PHYSICAL REVIEW D, VOLUME 58, 045016

Supersymmetric sum rules for electromagnetic multipoles

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(Received 17 March 1998; published 28 July 1998)

We derive model-independent, nonperturbative supersymmetric sum rules for the magnetic and electric multipole moments of any theory with $N=1$ supersymmetry. We find that in any irreducible $N=1$ supermultiplet the diagonal matrix elements of the l-multipole moments are completely fixed in terms of their off-diagonal matrix elements and the diagonal $(l-1)$-multipole moments. [S0556-2821(98)00716-4]

PACS number(s): 12.60.Jv, 11.30.Pb, 11.55.Hx

I. INTRODUCTION

Supersymmetry imposes constraints on the magnetic moments of the particle states [1,2]. These constraints are model independent, valid for any massive $N=1$ and $N=2$ supermultiplet. They are also in agreement with the results of Ferrara and Remiddi [3], who showed that $g=2$ to all orders in perturbation theory for any $N=1$ chiral multiplet, and those of Bilchak, Gastmans, and Van Proeyen [4], who demonstrated that when spin-1 fields are present supersymmetry does not necessarily demand $g=2$, but nevertheless leads to a relation between the g factors of the spin-1/2 and spin-1 particles of the superspin-1/2 multiplet.

The model-independent magnetic dipole moment sum rules were derived in [1] by noting that supersymmetry relates the matrix elements of the conserved electromagnetic current within the various states of a general massive supermultiplet. By selecting the magnetic dipole term in the multipole expansion of the electromagnetic current, the authors of [1] found, for the gyromagnetic ratios, the following sum rule:

$$
\begin{aligned}
g_{j+1/2}&=2+2jh_j,\\
g_j&=2+(2j+1)h_j,\\
g_{j-1/2}&=2+(2j+2)h_j. \qquad (1)
\end{aligned}
$$

Note that j is the superspin labeling the massive supermultiplet which contains states of spins $(j+\frac{1}{2},j,j,j-\frac{1}{2})$. Since both spin-$j$ states have identical gyromagnetic ratios, we see that all the g factors are determined in terms of a single real number h_j, corresponding to an off-diagonal magnetic dipole matrix element between the $j+\frac{1}{2}$ and $j-\frac{1}{2}$ states of the supermultiplet. In the special cases $j=0,1/2$, the sum rules read

$$
\begin{aligned}
&g_{1/2}=2 && (j=0),\\
&g_1=2+h_{1/2},\quad g_{1/2}=2+2h_{1/2} && (j=\tfrac{1}{2}). \qquad (2)
\end{aligned}
$$

Notice that chiral multiplets $(j=0)$ have a fixed gyromagnetic ratio $g=2$.

In this paper, we generalize the above gyromagnetic ratio sum rule to encompass higher multipole moments (both electric and magnetic). This is easily done by working to all orders in the momentum transfer in the appropriate electromagnetic matrix elements. The resulting multipole sum rules have a similar structure as Eqs. (1) and take the form

$$
\begin{aligned}
\mathcal{T}_{j+1/2}^{(l)^{(e,m)}}&=\mp\frac{1}{M}\,\mathcal{T}_j^{(l-1)^{(m,e)}}+\frac{2j+1-l}{l}\,\mathcal{H}_j^{(l)^{(e,m)}},\\
\mathcal{T}_{j}^{(l)^{(e,m)}}&=\mp\frac{1}{M}\,\mathcal{T}_j^{(l-1)^{(m,e)}}+\frac{2j+1}{l}\,\mathcal{H}_j^{(l)^{(e,m)}},\\
\mathcal{T}_{j-1/2}^{(l)^{(e,m)}}&=\mp\frac{1}{M}\,\mathcal{T}_j^{(l-1)^{(m,e)}}+\frac{2j+1+l}{l}\,\mathcal{H}_j^{(l)^{(e,m)}}, \qquad (3)
\end{aligned}
$$

where the electric and magnetic l-pole generalization of g is denoted by $\mathcal{T}_j^{(l)^{(e,m)}}$ and is defined in Eq. (21) below.[1] These sum rules indicate the general structure imposed by supersymmetry that the electric (magnetic) l-pole moments are completely determined solely in terms of a single magnetic (electric) $(l-1)$-pole moment and the real quantity $\mathcal{H}^{(l)}$ parametrizing an off-diagonal transition between the spin $j\pm\frac{1}{2}$ states of the multiplet. Note that the upper and lower signs in Eqs. (3) and subsequent equations correspond to the first and second entries in, e.g., (e,m). This difference in sign between the electric and magnetic sum rules may be

