Dynamic wormholes, antitrapped surfaces, and energy conditions

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It is by now apparent that topology is too crude a tool to accurately characterize a generic traversable wormhole. In two earlier papers we developed a complete characterization of generic but static traversable wormholes, and in the present paper extend the discussion to arbitrary time-dependent (dynamical) wormholes. A local definition of a wormhole throat, free from assumptions about asymptotic flatness, symmetries, future and past null infinities, embedding diagrams, topology, and even time dependence is developed that accurately captures the essence of what a wormhole throat is, and where it is located. Adapting and extending a suggestion due to Page, we define a wormhole throat to be a marginally anti-trapped surface, that is, a closed two-dimensional spatial hypersurface such that one of the two future-directed null geodesic congruences orthogonal to it is just beginning to diverge. Typically a dynamic wormhole will possess *two* such throats, corresponding to the two orthogonal null geodesic congruences, and these two throats will not coincide (though they do coalesce into a single throat in the static limit). The divergence property of the null geodesics at the marginally anti-trapped surface generalizes the ''flare-out'' condition for an arbitrary wormhole. We derive theorems regarding violations of the null energy condition (NEC) at and near these throats and find that, even for wormholes with arbitrary time dependence, the violation of the NEC is a generic property of wormhole throats. We also discuss wormhole throats in the presence of fully antisymmetric torsion and find that the energy condition violations *cannot* be dumped into the torsion degrees of freedom. Finally by means of a concrete example we demonstrate that even temporary suspension of energy-condition violations is incompatible with the flare-out property of dynamic throats. [S0556-2821(98)04916-9]

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I. INTRODUCTION

Traversable Lorentzian wormholes [1–3] have often been viewed as intrinsically topological objects, with the topological nature of their spatial sections revealed graphically by means of embedding diagrams and ''shape'' functions as either ''handles'' in spacetime (intra-universe wormholes joining two distant regions of the same universe) or as ''bridges'' (inter-universe wormholes linking two distinct spacetimes). Both of these types of wormhole give rise to the notion of multiply connected universes and spatio-temporal networks possessing a non-trivial topology [4]. More often than not, global geometric constraints are imposed on the wormhole, as well as symmetry properties. For example, the static Morris-Thorne inter-universe wormhole is an example of this more restrictive class in that it requires exact spherical symmetry and the existence of two asymptotically flat regions in spacetime [1]. As we have previously argued [5,6] there are many other spacetime configurations and geometries that one might still quite reasonably want to classify as wormholes that either do not possess any asymptotically flat regions, or have trivial topology, or exhibit both these features. An example of the former is provided by the Hochberg-Popov-Sushkov self-consistent semi-classical wormhole (which is a wormhole of the inter-universe type joining up two spacetimes with no asymptotically flat spatial regions) [7]. Examples of the topologically trivial wormholes [3] are provided by a closed Friedmann-Robertson-Walker (FRW) spacetime joined to an ordinary Minkowski spacetime by a narrow neck or two closed FRW spacetimes joined by a bridge [5,6].

The only difference between these two classes of wormholes (i.e. bridges and handles *versus* topologically trivial) arises at the level of *global* geometry and *global* topology. This suggests that it is important to identify a fundamental *local* property that can be used to characterize what one means by a wormhole, an intrinsic property to be abstracted from the broad phylum of wormholes which can then be used to unambiguously define what is meant by a wormhole. Indeed, the local physics, that which is operative near the ''throat'' of the wormhole, is insensitive to global properties and indicates that a local definition of what is meant by a wormhole throat is called for. This definition should be based solely on local properties and be free from technical assumptions about asymptotic flatness, future and past null infinities, global hyperbolicity, symmetries, embeddings and topology.

In two previous papers [5,6] we have performed such an analysis for static traversable wormholes. In this paper, we lift the static restriction and shall investigate the generic (not necessarily static) traversable wormhole. We make no assumptions about symmetries, spherical or otherwise, nor do we assume the existence of asymptotically flat regions. To proceed, we first have to define exactly what we mean by a

