2D gravity and the extended formalism

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The role of $SL(2,\mathbb{R})$ symmetry in two-dimensional gravity is investigated in the context of the extended Hamiltonian formalism. Using our results we clarify previous works on the subject. [S0556-2821(98)02216-4]

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I. INTRODUCTION

The analysis of the symmetries and quantization of the induced two-dimensional gravity model (2D gravity) proposed by Polyakov [1] has received the attention of several authors [3–5]. In the original work a ''residual'' symmetry appeared when the model was studied in the light-cone gauge [1], the generators satisfying an $SL(2,\mathbb{R})$ algebra. In subsequent papers this feature was approached with a variety of techniques. An important idea arising from these works is that to understand $SL(2,\mathbb{R})$ symmetry a gauge-independent analysis is fundamental, trying to confirm that this invariance is something basic in 2D gravity. The first gauge-invariant formulation [3] arrived at the conclusion that the $SL(2,\mathbb{R})$ algebra, realized by generalized currents, made sense as a symmetry only in the light-cone gauge. In Ref. [4] the problem was studied in the context of improper gauge transformations, but the results were not completely conclusive. Finally, in [5], working with the canonical Hamiltonian formalism, it was concluded that $SL(2,\mathbb{R})$ symmetry arose on the classical level only when the x^+ coordinate was taken as time.

In this work we propose to clarify this problem working with the *extended* Hamiltonian formalism. Adopting strictly this technique, we show that it is possible to understand the role and origin of $SL(2,\mathbb{R})$ symmetry when we impose a gauge fixing. Instead, what it is usually found in the literature is the *direct* injection of the gauge conditions in the original action, an approach that makes it impossible to elucidate the role of any residual symmetry.

The paper is organized as follows. In Sec. II we give a short description and comments on the approaches found in the literature. In Sec. III we present the fundamental ideas behind the extended Hamiltonian formulation and how it works with the problem of residual symmetries. Section IV shows our results when we apply the method to the induced gravity model. Finally, in Sec. V we present our conclusions.

II. GAUGE-INVARIANT AND REDUCED PHASE-SPACE APPROACHES

This section is devoted to a brief description of Polyakov's induced gravity model and of previous works related to $SL(2,\mathbb{R})$ symmetry, distiguishing basically two approaches: the gauge-invariant approach and the reduced phase-space approach.

The two-dimensional induced gravity model [1] has a rich gauge structure. In order to take advantage of this feature [to find physical solutions or to understand the role of $SL(2,\mathbb{R})$ invariance, for instance] it is important to have first a consistent gauge-invariant formulation.

The basic ideas in the gauge-invariant analysis begin to acquire shape when we manage to write the action as a local functional [introducing an auxiliary scalar field $\phi(x)$] [3]:

$$S=\int d^2x\sqrt{-g}(-\phi\Box\phi-\alpha R\phi+\alpha^2\beta), \tag{1}$$

where R is the two-dimensional scalar curvature and

$$\alpha^2=8k-\frac{1}{12\pi}\beta=-\mu^2\left(\frac{2k}{\alpha^2}\right), \tag{2}$$

k being a function of the central charge of the original model (gravity coupled to matter) and μ the cosmological constant.

Starting with Eq. (1) it is possible to construct the classical Hamiltonian formulation. The diffeomorphism invariance present in this model implies well-known expressions for the canonical Hamiltonian density H_c and primary (first class) constraints π^{00} and π^{01}, $\pi^{\mu\nu}$ being the momenta canocally conjugated to the metric components $g_{\mu\nu}$,

$$H_c=-\frac{\sqrt{-g}}{g_{11}}\phi_3+\frac{g_{01}}{g_{11}}\phi_4, \tag{3}$$

$$\pi^{00}=\phi_1\approx 0, \tag{4a}$$

$$\pi^{01}=\phi_2\approx 0, \tag{4b}$$

where ϕ_3 and ϕ_4 are secondary (first class) constraints, that follow from the time consistency of Eqs. (4),

$$\phi_3=\frac{1}{2}\left(\phi'^2-\frac{4}{\alpha^2}(g_{11}\pi^{11})^2-\frac{4}{\alpha}(g_{11}\pi^{11})\pi -\alpha\frac{g'_{11}}{g_{11}}+2\alpha\phi''+\alpha^2\beta g_{11}\right), \tag{5a}$$

