Duality and superconvergence relation in supersymmetric gauge theories

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We investigate the phase structures of various $N=1$ supersymmetric gauge theories including even the exceptional gauge group from the viewpoint of superconvergence of the gauge field propagator. Especially we analyze in detail whether or not a new type of duality recently discovered by Oehme in SU(N_c) gauge theory coupled to fundamental matter fields can be found in more general gauge theories with more general matter representations. The result is that in the cases of theories including matter fields in *only* the fundamental representation, Oehme's duality holds but otherwise it does not. In the former case, a superconvergence relation might give a good criterion to describe the interacting non-Abelian Coulomb phase without using information from dual magnetic theory. The problem of the gauge dependence of the results is also disscussed in the last section. [S0556-2821(98)01212-0]

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I. INTRODUCTION

Quark confinement is one of the most mysterious properties in quantum field theory. In spite of its obvious existence in experiment, the mechanism of confinement has not been elucidated yet. Of course there have been many challenges to understanding this phenomena. The lattice formulation of QCD, originally proposed by Wilson [1], is one of them. On the other hand, chiral symmetry breaking is another problem which must be solved. In nonsupersymmetric gauge theory, it is considered that these two phenomena, confinement and chiral symmetry breaking, are deeply connected and occur simultaneously in a certain QCD parameter region (strong gauge coupling or small number of quark flavors, etc.).

Our main interest here is how the phase structure of QCD changes as the number of quark flavors increases. Naively we expect the following picture: when the number is small, the theory is in a confinement phase due to its asymptotic freedom property. As the number of flavors is increasing, quarks are deconfined and chiral symmetry is restored before the theory becomes asymptotically nonfree.

In the intermediate region corresponding to no confinement but asymptotic freedom, we can realize the scale invariant theory because it is expected there exists a nontrivial IR fixed point in this region. The critical value of N_f in SU(3) QCD where quarks are deconfined and chiral symmetry is restored has been evaluated by many authors. Banks and Zaks first pointed out the existence of such a fixed point [2]. They evaluated the first two coefficients of the perturbatively expanded β function which are both gauge and renormalization scheme independent [3] . They showed $N_f^{\rm crit}=8.05$. Also $N_f^{\rm crit}=7$ has been obtained in lattice QCD calculation [4]. On the other hand, Oehme and Zimmermann expected $N_f^{\rm crit}=10$ using what they called a superconvergence relation [5]. In this relation, the anomalous dimension of the gauge field as well as the β function plays an important role. In this way, all the values of $N_f^{\rm crit}$ in different approaches do not coincide with each other.

Recently Oehme applied his superconvergence argument to $N=1$ supersymmetric SU(N_c) gauge theory and compared it with already known results [6]. The phase structure of this theory had already been investigated in detail with the help of the so-called ''electric-magnetic'' duality and holomorphy by Seiberg *et al.* [7]. Seiberg has insisted on the existence of an interval corresponding to the interacting non-Abelian Coulomb phase or conformal window where the theory becomes scale invariant. Consequently, Oehme showed quantitative agreement between his argument and the results from Seiberg's duality. Moreover, he found the important relationship between the original electric SUSY theory and the dual magnetic one, which might be interpreted as a new type of duality. Since superconvergence arguments can apply for *both* SUSY and non-SUSY theories, the comparison with exact results by Seiberg *et al.* is very significant.

In this paper, we apply his method to various supersymmetric gauge theories with other gauge groups and other matter contents and check whether the relation he found holds or not in those cases. In Sec. II, we review the concept of superconvergence of the gauge field propagator in the nonsupersymmetric case. In Sec. III, we extend the method in the previous section to the supersymmetric cases and check the Oehme's duality. Section IV is devoted to summary and discussions, where the problem of the gauge dependence of our results is also disscussed. In the Appendixes, some basic equations are followed.

II. SUPERCONVERGENCE RELATION

Here we shall consider the asymptotic behavior of the gluon propagator at large momentum with the help of the renormalization group (RG) analysis.