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[1] More precisely these sum rules hold for the generic case $2j\geqslant l+1$, where j denotes the superspin. Note that $\mathcal{T}_j^{(l)^{(e,m)}}$ is meaningless whenever $l>2j$, as may be inferred from Eq. (21).

understood intuitively from electromagnetic duality which exchanges electric and magnetic fields, $\vec{E}\to\vec{B}$ and $\vec{B}\to-\vec{E}$.

When $l=1$, the magnetic part of the sum rules (3) reduces to the result of Ferrara and Porrati, Eq. (1), since g is defined as a ratio: $\mathcal{T}_j^{(1)^{(m)}}=g_je/2M$ and $\mathcal{T}_j^{(0)^{(e)}}=e$. Furthermore, just as for the magnetic dipole moments, we note that setting $\mathcal{H}_j^{(l)^{(e,m)}}=0$ yields the ''preferred'' value for the l-pole moments,

$$\mathcal{T}_{j+1/2}^{(l)^{(e,m)}}=\mathcal{T}_j^{(l)^{(e,m)}}=\mathcal{T}_{j-1/2}^{(l)^{(e,m)}}=\mp\frac{1}{M}\,\mathcal{T}_j^{(l-1)^{(m,e)}}, \qquad (4)$$

generalizing the notion of $g=2$ as the preferred value of the gyromagnetic ratio.

II. DERIVATION OF THE SUM RULES

Derivation of the sum rules (3) follows the method of [1] and involves the transformation properties of a conserved current J_μ that commutes with the $N=1$ supersymmetry algebra. The main complication in obtaining the present results is the requirement of working to all orders in the multipole expansion (and, as a result, having to keep track of higher-order-in-momentum transfer terms in the matrix elements).

Recall that the $N=1$ algebra has the form $\{Q_\alpha,\overline{Q}_\beta\}=2(\gamma^\mu)_{\alpha\beta}P_\mu$, where $\overline{Q}=Q^TC$ is the Majorana conjugate and C is the charge conjugation matrix obeying $C\gamma^\mu C^{-1}=-\gamma^{\mu T}$ and $C^2=-1$. For a massive single-particle state, we may work in the rest frame $P^\mu=(M,0,0,0)$. Defining chiralities

$$\gamma_5 Q^{\substack{L\\R}}=\pm Q^{\substack{L\\R}} \qquad (5)$$

and helicities

$$\gamma^{12}Q_{\pm1/2}=\mp iQ_{\pm1/2}, \qquad (6)$$

the supersymmetry algebra can be recast as follows:

$$\{Q_{1/2}^L,Q_{-1/2}^R\}=2M, \quad \{Q_{-1/2}^L,Q_{1/2}^R\}=2M, \qquad (7)$$

while the remaining anticommutators vanish.[2] We may rescale the supercharges according to $q_{\pm1/2}^{L,R}=1/\sqrt{2M}Q_{\pm1/2}^{L,R}$ to recover the Clifford algebra for two fermionic degrees of freedom. One can then construct its irreducible representations by starting with a superspin-j Clifford vacuum $|j\rangle$, annihilated by $q_{\pm1/2}^L$ and acting on it with the creation operators $q_{\pm1/2}^R$. As a result, we see that the representation has dimension $(2j+1)\times2^2$, where $2j+1$ is the degeneracy of the original spin-j state. The spins of the states are given by the addition of angular momenta, $j\times[(1/2)+2(0)]$, giving states of spins $j-\frac{1}{2}$, j, and $j+\frac{1}{2}$ with degeneracies 1, 2, and 1.

Since the supercharges $Q_{\pm1/2}^{L,R}$ are operators of spin 1/2, this leads to a shorthand notation for labeling the states of a massive $N=1$ multiplet in the following manner: the spin-j Clifford vacuum is denoted by $|0\rangle$: acting on this state with the normalized supercharge $q_{1/2}^R$ or $q_{-1/2}^R$ then results in the spin ''up'' or ''down'' states $|\uparrow\rangle$ or $|\downarrow\rangle$, respectively. The action of two q's on the Clifford vacuum is denoted by $|\updownarrow\rangle$.