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wormhole and we find, just as in the treatment of the generic *static* case [5,6], that there is a natural local *geometric* (not topological) characterization of the existence and location of a wormhole ''throat.'' This characterization is developed in the language of the expansion of null geodesic congruences propagating outward from, and orthogonal to, closed two-dimensional spatial hypersurfaces (denoted Σ). The congruence is subject to a ''flare-out'' condition that suitably generalizes that of the Morris-Thorne analysis. But, unlike that earlier definition [1,2], ours makes no reference to embeddings or shape functions. In this language, the spatial hypersurface in question will be a wormhole throat provided the expansion $\theta_{\pm}$ of one of the two orthogonal null congruences vanishes on that surface: $\theta_{+}=0$ and/or $\theta_{-}=0$, and if the rate-of-change of the expansion along the *same* null direction ($u_{\pm}$) is positive-semi-definite at the surface: $d\theta_{\pm}/du_{\pm}\geqslant 0$. This latter constraint is precisely the ''flare-out'' condition generalized to an arbitrary wormhole. These two conditions on the expansion define the throat to be a minimal hypersurface, i.e., an extremal surface of minimal area (with respect to deformations in the appropriate $u_{\pm}$ null direction). Thus, a wormhole throat is a *marginally anti-trapped surface*. Historically, Page [8] was the first to suggest that under suitable circumstances a wormhole throat could be viewed as an anti-trapped surface in spacetime, and we shall soon see that this definition promises to be the most efficient and most physical framework for generalizing the concept of throat to the fully arbitrary and dynamic case.

While this definition captures the intuitive concept of a throat admirably, there can be cases calling for slight definitional refinements, for example, when $d\theta_{\pm}/du_{\pm}>0$ is strictly positive on the throat, in which case we are dealing with a *strongly anti-trapped surface*, as well as other cases for which weaker, averaged notions of flare-out will suffice.

In general, the vanishing of the independent expansions $\theta_{+}=0$ and $\theta_{-}=0$ will take place on two distinct hypersurfaces. Thus (dynamical) wormholes generally possess *two* throats provided each hypersurface is individually flared-out: $d\theta_{+}/du_{+}\geqslant 0$ on Σ_{u+}, and $d\theta_{-}/du_{-}\geqslant 0$ on Σ_{u-}. Of course, the two throats must (and they do) coincide in the static limit.

With these definitions in place, we move on to develop a number of theorems about the existence of matter at or near the throat(s) violating the null energy condition (NEC). These theorems make repeated use of the Raychaudhuri equation for the expansions $\theta_{\pm}$. These results are local and pointwise, in distinction to energy conditions obtained by averaging over inextendible null geodesics, which are global in nature. These energy theorems generalize the original Morris-Thorne result by demonstrating unequivocally that the NEC is generically violated at some points on or *near* the two-dimensional hypersurface comprising the throat(s). This is an important result since these theorems hold for an arbitrary dynamic or static wormhole irrespective of symmetries or other global concerns and demonstrate that the energy condition violations are truly generic. Our results are (of course) also completely in accord with the topological censorship theorem of Friedman, Schleich, and Witt [9].

The striking nature of the violations of the null energy condition discovered for the Morris-Thorne wormhole [1–3], has led numerous authors to try and find ways of evading or minimizing these violations. Most of these attempts focus on alternative gravity theories, be they Brans-Dicke, dilaton gravity, higher-derivative theories, etc. What all these extensions of Einstein gravity ''accomplish'' from a practical point of view is to provide one with additional degrees of freedom (beyond the metric), which under certain circumstances can be coerced into absorbing the energy condition violations (leaving the remaining ordinary-matter sector free to satisfy the classical energy conditions). Nevertheless, the total effective stress energy tensor will violate the null energy condition at or near the throat, so sweeping the unavoidable energy condition violations into a specific sector does not make the problem go away. This important but often overlooked point has been treated in some detail in [6]. (We would be remiss in not warning the reader that a sizable fraction of the published papers claiming to build wormholes without violating the energy conditions suffer from severe technical problems, and are often internally inconsistent.)

Similar comments apply of course to gravity plus torsion, although theories with torsion are distinguished from other variants of gravity by the fact that non-zero torsion gives rise to a non-trivial contribution to the Raychaudhuri equation which *cannot* be absorbed into an effective total stress energy tensor. Moreover, torsion appears naturally (and unavoidably) in theories of gravity based on low-energy closed string theories. These facts make it of interest to treat the torsion case separately and in some detail to assess the ability of torsion to defocus (null) geodesics and to check the status of the NEC for throats in the presence of torsion. We find that totally antisymmetric torsion actually *promotes* the energy condition violation at the throat (but helps to lessen it away from the throat by generating twist). Other attempts to get around the energy-condition violations have led to considerations of time-dependent wormholes. In this domain, it is indeed possible to temporarily suspend the violations, but only at the heavy expense of totally destroying the flare-out properties of the throat.