First of all, following Oehme and Zimmermann [5], we introduce the operator

$$A^a_{\mu\nu}\equiv\partial_\mu A^a_\nu-\partial_\nu A^a_\mu, \qquad (1)$$

where $A^a_\mu(\mu=1,\ldots,4$, and $a=1,\ldots,N_c^2-1)$ is the SU(N_c) gauge field whose two point function is generally given by

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$$\langle 0|TA^a_\mu(x)A^b_\nu(y)|0\rangle = \int d^4k e^{-ip\cdot(x-y)}\frac{\delta^{ab}}{i}\left[\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)D(k^2)+\alpha\frac{k_\mu k_\nu}{k^4}\right]. \tag{2}$$

α is the gauge parameter.

Using Eqs. (1) and (2), we obtain

$$\langle 0|TA^a_{\mu\nu}(x)A^b_{\rho\sigma}(y)|0\rangle = \int d^4k e^{-ip\cdot(x-y)}\frac{\delta^{ab}}{i}(k_\mu k_\rho\delta_{\nu\sigma} - k_\mu k_\sigma\delta_{\nu\rho} - k_\nu k_\rho\delta_{\mu\sigma} + k_\nu k_\sigma\delta_{\mu\rho})D(k^2). \tag{3}$$

Scalar function $D(k^2)$ is called the ''transverse gluon structure function.'' Below we shall restrict our consideration to the asymptotic behavior of this function $D(k^2)$ at large momentum.

For that purpose, first we give the normalization condition for $D(k^2)$ as follows:

$$k^2D(k^2)=1 \quad \text{at} \quad k^2=\mu^2, \tag{4}$$

where μ is the normalization point. Then we can write the structure function in the form

$$D=D(k^2,g,\mu^2). \tag{5}$$

g is the gauge coupling constant.

Now, for convenience, let us introduce the dimensionless function R defined by

$$R\equiv k^2D(k^2,g,\mu^2)=R\left(\frac{k^2}{\mu^2},g\right). \tag{6}$$

Then the Callan-Symanzik equation (RG equation) for the R function in the Landau gauge $(\alpha=0)$ is given as

$$u\frac{\partial R(u,g)}{\partial u}=\beta(g^2)\frac{\partial R(u,g)}{\partial g^2}+\gamma(g^2)R(u,g), \tag{7}$$

where $u\equiv k^2/\mu^2$ and $\beta(g^2)$ and $\gamma(g^2)$ are the β function and the anomalous dimension of the gluon field, respectively. In the region of the small gauge coupling constant (i.e., at large momentum), they are of the form

$$\beta(g^2)=g^4(\beta_0+\beta_1g^2+\cdots),$$
$$\gamma(g^2)=g^2(\gamma_0+\gamma_1g^2\cdots), \tag{8}$$

where

$$\beta_0=-\frac{1}{16\pi^2}\left(\frac{11}{3}N_c-\frac{2}{3}N_f\right),$$

$$\gamma_0=-\frac{1}{16\pi^2}\left(\frac{11}{6}N_c-\frac{4}{3}N_f\right). \tag{9}$$

N_f denotes the number of quark flavors.

Oehme and Zimmermann solved Eq. (7) in the following way (see Appendix A):

$$R\left(\frac{k^2}{\mu^2},g\right)=R(1,Q)\ \exp\left[\int_{g^2}^{Q^2}dx\,\gamma(x)\beta^{-1}(x)\right]. \tag{10}$$

Here the effective RG-invariant coupling constant $Q(k^2/\mu^2,g)$ is defined through the equation

$$u\frac{\partial Q^2(u,g)}{\partial u}=\beta(g^2)\frac{\partial Q^2(u,g)}{\partial g^2}. \tag{11}$$

For large momentum, we can show that

$$Q^2\left(\frac{k^2}{\mu^2},g\right)\approx-\frac{1}{\beta_0\ln(k^2/\mu^2)}. \tag{12}$$

β_0 is the first coefficient of the β function defined in Eq. (9).