For $N=1$ supersymmetry, any conserved current commuting with the supersymmetry generators must belong to a real linear multiplet. The components of a real linear multiplet multiplet are $(C(x),\zeta(x),J_\mu(x))$, where $C(x)$ is a real scalar and $\zeta(x)$ a Majorana spinor. As a result of current conservation, $\partial^\mu J_\mu=0$, the multiplet consists of four fermionic and four bosonic degrees of freedom. The transformation properties of the components under a supersymmetry variation are given by

$$\delta C=i\bar{\epsilon}\gamma_5\zeta, \quad \delta\zeta=i(\gamma^\lambda J_\lambda+i\gamma_5\gamma^\lambda\partial_\lambda C)\epsilon,$$
$$\delta J_\mu=-\bar{\epsilon}\gamma_\mu{}^\lambda\partial_\lambda\zeta. \qquad (8)$$

It follows that two successive supersymmetry transformations on the conserved current J_μ gives

$$\delta_\eta\delta_\epsilon J_\mu=i\bar{\epsilon}\gamma_\mu{}^\nu\gamma^\rho(\partial_\nu J_\rho-i\gamma_5\partial_\nu\partial_\rho C)\eta. \qquad (9)$$

The matrix elements of this equation between single-particle states which belong to the same $N=1$ multiplet give rise to sum rules for the electromagnetic multipoles of the particle states.

To obtain the connection between the matrix elements of J_μ and the terms in the multipole expansion, we first recall the standard definitions (see, e.g., [5]) for the electric l-pole moments,

$$Q_{i_1i_2\cdots i_l}^{(l)}=\int d^3x(x_{i_1}x_{i_2}\cdots x_{i_1})J_0(x)-\text{trace}, \qquad (10)$$

and the magnetic l-pole moments,

$$M_{i_1i_2\cdots i_l}^{(l)}=-\frac{1}{l+1}\int d^3x(x_{i_1}x_{i_2}\cdots x_{i_l})\vec{\nabla}\cdot[\vec{x}\times\vec{J}(x)]-\text{trace}. \qquad (11)$$

While ordinarily defined in terms of spherical tensors (see, e.g., [6]), the above multipole moments, expressed as Cartesian tensors, are more naturally related to the expansions for the matrix elements of J_μ,

[2]To fix our phase conventions, we work in the Dirac representation for the γ matrices and take $\gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3$ and $C=i\gamma^0\gamma^2$. The spinors then decompose as $\sqrt{2}Q_\alpha^T=Q_{1/2}^L[1\,0\,1\,0]+Q_{-1/2}^L[0\,1\,0\,1]+Q_{1/2}^R[-1\,0\,1\,0]+Q_{-1/2}^R[0\,1\,0\,-1]$.

$$\langle j',m',\vec{p}|J_0|j,m,0\rangle=\sum_{l=0}^{\infty}\frac{1}{l!}(ip)_{i_1}(ip)_{i_2}\cdots(ip)_{i_l}\langle j',m',0,|T^{(l)^e}_{i_1i_2\cdots i_l}|j,m,0\rangle,$$

$$\langle j',m',\vec{p}|J_i|j,m,0\rangle=p_i\langle j',m',0,|\Lambda|j,m,0\rangle-i\epsilon_{ijk}p_j\sum_{l=1}^{\infty}\frac{1}{l!}(ip)_{i_2}(ip)_{i_3}\cdots(ip)_{i_l}\langle j',m',0,|T^{(l)^m}_{ki_2\cdots i_l}|j,m,0\rangle. \tag{12}$$

In particular, the *traceless* components of $T^{(l)^{(e)}}$ and $T^{(l)^{(m)}}$ correspond exactly to $Q^{(l)}$ and $M^{(l)}$, respectively. Note that the matrix elements of Λ are completely determined by current conservation, $\Lambda=-[(E-M)/p^2]J_0$.