Since the Raychaudhuri equation with torsion is not standard textbook fare, we include a brief resume of torsion in Sec. II to establish the notation used in the rest of the paper and provide a simple derivation of the generalized Raychaudhuri and the companion twist equations corresponding to the two independent null congruences in Sec. III. We then define wormhole throats in terms of the expansions in Sec. IV and prove the coalescence of the two throats in the static limit. Armed with these definitions, we go on to derive the energy condition theorems for wormholes in normal spacetime as well as in the presence of torsion in Sec. V. Worked examples of dynamic wormholes are provided in Sec. VI where, among other things, we show how the temporal suspension of energy-condition violations eradicates the throat. Conclusions and a discussion of our results are collected in Sec. VII.

II. GEOMETRIC PRELIMINARIES: SPACETIMES WITH TORSION

In preparation for the derivation of the Raychaudhuri equation governing the expansion in the presence of torsion, and to establish the notation to be used throughout, we collect here a few basic definitions and identities which will prove useful later on. (Basic definitions regarding torsion can be gleaned from [10–12], while an overview of torsion in the string theory context can be extracted from [13].)

As we are interested in keeping our discussion as general as possible, we endeavor to work in a coordinate-free language and to this end, shall make use of the abstract index notation, later specializing when and if needed, to explicit coordinate systems. For the time being then, the use of lower case latin letters designates abstract indices: $(a,b,c, \ldots)$ and run from 0 to 3. (See Wald [14] for a discussion of the subtleties associated with the use of ''abstract indices.'') Let v_a be a covariant vector, its covariant derivative is

$$\nabla_a v_b = \partial_a v_b - C^c_{ab} v_c , \tag{1}$$

where C^b_{ac} denotes the connection of the underlying four-dimensional spacetime. In principle, the connection can be any ''tensor'' field guaranteeing that the covariant derivative (1) based upon it satisfies all the usual properties (linear, Leibnitz, etc.) [14]. However, we will not impose the torsion-free condition, which means that the (total) connection can be decomposed as

$$C^c_{ab} = \Gamma^c_{ab} + H^c_{ab} , \tag{2}$$

where $C^c_{(ab)} = \frac{1}{2}(C^c_{ab} + C^c_{ba}) = \Gamma^c_{ab}$ is the ordinary symmetric Christoffel connection, depending on the metric in the usual way, while $C^c_{[ab]} = \frac{1}{2}(C^c_{ab} - C^c_{ba}) = H^c_{ab}$ defines the torsion, which is manifestly anti-symmetric in its two lower indices.

Due to the mixed symmetry of the connection, the commutator of the covariant derivative, which is used to define the curvature tensor, works out to be

$$\begin{aligned}[\nabla_a, \nabla_b] v_c &= (-2\partial_{[a} C^d_{b]c} + 2C^e_{[a|c} C^d_{b]e}) v_d - 2C^e_{[ab]} \nabla_e v_c \\ &= \overline{R}_{ab,c}{}^d(C) v_d - 2H^e_{ab} \nabla_e v_c , \end{aligned}\tag{3}$$

where

$$\overline{R}_{ab,c}{}^d(C) = -2\partial_{[a} C^d_{b]c} + 2C^e_{[a|c} C^d_{b]e} , \tag{4}$$

is the associated curvature tensor. The vertical bar within the antisymmetrization brackets indicates that one is to antisymmetrize over the pair a and b, but not c. We have distinguished the curvature with an overbar in order to emphasize that this tensor is not the ordinary Riemann tensor, unless the torsion vanishes identically. It *is* however the curvature associated with a general connection C. We note that the derivative of a vector couples directly to the torsion, as evidenced by the second term in the above identity (3). The torsion also shows up explicitly (and implicitly in the covariant derivatives) in the commutator of two vector fields:

$$[v,w]^b = v^a \nabla_a w^b - w^a \nabla_a v^b - 2v^a w^c H^b_{ac} . \tag{5}$$

Although Eq. (4) is not the standard Riemann tensor, it is related to it as follows:

$$\overline{R}_{ab,c}{}^d(C) = R_{ab,c}{}^d(\Gamma) - (\tilde{\nabla}_a H^d_{bc} - \tilde{\nabla}_b H^d_{ac}) + 2H^e_{[a|c} H^d_{b]e} , \tag{6}$$

where the covariant derivatives of the torsion are calculated with respect to the symmetric (Christoffel) part of the connection only; that is,

$$\tilde{\nabla}_a H^d_{bc} = \partial_a H^d_{bc} - \Gamma^e_{ab} H^d_{ec} - \Gamma^e_{ac} H^d_{be} + \Gamma^d_{ae} H^e_{bc} . \tag{7}$$