$$\phi_4=\pi\phi'-2g_{11}\pi^{11'}-\pi^{11}g_{11'}, \tag{5b}$$

and π is the momentum canonically conjugated to the scalar field $\phi(x)$. The set of first class constraints showed above represents, as usual, the Hamiltonian generators of diffeomorphism invariance. An important feature here is that it is possible in this context to obtain some information about the residual $SL(2,\mathbb{R})$. In [3], Abdalla *et al.* proposed the construction of a generalization of the light-cone gauge currents $[J(x)]$,

$$J^{+}=\frac{1}{g_{11}}(\phi_2-\phi_1)+\frac{1}{2}\alpha^2\beta, \tag{6a}$$

$$J^{0}=j^{0}-x^{-}J^{+}, \tag{6b}$$

$$j^{0}=\sqrt{2}\left[g_{11}\left(\pi^{11}+\frac{\alpha}{2}\frac{\phi'}{g_{11}}\right)+\frac{\alpha}{2}\left(\pi-\frac{\alpha}{2}\frac{g'_{11}}{g_{11}}-\phi'\right)\right], \tag{6c}$$

$$J^{-}=j^{-}-2x^{-}J^{0}-(x^{-})^2J^{+}, \tag{6d}$$

$$j^{-}=\alpha^2(g_{11}+1), \tag{6e}$$

which satisfy, as their light-cone partners, the well-known $SL(2,\mathbb{R})$ algebra

$$\{J^a(x),J^b(y)\}=-2\sqrt{2}\epsilon^{abc}\eta_{cd}J^d(x)\delta(x-y). \tag{7}$$

The crucial point here is that these generalized currents represent symmetry generators *only* in the light-cone gauge, the very $SL(2,\mathbb{R})$ symmetry, playing no role in other gauges and therefore losing their gauge-independent nature.

A different approach to this problem was tried in [4] (a reduced phase-space formulation). The basic idea was that $SL(2,\mathbb{R})$ symmetry can be interpreted as an inproper gauge transformation of the action (1). An improper gauge transformation [6] appears when the generators of the local symmetries (G) need extra terms (F) in order to define unambiguously the field's variations under the action of $\overline{G}$,

$$\overline{G}(\epsilon)=G(\epsilon)+F(\epsilon), \tag{8}$$

where ϵ are the parameters of the gauge transformation. In the case of Polyakov's induced gravity the G's are simply linear combinations of the first class constraints (4) and (5). On the other hand, F is given by

$$F=a_1l_1+a_2l_2+a_3l_3, \tag{9}$$

where

$$\epsilon(x_-,x_+)=a_1(x_+)+x_-a_2(x_+)+(x_-)^2a_3(x_+) \tag{10}$$

and

$$l_1=\partial_-,\quad l_2=x^{-}\partial_--1,\quad l_3=(x^{-})^2\partial_--2x^{-}. \tag{11}$$

The problem is that although the l_i's obey an $SL(2,\mathbb{R})$ algebra, it was not clear why these quantities had to be associated with the generators of the residual $SL(2,\mathbb{R})$ symmetry (as the authors recognize [4]).

III. RESIDUAL SYMMETRIES AND THE EXTENDED FORMALISM

In the previous section we have seen that the interpretation for the presence of $SL(2,\mathbb{R})$ symmetry in the induced gravity model is full of drawbacks. These problems can be effectively solved if we analyze our model using the Hamiltonian extended formalism [2]. This formulation works with Dirac's idea that the maximum of information about symmetries in a gauge theory can be obtained if we consider as a basic ingredient the so-called extended action

$$S_e=\int(p_n\dot{q}^n-H-\lambda^a\phi_a-\lambda^\alpha\chi_\alpha)dt, \tag{12}$$

the ϕ being the first class constraints and χ are the second class ones (the λ^a represent their respective Lagrange multipliers).

The formalism gives the following expressions for the canonical gauge structure:

$$\{\phi_a,\phi_b\}=C^c_{ab}\phi_c+T^{\alpha\beta}_{ab}\chi_\alpha\chi_\beta, \tag{13a}$$

$$\{\phi_a,\chi_\alpha\}=C^b_{a\alpha}\phi_b+C^\beta_{a\alpha}\chi_\beta, \tag{13b}$$

$$\{H,\phi_a\}=V^b_a\phi_b+V^{\alpha\beta}_a\chi_\alpha\chi_\beta, \tag{13c}$$

$$\{H,\chi_\alpha\}=V^b_\alpha\phi_b+V^\beta_\alpha\chi_\beta, \tag{13d}$$

the structure functions C, T, and V being fundamental for our purposes. The gauge transfomations are given by