The multipole moment sum rules are derived by taking the double-supersymmetry variation of the conserved current J_μ,

$$\delta_\eta\delta_\epsilon J_\mu=[\bar{\eta}Q,[\bar{\epsilon}Q,J_\mu]]$$
$$=\bar{\eta}Q\bar{\epsilon}QJ_\mu-\bar{\eta}QJ_\mu\bar{\epsilon}Q-\bar{\epsilon}QJ_\mu\bar{\eta}Q+J_\mu\bar{\epsilon}Q\bar{\eta}Q, \tag{13}$$

and evaluating it between single-particle states $\langle\alpha|$ and $|\beta\rangle$. Since the supercharge Q generates superpartners ($Q|\alpha\rangle\sim|\tilde{\alpha}\rangle$), this expression relates matrix elements of J_μ between different states of a supermultiplet in terms of $\delta_\eta\delta_\epsilon J_\mu$, which is given by Eq. (9). The electromagnetic l-pole sum rules then follow by using Eq. (12) to expand the matrix elements in terms of multipoles and then by collecting terms of order p^l. We note that an important simplification occurs since we are only interested in sum rules on the static multipole moments. This means in practice that all terms depending explicitly on the contracted momentum p^2 may be ignored, as they do not contribute to the static l-pole moments (and instead correspond to the trace terms in $T^{(l)^{(e,m)}}$).[3]

The general double-supersymmetry variation procedure is simplified in practice by choosing the global supersymmetry transformation parameters η and ϵ in such a way that several terms on the right-hand side of Eq. (13) act as annihilation operators on the initial or final states and hence may be dropped. In particular, by choosing $\eta_L=0$, we find

$$\langle\alpha,\vec{p}|\delta_{\eta_R}\delta_\epsilon J_\mu|\beta,0\rangle=\langle\alpha,\vec{p}|J_\mu\bar{\epsilon}Q\bar{\eta}_RQ|\beta,0\rangle$$
$$-\langle\alpha,0|\bar{\epsilon}Q^{(p)}L^{-1}(\vec{p})J_\mu\bar{\eta}_RQ|\beta,0\rangle, \tag{14}$$

where $Q^{(p)}$ denotes the Lorentz boost of Q, namely, $Q^{(p)}=L^{-1}(\vec{p})QL(\vec{p})$, and $|\alpha,\vec{p}\rangle=L(\vec{p})|\alpha,0\rangle$.

By further choosing $\epsilon_L=0$ and noting from Eq. (9) that $\delta_{\eta_R}\delta_{\epsilon_R}J_\mu=0$, we easily obtain the ''vanishing'' sum rule

$$\langle\alpha,\vec{p}|J_\mu\bar{\epsilon}_RQ\bar{\eta}_RQ|\beta,0\rangle=0. \tag{15}$$

This demonstrates that *all* matrix elements of the electromagnetic current vanish between states $|0\rangle$ and $|\updownarrow\rangle$ and, hence, that there are no off-diagonal moments between the two spin-j states of the supermultiplet.

If instead we choose $\epsilon_R=0$ and make use of the fact that Q transforms as a spinor,

$$Q^{(p)}=e^{1/2\omega^i\gamma^{0i}/2}Q=\sqrt{\frac{E+M}{2M}}\left(I+\frac{p^i}{E+M}\gamma^{0i}\right)Q, \tag{16}$$

we obtain from Eq. (14) the expression

$$\langle\alpha,\vec{p}|i\bar{\epsilon}_L\gamma_\mu{}^\nu\gamma^\lambda\partial_\nu(J_\lambda-i\gamma_5\partial_\lambda C)\eta_R|\beta,0\rangle$$
$$=2M(\bar{\epsilon}_L\gamma^0\eta_R)\langle\alpha,\vec{p}|J_\mu|\beta,0\rangle$$
$$-\sqrt{\frac{E+M}{2M}}\langle\alpha,0|\bar{\epsilon}_LQL^{-1}(\vec{p})J_\mu\bar{\eta}_RQ|\beta,0\rangle$$
$$-\frac{p^i}{\sqrt{2M(E+M)}}\langle\alpha,0|\bar{\epsilon}_L\gamma^{0i}QL^{-1}(\vec{p})J_\mu\bar{\eta}_RQ|\beta,0\rangle. \tag{17}$$