This identity (6) suggests that the torsion can be regarded as a dynamic field propagating over a normal Riemannian spacetime, i.e., may either be regarded as fundamentally geometric, as part and parcel of the connection (2), or as a ''matter'' tensor field in a spacetime with a conventional symmetric connection. We can make this latter association more precise by writing the action from which we will infer the corresponding equations of motion. We form the equivalent of the Einstein-Hilbert action for the generalized curvature and allow for the presence of ordinary matter (every other dynamical field imaginable except for the metric and torsion):

$$S = -\frac{1}{16\pi}\int d^4x \sqrt{-g}\,\overline{R}(C) + \int d^4x \sqrt{-g}\,\mathcal{L}_{matter} , \tag{8}$$

where the generalized scalar curvature is $\overline{R}(C) = g^{ac}\overline{R}_{ab,c}{}^b(C)$ and is related to the scalar of Riemannian curvature via

$$\overline{R}(C) = R(\Gamma) - g^{bc}\tilde{\nabla}_b H^a_{ac} - H_{abc}H^{abc}, \tag{9}$$

which follows from (6) and using the covariant constancy of the metric $\tilde{\nabla}_a g_{bc} = 0$, with respect to $\tilde{\nabla}$. (Mathematically, it is possible to consider even more general affine connections for which the covariant derivative of the metric is not zero. The most general such affine connection is then a linear combination of the Christoffel connection, the torsion tensor, and a ''non-metricality tensor.'' We will not generalize our analysis to this level of abstraction as little seems to be gained, and there are good physics reasons for keeping the covariant derivative of the metric zero.)

Thus far, we have kept the treatment of the torsion part of the connection completely general. If we now identify the torsion with the totally anti-symmetric rank-three field strength $\mathbf{H} = d\mathbf{A}$, where $\mathbf{A}$ is a two-form potential, or in terms of components

$$H_{abc} = \partial_a A_{bc} + \partial_b A_{ca} + \partial_c A_{ab} , \tag{10}$$

then we have an explicit realization of torsion that is known to arise naturally in closed string-theoretic low energy gravity [13,15,16]. In this particular incarnation as an antisymmetric rank-three tensor, the torsion is also known as the Kalb-Ramond field. From here on, when we refer to torsion, it will be of this form.

The equations of motion now follow immediately upon varying the full action (8) with respect to the metric, torsion, and whatever other matter fields may be present. The equation of motion for the metric is given by

$$G_{ab}(\Gamma)=\left(R_{ab}-\frac{1}{2}g_{ab}R\right)$$
$$=8\pi T_{ab}+3H_{ade}H_b^{de}-\frac{1}{2}g_{ab}\ H_{cde}H^{cde}, \quad (11)$$

where T_{ab} is the complete stress-energy tensor for the matter fields. We see that although H originates from the connection, it can also be treated as simply an additional species of matter and can therefore be shifted into an effective matter stress tensor. However, it is of more than academic interest not to do so at this stage. When we come to consider the expansion and twist of (null) geodesic congruences in spacetimes with torsion, we will find that the torsion makes explicit non-dynamic contributions to the differential equations for the expansion and twist that cannot be re-defined away, as it were, by invoking the equations of motion, or by redefining the total effective stress energy tensor. Thus, it will be of interest to see what influence the torsion may have to focus and defocus bundles of null geodesics. The equation of motion for the torsion that follows from varying Eq. (8) is simply that

$$\tilde{\nabla}_a H^{abc}=0. \quad (12)$$

Using the metric equation (11), it follows that the Ricci tensor obeys the equation

$$R_{ab}=8\pi\left[T_{ab}-\frac{1}{2}g_{ab}T\right]+3H_{ade}H_b^{de}-g_{ab}\ (H_{ade}H^{ade}), \quad (13)$$

while the scalar curvature is

$$R=-8\pi T-H_{ade}H^{ade}. \quad (14)$$

III. NULL GEODESIC CONGRUENCES

We start by considering a compact two-dimensional hypersurface that is both orientable and embedded into spacetime in a two-sided manner in such a way that the induced two metric is spacelike. To discuss the null geodesic congruences orthogonal to this surface, we shall, following the description of Carter [17] begin by introducing a future-directed ''outgoing'' null vector l_+^a, a future-directed ''ingoing'' null vector l_-^a and a spatial orthogonal projection tensor γ^{ab} satisfying the following relations:

$$l_+^a l_{+a}=l_-^a l_{-a}=0,$$
$$l_+^a l_{-a}=l_-^a l_{+a}=-1,$$
$$l_\pm^a \gamma_{ab}=0,$$
$$\gamma_c^a\gamma^{cd}=\gamma^{ad}. \quad (15)$$