Now we would like to obtain the asymptotic behavior of the R function given by Eq. (10). By substituting Eqs. (8) and (9) into Eq. (10), we get the following result (see Appendix B):

$$R\left(\frac{k^2}{\mu^2},g\right)=\left(\frac{Q^2}{g^2}\right)^{\gamma_0/\beta_0}\exp\left[\int_{g^2}^{Q^2}dx\,\tau(x)\right],$$
$$\approx C_V\left(\ln\frac{k^2}{\mu^2}\right)^{-\gamma_0/\beta_0}, \tag{13}$$

where

$$C_V=(g^2|\beta_0|)^{-\gamma_0/\beta_0}\exp\left[\int_{g^2}^{Q^2}dx\,\tau(x)\right]. \tag{14}$$

Here the function $\tau(x)$ is the regular part of $\gamma(x)/\beta(x)$ at $x=0$.

Thus leading asymptotic behavior of the R function at large momentum was determined. Similarly, for the D function, we find

$$D_{\text{asymp}}(k^2)\approx C_Vk^{-2}\left(\ln\frac{k^2}{\mu^2}\right)^{-\gamma_0/\beta_0}. \tag{15}$$

We can conclude here that the asymptotic behavior of the transverse gluon propagator drastically changes due to the sign of γ_0 (or, equivalently, the sign of γ_0/β_0). If $\gamma_0<0$, it converges. If $\gamma_0>0$, it diverges, where we have assumed the asymptotic freedom of the theory which means $\beta_0<0$.

Let us consider such a case where β_0 and γ_0 are both negative, i.e., the ratio γ_0/β_0 is positive. For the scalar part of the transverse gluon propagator $D(k^2)$, we can apply the Lehmann representation:

$$D(k^2)=\int_0^\infty dm^2\frac{\rho(m^2)}{m^2-k^2}, \tag{16}$$

where $\rho(k^2)$ is called the spectral function and is given as the absorptive part of the D function:

$$\pi\rho(k^2)=\mathrm{Im}D(k^2)=k^{-2}\mathrm{Im}R(k^2). \tag{17}$$

Therefore, for the limit $k^2\to-\infty$, we find

$$\rho_{\mathrm{asymp}}\approx-\frac{\gamma_0}{\beta_0}C_V k^{-2}\left(\ln\frac{k^2}{\mu^2}\right)^{-\gamma_0/\beta_0-1}. \tag{18}$$

Combining Eqs. (15), (16) and (18) with a kind of sum rule called the superconvergence relation, we can obtain (see Appendix C)

$$\int_0^\infty dm^2\rho(m^2)=0 \quad \text{for } \frac{\gamma_0}{\beta_0}>0. \tag{19}$$

We can show that the superconvergence relation obtained here gives some circumstantial evidence for color confinement. In Ref. [8], it was said that the superconvergence relation is connected with various interpretations of color confinement, such as metric cancellation, the bag model picture, and the area law behavior of the Wilson loop in lattice QCD.

On the contrary, in the region where the superconvergence relation does not hold (i.e., $\beta_0<0$ and $\gamma_0>0$), quarks are deconfined and chiral symmetry restored. Moreover, it is expected that there exists a nontrivial IR fixed point in this region, preventing the gauge coupling from becoming strong enough to confine quarks or cause them to condense. In the following section, we will restrict our considerations to this interval in supersymmetric gauge theories.

III. DUALITY AND SUPERCONVERGENCE RELATION IN VARIOUS $N=1$ SUSY GAUGE THEORIES

In the previous section, we discussed the superconvergence relation (a kind of sum rule for the spectral function of the transverse gluon propagator) and commented on its relation with color confinement. There, the essential point was that there exists a region in which we have asymptotic freedom of the theory but no confinement. In such a region, the theory may have a nontrivial infrared fixed point at a nonvanishing value of the gauge coupling.

