilon^{-1/3} + \mathcal{O}(\epsilon^0). \tag{88}$$

As discussed in Sec. V, the quantity of interest is the inverse Laplace transform of the universal part of τ, given in this case by

$$\tilde{\tau}(L,A) = \frac{(1+2\sqrt{7})^2 g_c}{50\sqrt{2}\pi} L^{2/3} A^{-5/3} e^{-L^2/A}. \tag{89}$$

Here, we have introduced rescaled area and boundary length parameters as in Eq. (45).

It is much easier to compute the critical expansion of $\lambda(h)$ since it is directly related to $\phi(h)$ via

$$\lambda(h) = \partial_g \phi(h). \tag{90}$$

This gives an expansion

$$\lambda = \frac{2^{2/3}(-1+2\sqrt{7})(1+(-1+\sqrt{7})e^{2h} + (2+\sqrt{7})e^{4h})\Lambda(Z,T)}{81 g_c e^{2h}(1+e^{2h})} \epsilon^{-2/3} + \mathcal{O}(\epsilon^{-1/3}), \tag{91}$$

where

$$\Lambda(Z,T) = \frac{(Z-\sqrt{Z^2-4T})^{1/3} - (Z+\sqrt{Z^2-4T})^{1/3}}{\sqrt{Z^2-4T}}. \tag{92}$$

The leading term in λ has an inverse Laplace transform

$$\tilde{\lambda}(L,A) = \frac{(-1+2\sqrt{7})}{5\times 3^{7/2}\pi g_c} L^{1/3} A^{-4/3} e^{-L^2/A}. \tag{93}$$

The bulk magnetization in the continuum limit is then given by

$$\langle M\rangle = \frac{\tilde{\tau}(L,A)}{\tilde{\lambda}(L,A)} = L^{1/3} A^{-1/3}. \tag{94}$$

The numerical values of the coefficients of $\tilde{\tau}$ and $\tilde{\lambda}$ have exactly cancelled. Although it may appear that the magnetization can be greater than one, this formula is valid in the continuum limit, for which $A \sim L^2 \gg 1$. We also notice that this form of the magnetization is independent of h except for the dependence on the scaling factor $\alpha(h)$ incorporated in L. At $h=0$, this magnetization is discontinuous and vanishes. One nice feature of this result is that it correctly reproduces the scaling behavior expected after the magnetization operator has been gravitationally dressed according to the Knizhnik-Polyakov-Zamolodchikov (KPZ) and David-Distler-Kawai (DDK) description of Liouville theory [21,22]. The gravitationally dressed scaling dimension of the bulk magnetization field is $\Delta = 1/6$. By analogy with the flat space theory we expect that the bulk magnetization should scale as $\langle M\rangle \sim d^{-2\Delta}$ where d is a measure of the distance from the boundary. This is precisely the behavior seen in (94), since A/L has dimensions of length.

VIII. DISCUSSION

Let us summarize the implications of the results we have derived in the previous sections.

A. Renormalization group flow

In Sec. IV we computed the disk amplitude in the presence of a boundary magnetic field as a function of the disk area a, the boundary length l, and the boundary field h. We discovered that the result could be written in terms of the two variables $A = a/5$ and $L = \alpha(h) l$, in which case the disk amplitude took on precisely the form of the analogous function when the boundary conditions are conformally invariant. Thus, the effect of a boundary field on this amplitude

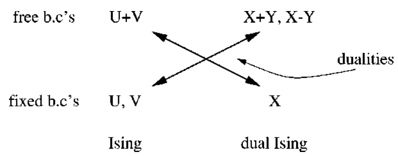

FIG. 4. The duality map that interchanges free and fixed boundary conditions. Note that for the Ising model, there are two fixed states and only one free state, whereas the opposite is true of the dual model.

amounts to a rescaling of the boundary length by an h-dependent factor. In Sec. VII, meanwhile, we found that the bulk magnetization as a function of a, l and h could also be expressed in terms of A and L, and that its functional form was the same as in the presence of an infinite boundary field.