In terms of these null vectors and projector, we can decompose the full spacetime metric (indeed, any tensor) uniquely:

$$g_{ab}=\gamma_{ab}-l_{-a}l_{+b}-l_{+a}l_{-b}. \quad (16)$$

Physically, this decomposition leads to a parametrization of spacetime points in terms of two spatial coordinates (typically denoted x) plus two null coordinates [$u_\pm$, or sometimes (u,v)]. (We do not want to prejudice matters by taking the words ''outgoing'' and ''ingoing'' too literally, since outside and inside do not necessarily make much sense in situations of nontrivial topology. The critical issue is that the spacelike hypersurface must have two sides and $+$ and $-$ are just two convenient labels for the two null directions.)

We consider the tensor fields defined by the covariant derivative of the future-directed null vectors (there is one such tensor field for each null congruence)

$$B_{ab}^{\pm}\equiv\nabla_b l_{\pm a}, \quad (17)$$

and ask for their rate of change along the corresponding null geodesic parametrized with affine parameter $u_\pm$:

$$\begin{aligned}\frac{dB_{ab}^{\pm}}{du_\pm}&\equiv l_\pm^c\ \nabla_c B_{ab}^{\pm}=l_\pm^c\ \nabla_c\nabla_b l_{\pm a}\\&=l_\pm^c\ \nabla_b\nabla_c l_{\pm a}+l_\pm^c[\nabla_c,\nabla_b]l_{\pm a}\\&=-\nabla_b l_\pm^c\ \nabla_c l_{\pm a}+l_\pm^c[\nabla_c,\nabla_b]l_{\pm a}\\&=-B^{\pm c}{}_b\ B_{ac}^{\pm}+\overline{R}_{cb,a}{}^d(C)l_{\pm d}l_\pm^c-2l_\pm^c H_{cb}B_{ae}^{\pm}.\end{aligned} \quad (18)$$

This uses the fact that the parallel transport of a tangent vector along its corresponding geodesic vanishes: $l_\pm^c\nabla_c l_{\pm b}=0$ (see technical comment below dealing with non-affine parametrizations), plus the commutator identity in Eq. (3).

In contrast to the case of timelike geodesics, the tensor field $B_{ab}^{\pm}$ is not purely spacelike but has in addition, mixed null-spacelike components:

$$\begin{aligned}\nabla_a l_{+b}&=\gamma_a^c\gamma_b^d\nabla_c l_{+d}-l_{+b}\gamma_a^d l_-^c\nabla_d l_{+c}\\&=v_{ab}^{+}-l_{+b}\gamma_a^d l_-^c\ \ \nabla_d l_{+c},\end{aligned} \quad (19)$$

and

$$\begin{aligned}\nabla_a l_{-b}&=\gamma_a^c\gamma_b^d\nabla_c l_{-d}-l_{-b}\gamma_a^d l_+^c\nabla_d l_{-c}\\&=v_{ab}^{-}-l_{-b}\gamma_a^d l_+^c\nabla_d l_{-c}.\end{aligned} \quad (20)$$

which define the purely spatial tensors $v_{ab}^{\pm}=\gamma_a^c\gamma_b^d\nabla_c l_{\pm d}$, which admit the further decomposition as follows ($\gamma^{ab}\gamma_{ba}=2$):

$$v_{ab}^{\pm}=\frac{1}{2}\theta_\pm\gamma_{ab}+\sigma_{ab}^{\pm}+\omega_{ab}^{\pm}, \quad (21)$$

$$\theta_\pm=\gamma^{ab}v_{ab}^{\pm}=g^{ab}\nabla_a l_{\pm b}, \quad (22)$$

$$\sigma_{ab}^{\pm}=v_{(ab)}^{\pm}-\frac{1}{2}\theta_\pm\gamma_{ab}, \quad (23)$$

$$\omega_{ab}^{\pm}=v_{[ab]}^{\pm}, \quad (24)$$

where $\theta_\pm$ is the trace of $v^\pm_{ab}$ and provides the measure of the instantaneous expansion of the cross-sectional area of a bundle of null geodesics, while $\sigma^\pm_{ab}$ and $\omega^\pm_{ab}$ denote the shear, and twist, respectively, and are also purely spatial tensors.