Equation (17) can be simplified significantly if we ignore p^2 (i.e., trace) terms which do not contribute to the electromagnetic multipole sum rules. After some manipulation, the time and space components of Eq. (17) can be written as follows:

$$\frac{1}{2M}\langle\alpha,0|\bar{\epsilon}_LQL^{-1}(\vec{p})J_0\bar{\eta}_RQ|\beta,0\rangle$$
$$=(\bar{\epsilon}_L\gamma^0\eta_R)\langle\alpha,\vec{p}|J_0|\beta,0\rangle$$
$$-i\epsilon_{ijk}\frac{p^j}{2M}(\bar{\epsilon}_L\gamma^k\eta_R)\langle\alpha,\vec{p}|J_i|\beta,0\rangle,$$

$$\frac{1}{2M}\langle\alpha,0|\bar{\epsilon}_LQL^{-1}(\vec{p})J_i\bar{\eta}_RQ|\beta,0\rangle$$
$$=(\bar{\epsilon}_L\gamma^0\eta_R)\langle\alpha,\vec{p}|J_i|\beta,0\rangle$$
$$-i\epsilon_{ijk}\frac{p^j}{2M}(\bar{\epsilon}_L\gamma^k\eta_R)\langle\alpha,\vec{p}|J_0|\beta,0\rangle, \tag{18}$$

where we have omitted terms explicitly proportional to p^2. Note in particular that matrix elements of C do not enter.

[3]In principle, supersymmetry would give complete relations between electromagnetic form factors $T^{(l)^{(e,m)}}(p^2)$ of superpartners. However, in this case it appears the moments of the ''auxiliary field'' C enter in a nontrivial manner.

We now use the explicit multipole expansion of the matrix elements, Eqs. (12), and equate terms of the same order in $\vec{p}$. Because of the explicit factor of p^j in Eqs. (18), we see that multipole terms of order l and $l-1$ are explicitly related; heuristically, Eq. (18) states that the l-pole moment of a superpartner is given by the same l-pole moment of the original state plus a correction based on the opposite (electric/magnetic) $(l-1)$-pole. Explicitly, at order $p_{i_1}p_{i_2}\cdots p_{i_l}$, we find

$$\begin{aligned}
&\frac{1}{2M}\langle\alpha,0|\overline{\epsilon_L}QT^{(l)^{(e,m)}}_{i_1i_2\cdots i_l}\overline{\eta_R}Q|\beta,0\rangle\\
&\quad=(\overline{\epsilon_L}\gamma^0\eta_R)\langle\alpha,0|T^{(l)^{(e,m)}}_{i_1i_2\cdots i_l}|\beta,0\rangle\\
&\quad\mp\frac{l}{2M}(\overline{\epsilon_L}\gamma^{i_1}\eta_R)\langle\alpha,0|T^{(l-1)^{(m,e)}}_{i_2i_3\cdots i_l}|\beta,0\rangle\\
&\quad\pm\frac{l}{2M}\frac{l-1}{2l-1}\delta_{i_1i_2}(\overline{\epsilon_L}\gamma^j\eta_R)\langle\alpha,0|T^{(l-1)^{(m,e)}}_{ji_3\cdots i_l}|\beta,0\rangle,
\end{aligned}\tag{19}$$

where the indices $i_1,i_2,\ldots,i_l$ are to be explicitly symmetrized, and all tensor quantities are assumed traceless. Note that the last term in Eq. (19) is responsible for subtracting out the trace from the spin-1×spin-$(l-1)$ combination $(\overline{\epsilon_L}\vec{\gamma}\eta_R)\times T^{(l-1)^{(m,e)}}$.

Because of rotational invariance, each l-pole moment may be completely characterized by a single quantity—essentially a reduced matrix element according to the Wigner-Eckart theorem. In particular, for a single-particle state of spin j and z-component m, we define the reduced l-pole moment $\mathcal{T}^{(l)^{(e,m)}}_j$ by

$$\begin{aligned}
&\langle j,m'|T^{(l)^{(e,m)}}_{i_1i_2\cdots i_l}|j,m\rangle\\
&\quad=\mathcal{T}^{(l)^{(e,m)}}_j\langle j,m'|(J_{(i_1}J_{i_2}\cdots J_{i_l)}-\text{trace})|j,m\rangle.
\end{aligned}\tag{20}$$