On the other hand, Oehme recently applied the method of the superconvergence relation to $N=1$ supersymmetric gauge theory [SU(N_c) gauge theory with a fundamental chiral supermultiplet of N_f flavors] [6] and compared the result with the one Seiberg *et al.* already obtained using holomorphy and ''electric-magnetic'' duality [7]. Oehme insisted in Ref. [6] that his result is in quantitative agreement with Seiberg's duality argument. Moreover, he showed that there was an interesting relationship between the coefficients β_0 and γ_0 (defined in the previous section) of the original (electric) theory and those of the dual (magnetic) theory. This may be considered as a new type of duality.

Thus in this section, we first review the analysis by Oehme in Ref. [6] and then try to apply his method to other models of $N=1$ SUSY gauge theories, which have different gauge groups and different matter representations, to check Oehme's duality. That is the main purpose of this paper.

A. Oehme's analysis in $N=1$ SU(N_c) gauge theory with massless matter fields in the fundamental representation of N_f flavors

In this case, the one loop coefficients β_0 and γ_0 are given by

$$\beta_0=-\frac{1}{16\pi^2}(3N_c-N_f),$$

$$\gamma_0=-\frac{1}{16\pi^2}\left(\frac{3}{2}N_c-N_f\right). \tag{20}$$

If we demand $\beta_0<0$ (asymptotic freedom) and $\gamma_0>0$ (deconfinement), then we will have the following result:

$$\frac{3}{2}N_c<N_f<3N_c. \tag{21}$$

This interval corresponds to the one Seiberg called the interacting non-Abelian Coulomb phase or conformal window. The theory in this interval has a nontrivial infrared fixed point and becomes scale invariant.

On the other hand, as is well known, we have the dual magnetic description for the original electric theory in the region (21). The gauge group of dual theory is $G_{\mathrm{dual}}=\mathrm{SU}(N_f-N_c)$ with $N_f^{\mathrm{dual}}=N_f$ flavors of magnetic chiral superfields and a certain number of singlet massless superfields. In this dual theory, the one-loop coefficients β_0^d and γ_0^d are

$$\beta_0^d=-\frac{1}{16\pi^2}(2N_f-3N_c),$$

$$\gamma_0^d=-\frac{1}{16\pi^2}\left(\frac{1}{2}N_f-\frac{3}{2}N_c\right). \tag{22}$$

From Eqs. (20) and (22), we can extract the relation

$$\beta_0^d(N_f)=-2\gamma_0(N_f),$$

$$\beta_0(N_f)=-2\gamma_0^d(N_f). \tag{23}$$

This may be viewed as the new type of duality Oehme first discovered in Ref. [6]. Note here that the variable N_f on both sides refers to matter fields with different quantum numbers, i.e., one is electric, the other magnetic. Equation (23) might also be interpreted as follows: originally in Seiberg's duality argument the interval (21) was determined from the requirement of asymptotic freedom in both electric and magnetic theories, which means $\beta_0<0$ and $\beta_0^d<0$. We believe here, however, that we can find a set of parameters describing the interval (21) *only* from those of the original electric theory. Then anomalous dimension γ_0 might be a candidate. Below we shall give the same considerations to various gauge theories with various matter representations to check whether a new type of duality (23) proposed by Oehme is satisfied.

B. $G=\mathrm{SO}(N_c)$ with fundamental matters of N_f flavors

Next we shall investigate the case of $\mathrm{SO}(N_c)$ gauge theory with massless superfields of N_f flavors in the fundamental representation. In this case β_0 and γ_0 are given by [9]