Taken together, these results imply the existence of an RG flow to the conformally invariant boundary condition with an infinite boundary field; h is a relevant operator which goes to $\pm\infty$ in the infrared (in this context, as the disk area and length grow large). Further evidence is provided by the discontinuity in the rescaling function at $h=0$: any imposed boundary field, no matter how small, leads to magnetization in the bulk of the same form as that expected in the presence of fixed boundary conditions.

A related phenomenon has previously been derived for the flat-space Ising model on a half-plane geometry [7,8,9]. Again, one can compute the magnetization of a point in the bulk in the presence of a boundary field; however, rather than depending on the area of the surface and length of the boundary (both of which are infinite for the half-plane), the magnetization is a function of the distance from the boundary. (In quantum gravity, where we sum over all geometries, it would be conceivable but much more difficult to compute any quantity as a function of, say, minimum geodesic distance from the boundary. Computing anything ''at a fixed point'' is even more problematic, and not really well-defined in the absence of additional fields.) Chatterjee and Zamolodchikov [9] show that the asymptotic form of the bulk magnetization depends on the distance from the boundary as $y^{-1/8}$ in the presence of any nonzero boundary field, just as it does for fixed boundary conditions [23]. Our results demonstrate that this RG flow is preserved in an appropriate form after coupling the theory to quantum gravity.

B. The dual picture

As discussed in Sec. II, the Ising model on a random lattice can also be formulated in the dual picture, in terms of matrices X and Y, where X denotes an edge separating two equal spins and Y an edge separating two opposite spins. We have seen in the introduction that the change of variables

$$U\to\frac{1}{\sqrt{2}}(X+Y)$$
$$V\to\frac{1}{\sqrt{2}}(X-Y) \tag{95}$$

in the action for the Ising model leads to the dual action. Thus any calculations in the original model can be reinterpreted in terms of the dual model. We shall now discuss the duals of our results for the boundary and bulk magnetizations in the presence of a boundary magnetic field.

First, let us discuss the boundary conditions corresponding to the weights

$$\begin{aligned}\langle \mathrm{Tr}\, Q^n\rangle &= \langle \mathrm{Tr}(e^h(X+Y)+e^{-h}(X-Y))^n\rangle\\ &= \langle \mathrm{Tr}(\cosh(h)X+\sinh(h)Y)^n\rangle\end{aligned} \tag{96}$$

where we have dropped factors of $\sqrt{2}$ for notational simplicity. In the dual variables, $h=0$ corresponds to fixed boundary conditions, while $h=\pm\infty$ corresponds to the two types of free boundary conditions, $(X\pm Y)^n$. Thus, in this picture h plays the role of a ''boundary freedom field'' rather than a boundary magnetic field. In the limit $h\to 0$ the boundary condition is fixed (all X's), while in the limits $h\to\pm\infty$, the boundary conditions are a pair of free boundary conditions with different signs in the weights assigned to configurations with an odd number of Y's on the boundary. These fixed and free boundary conditions in the dual model are of course precisely the Kramers-Wannier duals of the free and fixed boundary conditions of the spin representation (see Fig. 4). Correspondingly [11], a spin operator in the original variables is transformed into a disorder operator in the dual variables.

Our results in Sec. VI for the boundary magnetization can thus be reinterpreted as a calculation of the expectation value of the boundary disorder operator. We see that as the boundary freedom field is increased, so the expectation value of the boundary disorder operator

$$\langle d\rangle_{n,k}=\frac{\langle \mathrm{Tr}(e^h(X+Y)-e^{-h}(X-Y))(e^h(X+Y)+e^{-h}(X-Y))^k\rangle}{\langle \mathrm{Tr}(e^h(X+Y)+e^{-h}(X-Y))^{k+1}\rangle}=\frac{\langle \mathrm{Tr}(\sinh(h)X+\cosh(h)Y)(\cosh(h)X+\sinh(h)Y)^k\rangle_n}{\langle \mathrm{Tr}(\cosh(h)X+\sinh(h)Y)^{k+1}\rangle_n} \tag{97}$$

tends to 1. At $h=0$ the boundary condition is fixed and so, by symmetry, the boundary disorder must vanish.