From these relations one may derive rate-of-change equations for the expansion, shear and twist with respect to the corresponding affine parameters $u_\pm$ starting from Eq. (18), though we shall be primarily interested in the rate of change of the expansion $\theta_\pm$ as this equation will play a fundamental role later on when we come to define a generic wormhole throat. So, taking the trace of Eq. (18) yields a generalized version of the Raychaudhuri equation (generalized as it contains the effects of torsion) for the two expansions [one for the (+) congruence, the other for the (−) congruence]:

$$\frac{d\theta_\pm}{du_\pm} = -\frac{1}{2}\theta_\pm{}^2 - \sigma^{\pm ab}\sigma_{\pm ab} + \omega^{\pm ab}\omega_{\pm ab} - R^d_c(\Gamma) l^c_\pm l_{\pm d} - 2H^d_{cb}B^{\pm b}{}_d l^c_\pm + H_{eac}H^{ead} l^c_\pm l_{\pm d}. \tag{25}$$

With a view to applications for deriving the energy conditions associated with generic wormhole throats, it is useful to have at hand the companion equation governing the rate of change of the twist along null geodesics. This is derived by going back to Eq. (18), antisymmetrizing on the free indices and projecting out the purely spatial part of the resulting equation. These two operations yield a generalization of the twist equation [again, one for the (+) congruence, the other for the (−) congruence]:

$$\frac{d\omega^\pm_{ba}}{du_\pm} = -\theta_\pm \omega^\pm_{ba} - 2\sigma^{\pm c}{}_{[a}\omega^\pm_{b]c} + \tilde{\nabla}_c H^d_{ab} l^c_\pm l_{\pm d} + H^e_{c[a}H^d_{b]e} l^c_\pm l_{\pm d} - 2l^{\pm c}H^e_{c[b}B^\pm{}_{a]e}. \tag{26}$$

The term linear in H that appears in both the expansion and twist equations is purely geometrical in origin, arising as it does, from the commutator of two torsion-bearing covariant derivatives (3). The other torsion contributions are dynamic in origin, as these arise instead directly from the action and equations of motion. These features distinguish the torsion from all other fields. Of course, in the absence of torsion, these reduce to the standard Raychaudhuri and twist equations, for $\theta_\pm$ and $\omega^\pm$, respectively [18,14].

Technical aside: if one is working with a non-affine parameterization for the null congruences, then the parallel transport equation becomes $l^c_\pm \nabla_c l_{\pm b} = \mathcal{K}_\pm l_{\pm b}$ where $\mathcal{K}_\pm = -l^a_\mp l^b_\pm \nabla_b l_{\pm a}$. The expansion is still given by the trace of the spatial part of $\nabla_a l_{\pm b}$ and we have that $\theta_\pm = \gamma^{ab}v^\pm_{ab} = g^{ab}\nabla_a l_{\pm b} - \mathcal{K}_\pm$. The Raychaudhuri equation (25) will then pick up an extra factor of $\mathcal{K}_\pm \theta_\pm$ [17].

IV. DEFINITION OF GENERIC WORMHOLE THROATS

Our aim is to provide a precise, local, and robust geometric definition of a (traversable) wormhole throat, equally valid for static as well as time-dependent wormholes. As a guide, we recall that in the generic but *static* case, the throat was defined as a two-dimensional hypersurface of minimal area [5,6]. The time independence allows one to locate that minimal hypersurface entirely within one of the constant-time three-dimensional spatial slices, and the conditions of extremality and minimality can be applied and enforced within that single time slice. For a static throat, variational principles involve performing arbitrary time-independent surface deformations of the hypersurface in the remaining spatial direction orthogonal to the hypersurface, which can always be taken to be locally Gaussian. By contrast, in the time-dependent case, it may not be possible to locate the entire throat within one time slice, as the dynamic throat is an extended object in spacetime, and the variational principle must be carried out employing surface deformations in the two independent *null* directions orthogonal to the hypersurface: say, δu_+ and δu_-. This, by the way, suggests why it is that the embedding of the spatial part of a wormhole spacetime in an Euclidean $\mathbf{R}^3$ is no longer a reliable operational technique for defining ‘‘flare-out’’ in the time-dependent case. Of course, in the static limit these two variations will no longer be independent and arbitrary deformations in the two null directions reduce to a single variation in the constant-time spatial direction (see below). Realizing that the time-dependent wormhole typically has two non-coincident throats was perhaps the major conceptual stumbling block to overcome in developing this formalism.