$$\beta_0=-\frac{1}{16\pi^2}[3(N_c-2)-N_f],$$

$$\gamma_0=-\frac{1}{16\pi^2}\left[\frac{3}{2}(N_c-2)-N_f\right]. \tag{24}$$

Requiring $\beta_0<0$ and $\gamma_0>0$, we obtain

$$\tfrac{3}{2}(N_c-2)<N_f<3(N_c-2). \tag{25}$$

This is the very same region corresponding to the conformal window discussed in Ref. [10]. Then corresponding dual magnetic theory is $G_{\mathrm{dual}}=\mathrm{SO}(N_f-N_c+4)$ gauge theory with fundamental matter fields of N_f flavors. Dual one-loop coefficients become as

$$\beta_0^d=-\frac{1}{16\pi^2}[2N_f-3(N_c-2)],$$

$$\gamma_0^d=-\frac{1}{16\pi^2}\left[\frac{1}{2}N_f-\frac{3}{2}(N_c-2)\right]. \tag{26}$$

Thus we can conclude

$$\beta_0^d(N_f)=-2\gamma_0(N_f),$$

$$\beta_0(N_f)=-2\gamma_0^d(N_f). \tag{27}$$

In this case we find Oehme's duality also holds.

C. $G=\mathrm{Sp}(2N_c)$ with fundamental matter fields of N_f flavors

This theory also has a magnetic description with certain values of N_c and N_f. Dual gauge theory is $G_{\mathrm{dual}}=Sp(2N_f-2N_c-4)$ with fundamental matter superfields of N_f flavors [11]. The one-loop coefficients of both theories are given as follows:

$$G=\mathrm{Sp}(2N_c)\quad \text{``electric''}\quad N=1\quad \text{SUSY},$$

$$\beta_0=-\frac{1}{16\pi^2}[3(2N_c+2)-2N_f],$$

$$\gamma_0=-\frac{1}{16\pi^2}\left[\frac{3}{2}(2N_c+2)-2N_f\right], \tag{28}$$

and

$$G=\mathrm{Sp}(2N_f-2N_c-4)\quad \text{``magnetic''}\quad N=1\quad \text{SUSY},$$

$$\beta_0^d=-\frac{1}{16\pi^2}[4N_f-3(2N_c+2)],$$

$$\gamma_0^d=-\frac{1}{16\pi^2}\left[N_f-\frac{3}{2}(2N_c+2)\right]. \tag{29}$$

From Eqs. (28) and (29), we get the results $\beta_0^d=-2\gamma_0$ and $\beta_0=-2\gamma_0^d$ in this case, too.

D. $G=G_2$ with fundamental matter fields of N_f flavors

This case may be a little nontrivial. Original electric gauge theory has $G=G_2$ with N_f flavors of massless superfields in the fundamental representation. In this case [12],

$$\beta_0=-\frac{1}{16\pi^2}(12-N_f),$$

$$\gamma_0=-\frac{1}{16\pi^2}(6-N_f). \tag{30}$$

Then the interval $6<N_f<12$ corresponds to the interacting non-Abelian Coulomb phase as discussed in Ref. [12]. A magnetic description exists in this interval. Dual magnetic gauge theory has $G_{\mathrm{dual}}=\mathrm{SU}(N_f-4)$ with fundamental N_f massless matter fields. The one-loop coefficients are given by

$$\beta_0^d=-\frac{1}{16\pi^2}[3(N_f-4)-N_f]=-\frac{1}{16\pi^2}(2N_f-12),$$

$$\gamma_0^d=-\frac{1}{16\pi^2}\left[\frac{3}{2}(N_f-4)-N_f\right]=-\frac{1}{16\pi^2}\left(\frac{1}{2}N_f-6\right). \tag{31}$$

Comparing Eq. (30) with Eq. (31), we can come to the conclusion that Oehme's duality is satisfied. This seems to be a rather nontrivial check for this duality.