The dualization of the calculation of the bulk magnetization goes along similar lines, and we conclude that the bulk disorder in the presence of a boundary freedom field is given by the expression

$$\langle D\rangle = L^{1/3}A^{-1/3} \tag{98}$$

except when $d=0$, when the bulk disorder also vanishes identically. There is a striking consequence of this result. In the Ising model in U, V variables, the coupling of a boundary magnetic field was seen to induce renormalization group flow from free to fixed boundary conditions. On the other hand, when a boundary freedom field is coupled in the dual formulation of the model, the renormalization group flow is from fixed to free boundary conditions. This change in the direction of the renormalization group flow is a natural consequence of the Kramers-Wannier duality of the system. The duality symmetry implies that the ground state degeneracies of the free and fixed states are swapped under the duality transformation, and so the reversal of RG flow is consistent with Affleck and Ludwig's g-theorem [12]. In general, one expects that the RG flow of the theory should be towards the conformal boundary condition with smaller degeneracy, so it is natural that the direction of the flow switches under the duality transformation. As we see, this result seems to hold in the theory equally well after coupling to quantum gravity.

C. The effects of gravity

On any fixed lattice, the introduction of an external magnetic field on the boundary leads to a direct effect on the Ising spins, as there are no other degrees of freedom with which to interact. It is therefore natural to expect that coupling to quantum gravity, which introduces the local geometry as an additional degree of freedom, will lead to quantitative changes in the response of the spins to the boundary field, and indeed this is what we have observed.

The calculation of the boundary magnetization in Sec. VI can be compared with the results that have been obtained in flat space by McCoy and Wu [7]. They find that the magnetization scales as $h \ln h$ for small h, and as a result, the magnetic susceptibility χ diverges at the critical temperature. On the other hand, we have seen in Eq. (64) that the magnetic susceptibility at the critical temperature is finite when the Ising model is defined on a random lattice. It seems likely that the exact numerical value (64) of the magnetic susceptibility χ depends on the discretization scheme we have used, and is not universal. However, we expect the fact that χ is a finite constant to be universal (independent of the specific discretization scheme chosen), although in the absence of calculations in alternative schemes, this remains a conjecture.

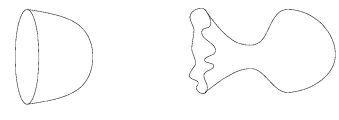

FIG. 5. The relative weight of geometries of different shapes changes with boundary magnetic field h. Roughly speaking, for large h, geometries of type (a) are suppressed relative to type (b).

The effect of gravity is therefore to soften the initial impact of the boundary field. It is natural to interpret this softening as being due to the interaction between the spins and the geometry; the coupling between spins and the boundary field changes the relative weighting of different geometries, which changes in turn the effect of the neighboring spins on any one boundary site, leading to a more gradual increase in the boundary magnetization as a function of boundary field.

A related aspect of our results is that the bulk magnetization is seen to decrease when a nonzero boundary magnetic field is increased (at least asymptotically, for large areas). At first this seems implausible, and indeed at a fixed point in flat space, the bulk magnetization cannot behave in this way. However, the sum over geometries provides a possible explanation for this unusual effect. The expectation value of a spin in the bulk naturally depends not only on the magnitude of the boundary field, but also on the average distance of the point from the boundary. Therefore, the decrease in the bulk magnetization can arise if the boundary field alters the relative weights of different geometries in such a way as to move a typical interior point further away from the boundary (as in Fig. 5).

We have therefore verified that a number of features of the Ising model in flat space are maintained in the presence of quantum gravity, while also demonstrating that the dynamical geometry does have a measurable effect. It would be interesting to check more directly that the explanations we have given for these phenomena are correct, for example by numerical simulation methods such as those described recently in [24].

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