A. Preliminaries

In the following, we set up and define the properties of throats in terms of the null congruences. Bear in mind that a throat will be characterized in terms of the behavior of a single set of null geodesics orthogonal to it. We define a wormhole throat Σ_{u+} (there is also one for the other null congruence) to be a closed two-dimensional hypersurface of minimal area taken in one of the constant-u_+ slices, where u_+ is an affine parameter suitable for parametrizing the future-directed null geodesics l_+ orthogonal to Σ_{u+}. All this means is that we imagine ‘‘starting’’ off a collection of light pulses along the hypersurface and we can always arrange the affine parametrizations of each pulse to be equal to some constant on the hypersurface; we take this constant to be zero. We wish to emphasize that there is a corresponding definition for the other throat Σ_{u-}. In the following, we define and develop the conditions that both hypersurfaces must satisfy individually to be considered as throats, and shall do so in a unified way by treating them together by employing the $\pm$ label. Our next task is to compute the hypersurface areas and impose the conditions of extremality and minimality directly and to express these constraints in terms of the expansion of the null geodesics. The area of $\Sigma_{u\pm}$ is given by

$$A(\Sigma_{u\pm}) = \int_{\Sigma_{u\pm}} \sqrt{\gamma}\, d^2x. \tag{27}$$

An arbitrary variation of the surface with respect to deformations in the null direction parametrized by $u_\pm$ is

$$\delta A(\Sigma_{u\pm}) = \int_{\Sigma_{u\pm}} \frac{d\sqrt{\gamma}}{du_\pm}\,\delta u_\pm(x)\; d^2x$$
$$= \int_{\Sigma_{u\pm}} \sqrt{\gamma}\,\frac{1}{2}\,\gamma^{ab}\frac{d\gamma_{ab}}{du_\pm}\,\delta u_\pm(x)\; d^2x. \qquad (28)$$

If this is to vanish for arbitrary variations $\delta u_\pm(x)$, then we must have that

$$\frac{1}{2}\,\gamma^{ab}\frac{d\gamma_{ab}}{du_\pm}=0, \qquad (29)$$

which expresses the fact that the hypersurface $\Sigma_{u\pm}$ is extremal.

This condition of hypersurface extremality can also be phrased equivalently and directly in terms of the expansion of the null congruences. The simplest way to do so is to consider the Lie derivative $\mathcal{L}_l^\pm$ acting on the full spacetime metric:

$$\mathcal{L}_l^\pm g_{ab} = l^c_\pm \nabla_c g_{ab} + g_{cb}\nabla_a l^c_\pm + g_{ac}\nabla_b l^c_\pm = \nabla_a l_{\pm b} + \nabla_b l_{\pm a}$$
$$= B^\pm{}_{ba} + B^\pm{}_{ab} = 2B^\pm{}_{(ab)}, \qquad (30)$$

with the second equality holding provided the metric is covariantly constant with respect to the full covariant derivative, which is in fact the case, even in the presence of arbitrary torsion. We now use the decomposition (16) of the spacetime metric and work out the Lie derivative using the Leibnitz rule:

$$B^\pm{}_{(ab)} = \frac{1}{2}\mathcal{L}_l^\pm g_{ab} = \frac{1}{2}\mathcal{L}_l^\pm(\gamma_{ab} - l_{-a}l_{+b} - l_{+a}l_{-b}),$$
$$= \frac{1}{2}\mathcal{L}_l^\pm \gamma_{ab} - \frac{1}{2}[l_{-a}\mathcal{L}_l^\pm l_{+b} + l_{+b}\mathcal{L}_l^\pm l_{-a} + (a\leftrightarrow b)],$$
$$(31)$$

from which, and using the properties in Eq. (15), implies

$$\theta_\pm = g^{ab}B^\pm{}_{(ab)} = \gamma^{ab}B^\pm{}_{(ab)} = \gamma^{ab}v^\pm_{ab} = \frac{1}{2}\,\gamma^{ab}\mathcal{L}_l^\pm \gamma_{ab}$$
$$= \frac{1}{2}\,\gamma^{ab}\frac{d\gamma_{ab}}{du_\pm}. \qquad (32)$$

So the condition that the area of the hypersurface be extremal is simply that the expansion of the null geodesics vanish at the surface: $\theta_\pm = 0$. To ensure that the area be *minimal*, we need to impose an additional constraint and shall require that $\delta^2 A(\Sigma_{u\pm}) \geqslant 0$. By explicit computation,

$$\delta^2 A(\Sigma_{u\pm}) = \int_{\Sigma_{u\pm}} \sqrt{\gamma}\left(\theta_\pm{}^2 + \frac{d\theta_\pm}{du_\pm}\right)\delta u_\pm(x)\,\delta u_\pm(x)\,d^2x$$
$$= \int_{\Sigma_{u\pm}} \sqrt{\gamma}\,\frac{d\theta_\pm}{du_\pm}\,\delta u_\pm(x)\,\delta u_\pm(x)\; d^2x \geqslant 0, \qquad (33)$$