The sum rules may now be established by examining the $i_1,i_2,\ldots i_l=3,3,\ldots,3$ components of Eq. (19). Furthermore, the spin-j angular momenta manipulations are simplified by picking the particular $m=j$ state in the matrix elements, in which case Eq. (20) may be reexpressed as

$$\langle j,j|T^{(l)^{(e,m)}}_{33\cdots3}|j,j\rangle=\frac{(2j)(2j-1)\cdots[2j-(l-1)]}{\binom{2l}{l}}\mathcal{T}^{(l)^{(e,m)}}_j.\tag{21}$$

With the same motivation we define the multipole transition moments $\mathcal{H}^{(l)^{(e,m)}}_j$ as

$$\begin{aligned}
&\langle j-\tfrac12,j-\tfrac12|T^{(l)^{(e,m)}}_{33\cdots3}|j+\tfrac12,j-\tfrac12\rangle\\
&\quad=\frac{1}{\sqrt{2j}}\frac{[(2j)(2j-1)\cdots(2j-(l-1))]}{\binom{2l}{l-1}}\mathcal{H}^{(l)^{(e,m)}}_j.
\end{aligned}\tag{22}$$

Recall that in Eq. (19) both $\langle\alpha,0|$ and $|\beta,0\rangle$ denote the spin-j Clifford vacuum state $|j,m,0\rangle$, which may be abbreviated as $|0\rangle$. By choosing the spinor parameters η_R and ϵ_L appropriately, we then relate the electromagnetic multipoles of the different members of the $N=1$ massive multiplet. With a total of two η_R and two ϵ_L parameters, we find

$$\begin{aligned}
\langle\uparrow|T^{(l)^{(e,m)}}_{33\cdots3}|\uparrow\rangle&=\langle0|T^{(l)^{(e,m)}}_{33\cdots3}|0\rangle\mp\frac{l}{2M}\frac{l}{2l-1}\langle0|T^{(l-1)^{(m,e)}}_{33\cdots3}|0\rangle,\\
\langle\downarrow|T^{(l)^{(e,m)}}_{33\cdots3}|\downarrow\rangle&=\langle0|T^{(l)^{(e,m)}}_{33\cdots3}|0\rangle\pm\frac{l}{2M}\frac{l}{2l-1}\langle0|T^{(l-1)^{(m,e)}}_{33\cdots3}|0\rangle,\\
\langle\uparrow|T^{(l)^{(e,m)}}_{33\cdots3}|\downarrow\rangle&=\mp\frac{l}{2M}\frac{l-1}{2l-1}\langle0|T^{(l-1)^{(m,e)}}_{-3\cdots3}|0\rangle,\\
\langle\downarrow|T^{(l)^{(e,m)}}_{33\cdots3}|\uparrow\rangle&=\mp\frac{l}{2M}\frac{l-1}{2l-1}\langle0|T^{(l-1)^{(m,e)}}_{+3\cdots3}|0\rangle,
\end{aligned}\tag{23}$$

where ± in the indices denote the combinations $x^1\pm ix^2$, and the states $|\uparrow\rangle$ and $|\downarrow\rangle$ are implicitly understood in terms of the Clebsch-Gordan combination of spin-1/2×spin-j. This is the main result of our paper. The matrix elements of the l-electric (magnetic) multipole moment between different members of the supermultiplet are given in terms of the matrix elements of the l-electric (magnetic) multipole moment and the $(l-1)$-magnetic (electric) multipole moment between the Clifford vacuum.

Finally by, carrying out the addition of the superspin j to the supersymmetry generated spin, we find the following sum rules:

$$\begin{aligned}
&\mathcal{T}^{(l)^{(e,m)}}_{j+1/2}=\mathcal{T}^{(l)^{(e,m)}}_j-\mathcal{H}^{(l)^{(e,m)}}_j,\quad \mathcal{T}^{(l)^{(e,m)}}_{j-1/2}=\mathcal{T}^{(l)^{(e,m)}}_j+\mathcal{H}^{(l)^{(e,m)}}_j,\\
&\mathcal{H}^{(l)^{(e,m)}}_j=\frac{l}{2j}\left[\mathcal{T}^{(l)^{(e,m)}}_j\pm\frac{1}{M}\mathcal{T}^{(l-1)^{(m,e)}}_j\right],
\end{aligned}\tag{24}$$

which may be written in a completely equivalent form as presented in Eq. (3). Note that both spin-j states carry identical l-pole moments, as may be established using the same argument as in [1].