$$\delta_\epsilon F=\epsilon^a\{F,\phi_a\}. \tag{14}$$

In order for the extended action to be invariant under Eq. (14) the Lagrange multipliers should transform as

$$\delta\lambda^a=\dot{\epsilon}^a+\lambda^c\epsilon^bC^a_{bc}-\epsilon^aV_b, \tag{15a}$$

$$\delta\lambda^\alpha=\lambda^c\epsilon^bT^{\alpha\beta}_{bc}\chi_\beta-\epsilon^bV^{\alpha\beta}_b\chi_\beta+\lambda^\beta\epsilon^bC^\alpha_{b\beta}. \tag{15b}$$

The important point here is that we can obtain a complete set of symmetries of the original action (total action) by simply imposing the gauge fixings in the Lagrange multipliers of the secondary constraints (λ^c) and we can insert them back into the extended action and get the total action. More important are the consequences that this gauge fixing has on the symmetries. Imposing these conditions on Eqs. (15) as

$$\lambda^c=0,\quad \delta\lambda^c=0, \tag{16}$$

we obtain the symmetries of the total action,

$$\delta\Phi(x)=\{\Phi(x),G\},\quad G=\mu^a\phi_a, \tag{17}$$

where the μ^a must preserve the gauge conditions (16).

A very instructive example of this method is the free Maxwell theory. The extended action reads

$$S_e=\int d^4(\pi^i\dot{A}_i+\pi^0\dot{A}_0-H-\lambda^1\phi_1-\lambda^2\phi_2), \tag{18}$$

where $\phi_1=\pi^0=0$ is the primary constraint and $\phi_2=\partial_i\pi^1=0$ is the secondary (Gauss law); both are first class.

The generator of the extended action invariances is

$$G=\int d^3x(\epsilon^1\phi_1+\epsilon^2\phi_2), \tag{19}$$

with independent gauge parameters ϵ. The commutation relations between the first class quantities are trivial in this case. The variations of Lagrange multipliers are [2]

$$\delta\lambda^1=\dot{\epsilon}^1, \quad \delta\lambda^2=\dot{\epsilon}^2-\epsilon^1. \tag{20}$$

The usual U(1) invariance of electromagnetism is recovered when we use the conditions (16):

$$\delta\lambda^2=0\Rightarrow\dot{\epsilon}^2=\epsilon^1. \tag{21}$$

IV. EXTENDED FORMULATION FOR 2D GRAVITY AND THE $SL(2,\mathbb{R})$ SYMMETRY

In this section we apply the method described above for the induced gravity model. The first step is to construct the extended action. We already know, from previous sections, the expressions for the canonical Hamiltonian and the constraint structure. So we obtain straightforwardly,

$$S_e=\int d^2x(\pi^{00}\dot{g}_{00}-2\pi^{01}\dot{g}_{01}-\pi^{11}\dot{g}_{11}-H_c-\lambda^i\phi_i). \tag{22}$$

Following Eqs. (13) we see that each of the first class constraints, ϕ_i, $i=1,\ldots,4$, will generate an independent local gauge transformation

$$G=\int dx(\epsilon_i\phi^i), \tag{23}$$

which leaves the extended action (18) invariant, given the correct transformations for the Lagrange multipliers. To obtain these transformations we must use first Eqs. (13) to find the structure functions C^c_{ab} and V^b_a (the others being zero because in this case we have just first class constraints). After some manipulations we find, explicitly,

$$\{\phi_3(x),\phi_4(y)\}=\int dz C^3_{34}(x,y,z)\phi(z), \tag{24}$$

where

$$C^3_{34}(x,y,z)=\delta(z-y)\delta'(z-y)+\delta(z-y)\delta'(x-z), \tag{25}$$

the other nonzero C's being $C^4_{33}(x,y,z)=C^4_{44}(x,y,z)$ with expressions identical to Eq. (25).