E. $G=\mathrm{SU}(N_c),\mathrm{SO}(N_c),\mathrm{Sp}(2N_c)$ with massless adjoint matter

A short time after the discovery of original non-Abelian dualities by Seiberg *et al.*, Kutasov extended Seiberg's argument to $\mathrm{SU}(N_c)$ gauge theory including not only the fundamental but also the adjoint matter superfield [13]. In this case the dual magnetic theory becomes an $\mathrm{SU}(2N_f-N_c)$ gauge theory with N_f massless matter fields in the fundamental representation and an adjoint field and a certain number of singlet massless matter fields. Then the one-loop coefficients in both theories are of the form

$$\beta_0=-\frac{1}{16\pi^2}(2N_c-N_f),$$

$$\gamma_0=-\frac{1}{16\pi^2}\left(\frac{1}{2}N_c-N_f\right) \tag{32}$$

and

$$\beta_0^d = -\frac{1}{16\pi^2}(2N_f - 3N_c),$$

$$\gamma_0^d = -\frac{1}{16\pi^2}\left(-\frac{1}{2}N_c\right). \qquad (33)$$

We find in this case Oehme's duality checked above does not hold.

A similar analysis can be done for $SO(N_c)$ and $Sp(2N_c)$ gauge groups with adjoint fields [14] and leads us to the same results, i.e., Oehme's duality condition is also not satisfied in these cases.

F. $G = SU(N_c)$ with an antisymmetric tensor field

As the final example, we investigate the theory of $G = SU(N_c)$ with an antisymmetric tensor field originally discussed in Ref. [15]. Remarkably the dual magnetic gauge group does not become simple in this case. And this theory is also attractive as a model of supersymmetry breaking [15].

The one-loop coefficients in the original electric theory are

$$\beta_0 = -\frac{1}{16\pi^2}(2N_c - N_f + 3),$$

$$\gamma_0 = -\frac{1}{16\pi^2}\left(\frac{1}{2}N_c - N_f + 3\right). \qquad (34)$$

For $N_f > 5$, dual magnetic gauge theory exists. It is represented as the product gauge group $SU(N_f-3)\otimes Sp(N_f-4)$ with five species of dual quark superfields: a field transforming as a fundamental under both groups, a conjugate antisymmetric tensor, fundamental and N_f antifundamentals of $SU(N_f-3)$, and N_c+N_f-4 fundamentals of $Sp(N_f-4)$.

Then one-loop coefficients in each group are given as

$$\beta_0^d = -\frac{1}{16\pi^2}\left(2N_f - \frac{9}{2}\right),$$

$$\gamma_0^d = -\frac{1}{16\pi^2}\left(\frac{1}{2}N_f - 3\right) \quad \text{in } SU(N_f-3) \qquad (35)$$

and

$$\beta_0^d = -\frac{1}{16\pi^2}\left(\frac{5}{2}N_f - \frac{1}{2}N_c - \frac{9}{2}\right),$$

$$\gamma_0^d = -\frac{1}{16\pi^2}\left(N_f - \frac{1}{2}N_c - \frac{3}{2}\right) \quad \text{in } Sp(N_f-4). \qquad (36)$$

Thus we conclude from these equations that Oehme's duality relation also does not hold in this final example.

IV. DISCUSSIONS

In this paper, we discussed the superconvergence relation following the original work by Oehme and Zimmermann and then investigated whether a new type of duality recently proposed by Oehme in $N=1$ supersymmetric gauge theory also holds in other gauge theories which have different gauge groups and different matter contents. As a result, we found that in the cases of gauge groups with matter fields *only* in the fundamental representation, Oehme's duality was satisfied while in those of theories including adjoint or antisymmetric tensor matter fields, it was not. The reason may be as follows: when we add an adjoint field to a theory, we can add a superpotential. The model without a superpotential will presumably flow in the infrared to a fixed point, while adding the superpotential drives the system to a new fixed point [13]. Kutasov's duality holds only in the model with a superpotential. On the other hand, in the superconvergence argument in which we evaluate β_0 and γ_0 in Eq. (8), we cannot distinguish the theory with a superpotential from the one without it because the contribution of the superpotential to the β function is a higher order one. Actually it appears not in β_0 but in β_1 in Eq.(8).