III. DISCUSSION

While the sum rules were derived for generic superspin j, it is important to realize that angular momentum selection rules forbid both diagonal $(\mathcal{T}^{(l)}_j)$ and nondiagonal $(\mathcal{H}^{(l)}_j)$ l-pole electromagnetic moments whenever $l>2j$. For $l=1$ (dipole moment), the magnetic sum rules reduces to that of Ref. [1], while the electric sum rule gives rise to the relation between electric dipole moments (EDM's):

$$d_{j+1/2}=d_j-\frac{d_j}{2j+1},\quad d_{j-1/2}=d_j+\frac{d_j}{2j+1},\tag{25}$$

where $\langle j,m|d_3^e|j,m\rangle = d_j m$.

The special cases $j=0$ and $j=1/2$ are noteworthy. For $j=0$ only dipole moments are allowed (for the spin-1/2 particle), in which case the gyromagnetic ratio of the spin-1/2 particle in the supermultiplet is $g=2$, as shown by Ferrara and Remiddi [3]. For $j=1/2$ (massive vector multiplet), only dipole and quadrupole moments are allowed. Robinett [7] and Bilchak, Gastmans, and Van Proeyen [4] showed that the electric quadrupole of the spin-1 particle is completely determined in terms of its anomalous magnetic dipole moment. Our sum rule reproduces this result. Indeed, by setting $j=1/2$ and $l=2$ in Eq. (3) we find the following relation between electric quadrupole and magnetic dipole:

$$\mathcal{T}_1^{(2)^{(e)}} = -\frac{1}{M}\,\mathcal{T}_{1/2}^{(1)^{(m)}}. \tag{26}$$

Since the conventional quantum definition of the electric quadrupole moment is given by $Q_j = \langle j,j|\int d^3x(3z^2 - r^2)J_0(x)|j,j\rangle$ and is related to $\mathcal{T}^{(2)^{(e)}}$ by $Q_j = j(2j-1)\mathcal{T}_j^{(2)^{(e)}}$, the above relation may in fact be rewritten as (cf. [4])

$$Q_1 = -(g_1 - 1)\,\frac{e}{M^2} \tag{27}$$

[where the g-factor sum rule (1) was also used]. This result can be understood in the following way. The action of a massive, charged vector multiplet W coupled to a real, massless vector multiplet V can be written in superfields as [1]

$$S = \left(\int d^2\theta\; W_\alpha^+ W^{\alpha-} + a\int d^4\theta\; W D_\alpha W^\dagger e^{-V} V^\alpha + \text{c.c.}\right) + M^2\int d^4\theta\; e^{-V} W^\dagger W. \tag{28}$$

Here M is the mass of W and a is an arbitrary constant; $W_\alpha^\pm$ and V_α are defined as in [1]. The term proportional to a is the only superfield expression that contributes to the magnetic dipole. Expanding in components, indeed, one finds a term proportional to

$$\int d^4x\; W^{\mu *} W^\nu F_{\mu\nu}. \tag{29}$$

The magnetic-dipole contribution comes by setting $\mu,\nu = i,j$ $(i,j=1,2,3)$. On the other hand, by setting $\mu=0$, $\nu=i$ $(i=1,2,3)$, one finds a contribution to the electric quadrupole, since on shell and at low momenta $\partial_\mu W^\mu = 0 \Rightarrow M W^0 \approx i\partial_i W^i$:

$$\int d^4x\; W^{0*} W^i F_{0i} = \frac{i}{M}\int d^4x\; W^{j*} W^i \partial_j F_{0i} + \cdots \quad (\text{on shell}). \tag{30}$$

No other quadrupole term can be written in superfields; therefore, the electric quadrupole is completely determined by the magnetic dipole, as explicitly found in [4] and implied by our sum rules.

ACKNOWLEDGMENTS

We would like to thank V. P. Nair and H. C. Ren for useful discussions. This work was supported in part by the Department of Energy under Contract No. DE-FG02-91ER40651-TASK B and by the NSF under Grant No. PHY-9722083.

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