We also have for the V's the following expressions:

$$\{H,\phi_3(x)\}=\int dzdy V^3_3(x,y,z)\phi_3(z)+\int dzdy V^4_3(x,y,z)\phi_4(z), \tag{26a}$$

$$\{H,\phi_4(x)\}=\int dzdy V^3_4(x,y,z)\phi_3(z)+\int dzdy V^4_4(x,y,z)\phi_4(z), \tag{26b}$$

where

$$V^3_3(x,y,z)=-\left\{\frac{\sqrt{-g}}{g_{11}}(y),\phi_3(x)\right\}\delta(x-z)+C^3_{43}(x,y,z)\frac{g_{01}}{g_{11}}(y), \tag{27a}$$

$$V^4_3(x,y,z)=\left\{\frac{g_{01}}{g_{11}}(y),\phi_3(x)\right\}\delta(x-z)+C^4_{33}(x,y,z)\frac{\sqrt{-g}}{g_{11}}(y), \tag{27b}$$

$$V^3_4(x,y,z)=-\left\{\frac{\sqrt{-g}}{g_{11}}(y),\phi_4(x)\right\}\delta(x-z)+C^3_{34}(x,y,z)\frac{\sqrt{-g}}{g_{11}}(y), \tag{27c}$$

$$V^4_4(x,y,z)=\left\{\frac{g_{01}}{g_{11}}(y),\phi_4(x)\right\}\delta(x-z)+C^4_{44}(x,y,z)\frac{g_{01}}{g_{11}}(y). \tag{27d}$$

The final step is our most important result. We can obtain the light-cone formulation going to the total action formulation imposing the conditions (16), which read, in our model,

$$\lambda^3=0=\lambda^4, \quad \delta\lambda^3=0=\delta\lambda^4, \tag{28}$$

into the secondary constraint's Lagrangian multipliers variations

$$\delta\lambda^3(x)=\dot{\epsilon}^3(x)+\int dy\{[\lambda^3(x)\epsilon^4(x)+\lambda^4(x)\epsilon^3(x)]+[\lambda^3(y)\epsilon^4(y)+\lambda^4(y)\epsilon^3(y)]\}\delta'(x-y)+\int dzdy[V^3_3(x,y,z)\epsilon^3(y)+V^3_4(x,y,z)\epsilon^4(y)], \tag{29}$$

$$\delta\lambda^4(x)=\dot{\epsilon}^4(x)+\int dy\{[\lambda^3(x)\epsilon^4(x)+\lambda^4(x)\epsilon^3(x)]+[\lambda^3(y)\epsilon^4(y)+\lambda^4(y)\epsilon^3(y)]\}\delta'(x-y)+\int dzdy[V^4_3(x,y,z)\epsilon^3(y)+V^4_4(x,y,z)\epsilon^4(y)]. \tag{30}$$

These relationships define restrictions on the gauge parameters $\epsilon(x)$ and the basic fields. Using these expressions in the original gauge transformations (diffeomophism invariance) for the basic fields $g_{\mu\nu}(x)$ and $\phi(x)$ [3] we obtain, after a tedious calculation $[g_{++}=\frac{1}{2}(g_{00}+2g_{01}+g_{11}),\ \epsilon^{\pm}=1/\sqrt{2}(\epsilon^{3}\pm\epsilon^{4})]$,

$$\delta\phi=\epsilon^{-}\partial_{-}\phi-\alpha\epsilon^{-}, \tag{31a}$$

$$\delta g_{++}=\epsilon^{-}\partial_{-}g_{++}-g_{++}\partial_{-}\epsilon^{-}-\partial_{+}\epsilon^{-}, \tag{31b}$$

which are exactly the transformations generated by the so-called $SL(2,\mathbb{R})$ currents (7) in the light-cone gauge.

We also verify that when we substitute these conditions on the extended action we obtain, following the method's prescription, the light-cone gauge action

$$S=\int dx(\pi_{\phi}\dot{\phi}+\pi_{g}\dot{g}_{++}-L_{lc}), \tag{32}$$

where L_{lc} is the light-cone gauge Lagrangian:

$$L_{lc}=\partial_{+}\phi\partial_{-}\phi+g_{++}(\partial_{-}\phi)^{2}-\alpha\partial_{-}g_{++}\partial_{-}\phi. \tag{33}$$

As we see the expressions (31) are obtained as a by-product of the extended formulation, making clear the role of the $SL(2,\mathbb{R})$ symmetry in the induced gravity model and the relation with the light-cone formulation as a whole.

V. CONCLUSIONS

In this work we have clarified the role of classical $SL(2,\mathbb{R})$ symmetry using the extended Hamiltonian formalism. This formulation leaves intact the separation between physical and spurius degrees of freedom, making the process of gauge fixing in induced gravity unambigous. In the early works mentioned, instead, the light-cone gauge conditons were injected directly on the original action, making obscure the origin and role of the $SL(2,\mathbb{R})$ symmetry as a residual symmetry. On the other hand, the gauge-independent formulations also mentioned here were inconclusive about this issue.

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