Also we have to consider the problem of the gauge dependence of our method [16]. Originally the gluon propagator was unphysical, because it depended on the gauge parameter α. In this paper, we chose Landau gauge ($\alpha=0$) for convenience. Certainly the value of β_0 does not depend on the specific gauge choice while that of γ_0 does. Therefore even if we obtain the result $\gamma_0<0$ in the Landau gauge, $\gamma_0>0$ might be realized when we move to other gauges. If so, our superconvergence argument based on the value of γ_0 might not be believed. We shall reconsider this point on another occasion.

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APPENDIX A: PROOF OF EQ. (10)

In this appendix, we prove Eq. (10). The starting point is

$$u\frac{\partial R(u,g)}{\partial u} = \beta(g^2)\frac{\partial R(u,g)}{\partial g^2} + \gamma(g^2)R(u,g). \qquad (A1)$$

Here we define $\tilde{R}$ as follows:

$$\tilde{R} \equiv \tilde{R}[Q^2(u,g),g] = R(u,g). \qquad (A2)$$

Then we find that Eq. (7) can be rewritten as

$$u\frac{\partial Q^2}{\partial u}\bigg|_g \frac{\partial \tilde{R}}{\partial Q^2}\bigg|_g = \beta\left(\frac{\partial \tilde{R}}{\partial g^2}\bigg|_u + \frac{\partial Q^2}{\partial g^2}\bigg|_u \frac{\partial \tilde{R}}{\partial Q^2}\bigg|_u\right) + \gamma\tilde{R}. \quad (A3)$$

On the other hand, the effective coupling constant Q is through Eq. (11): i.e.,

$$u\frac{\partial Q^2}{\partial u}\bigg|_g = \beta(g^2)\frac{\partial Q^2}{\partial g^2}u. \tag{A4}$$

Substituting relation (A4) into Eq. (A1), we obtain

$$0 = \beta(g^2)\frac{\partial \tilde{R}(Q^2,g^2)}{\partial g^2} + \gamma(g^2)\tilde{R}(Q^2,g^2), \tag{A5}$$

and we can easily solve this equation:

$$\tilde{R}(Q^2,g^2) = \tilde{R}(Q^2,Q^2)\exp\left[\int_{g^2}^{Q^2} dx\,\frac{\gamma(x)}{\beta(x)}\right]. \tag{A6}$$

Moreover, by definition,

$$R(1,g^2) = \tilde{R}[Q^2(1,g^2),g^2],$$

$$Q^2(1,g^2) = g^2. \tag{A7}$$

From these equations we obtain $\tilde{R}[Q^2(1,g^2),g^2] = R(1,Q^2)$ and finally we have the result

$$R\left(\frac{k^2}{\mu^2},g^2\right) = R(1,Q)\exp\left[\int_{g^2}^{Q^2} dx\,\frac{\gamma(x)}{\beta(x)}\right]. \quad \text{Q.E.D.} \tag{A8}$$

APPENDIX B: PROOF OF EQ. (13) WITH EQ. (14)

In this appendix, we evaluate Eq. (10) proved in Appendix A at large momentum and show that Eq. (13) is satisfied. For that purpose we first separate the integrand of Eq. (10):

$$\gamma(x)\beta^{-1}(x) = \frac{\gamma_0}{\beta_0 x} + \tau(x), \tag{B1}$$

where $\tau(x)$ corresponds to the regular part of $\gamma(x)\beta^{-1}(x)$ at nearly $x=0$. Note here that at large momentum, the functions $\beta(x)$ and $\gamma(x)$ are given by Eq. (8), respectively, in which g^2 is replaced by x.

Then it is not difficult to show Eqs. (13) and (14). In fact [here we put $R(1,Q)=1$]

$$\begin{aligned} R\left(\frac{k^2}{\mu^2},g^2\right) &= \exp\left[\int_{g^2}^{Q^2} dx\,\frac{\gamma_0}{\beta_0 x}\right]\exp\left[\int_{g^2}^{Q^2} dx\,\tau(x)\right], \\ &= \left(\frac{Q^2}{g^2}\right)^{\gamma_0/\beta_0}\exp\left[\int_{g^2}^{Q^2} dx\,\tau(x)\right], \\ &\approx \left[g^2|\beta_0|\ln\left(\frac{k^2}{\mu^2}\right)\right]^{-\gamma_0/\beta_0}\exp\left[\int_{g^2}^{Q^2} dx\,\tau(x)\right], \\ &\equiv C_V\left(\ln\frac{k^2}{\mu^2}\right)^{-\gamma_0/\beta_0}, \end{aligned} \tag{B2}$$

where

$$C_V = (g^2|\beta_0|)^{-\gamma_0/\beta_0}\exp\left[\int_{g^2}^{Q^2} dx\,\tau(x)\right]. \quad \text{Q.E.D.} \tag{B3}$$

APPENDIX C: PROOF OF SUPERCONVERGENCE RELATION (19)

In this appendix, we prove Eq. (19), the superconvergence relation which is the main subject of this paper. To do so, let us use Eqs. (15), (16), and (18) and combine them.

First multiplying Eq. (16) into k^2,

$$k^2 D(k^2) = \int_0^\infty dm^2\,\frac{k^2\rho(m^2)}{m^2-k^2}. \tag{C1}$$

Then we can show the following results:

$$\begin{aligned} &\lim_{k^2\to-\infty}[\text{LHS of Eq. (C1)}] \\ &\quad = \lim_{k^2\to-\infty} k^2 D(k^2) = \lim_{k^2\to-\infty} k^2 D_{\text{asymp}}(k^2) \\ &\quad = \lim_{k^2\to-\infty} C_V\left(\ln\frac{k^2}{\mu^2}\right)^{-\gamma_0/\beta_0} = 0 \quad \text{for } \frac{\gamma_0}{\beta_0} > 0, \end{aligned} \tag{C2}$$

on the other hand,

$$\begin{aligned} &\lim_{k^2\to-\infty}[\text{RHS of Eq. (C1)}] \\ &\quad = \lim_{k^2\to-\infty}\int_0^\infty dm^2\,\frac{k^2\rho(m^2)}{m^2-k^2} \\ &\quad = \lim_{k^2\to-\infty}\int_0^\infty dm^2\,\frac{-(m^2-k^2)\rho(m^2)+m^2\rho(m^2)}{m^2-k^2} \\ &\quad = -\int_0^\infty dm^2\rho(m^2) + \lim_{k^2\to-\infty}\int_0^\infty dm^2\,\frac{m^2\rho(m^2)}{m^2-k^2}. \end{aligned} \tag{C3}$$

Here we compute in detail more the integration in the last column:

$$\begin{aligned} &\lim_{k^2\to-\infty}\int_0^\infty dm^2\,\frac{m^2\rho(m^2)}{m^2-k^2} \\ &\quad = \lim_{k^2\to-\infty}\left(\int_0^{\Lambda^2} dm^2\,\frac{m^2\rho(m^2)}{m^2-k^2} + \int_{\Lambda^2}^\infty dm^2\,\frac{m^2\rho(m^2)}{m^2-k^2}\right), \\ &\quad = 0 + \lim_{k^2\to-\infty}\int_{\Lambda^2}^\infty dm^2\,\frac{m^2\rho_{\text{asymp}}(m^2)}{m^2-k^2}, \\ &\quad = \lim_{k^2\to-\infty}\int_{\Lambda^2}^\infty dm^2\,\frac{-(\gamma_0/\beta_0)C_V[\ln(m^2/\mu^2)]^{-\gamma_0/\beta_0-1}}{m^2-k^2} \\ &\quad = 0. \end{aligned} \tag{C4}$$

Compared with both sides of Eq. (C1), we have the result that

$$\int_0^\infty dm^2\rho(m^2) = 0. \quad \text{Q.E.D.} \tag{C5}$$

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