where we have used the extremality condition ($\theta_\pm = 0$) in arriving at this last inequality. For this to hold at the throat for arbitrary variations $\delta u_\pm(x)$, and since $(\delta u_\pm(x))^2 \geqslant 0$, we must have

$$\frac{d\theta_\pm}{du_\pm} \geqslant 0, \qquad (34)$$

in other words, the expansion of the cross-sectional area of the future-directed null geodesics must be locally increasing at the throat. This is the precise generalization of the Morris-Thorne ''flare-out'' condition to arbitrary wormhole throats. This makes eminent good sense since the expansion is the measure of the cross-sectional area of bundles of null geodesics, and a positive derivative indicates that this area is locally increasing or ''flaring-out'' as one moves along the null direction. Note that this definition is free from notions of embedding and ''shape'' functions. So in general, we have to deal with two throats: Σ_{u+} such that $\theta_+ = 0$ and $d\theta_+/du_+ \geqslant 0$ and Σ_{u-} such that $\theta_- = 0$ and $d\theta_-/du_- \geqslant 0$. We shall soon see that for static wormholes the two throats coalesce and this definition automatically reduces to the static case considered in [5,6]. The logical development in the present paper closely parallels that of the static case though there are many technical differences.

The conditions that a wormhole throat be both extremal and minimal are the simplest requirements that one would want a putative throat to satisfy and which may be summarized in the following definition (in the following, the hypersurfaces are understood to be closed and spatial). Since these definitions hold of course for both throats, we momentarily drop the distinction and suppress the $\pm$ label.

1. Definition: Simple flare-out condition

A two-surface satisfies the ''simple flare-out'' condition if and only if it is extremal, $\theta = 0$, *and also satisfies* $d\theta/du \geqslant 0$. The characterization of a generic wormhole throat in terms of the expansion of the null geodesics shows that any two-surface satisfying the simple flare-out condition is a *marginally anti-trapped surface*, where the notion of trapped surfaces is a familiar concept that arises primarily in the context of singularity theorems, gravitational collapse and black hole physics [14,18]. We hasten to point out however, that in the present context, identifying a wormhole throat as a marginally anti-trapped surface in no way, shape or form is meant to convey that we are dealing with horizons, apparent horizons, or singularities. Nor should this nomenclature suggest that wormholes are somehow allied with or are analogous to black holes or white holes. (For some special cases where wormholes do have applications in black hole physics, see [6].)

Generically, we would expect the inequality $\delta^2 A(\Sigma_u) > 0$ to be strict, so that the surface is truly a minimal (not just extremal) surface. This will pertain provided the inequality $d\theta/du > 0$ is a strict one for at least *some* points on the throat. This suggests the following definition.

2. Definition: Strong flare-out condition

A two-surface satisfies the ''strong flare-out'' condition at the point x if and only if it is extremal, $\theta=0$*, satisfies* $d\theta/du \geqslant 0$ *everywhere on the surface and if at the point x, the inequality is strict:*

$$\frac{d\theta}{du}>0. \tag{35}$$

If the latter strict inequality holds for all $x \in \Sigma_u$ in the surface, then the wormhole throat is seen to correspond to a *strongly anti-trapped surface*. Again, this terminology is not intended to convey any relation between wormholes and black holes. The physical distinction between simple and strong flare-out will become evident when we come to explore the consequences these definitions have on the energy conditions required to maintain a generic traversable wormhole throat. It is sometimes sufficient and convenient to work with a weaker, integrated form of the flare-out condition.

3. Definition: Averaged flare-out condition

A two-surface satisfies the ''averaged flare-out'' condition if and only if it is extremal, $\theta=0$*, and*

$$\int_{\Sigma_u} \sqrt{\gamma}\ \mathrm{sgn}\left(\frac{d\theta}{du}\right) d^2x>0, \tag{36}$$

where $\mathrm{sgn}(x)$ is the sign of x. This averaged flare-out condition places a constraint on the putative throat by asking that the extremal surface be outward flaring over at least half its area before one can be justified in calling it a wormhole throat. This definition has been carefully constructed to remain invariant under arbitrary affine reparametrizations of the null geodesic congruence. An apparently plausible alternative to the above, using the integral $\mathcal{I} \equiv \int_{\Sigma_u} \sqrt{\gamma}\ (d\theta/du) d^2x$, is deficient in that if the integrand $d\theta/du$ changes sign anywhere on the surface Σ then by appropriate affine reparametrizations of the null geodesic congruence the integral may be made arbitrarily positive or arbitrarily negative [19]. (Thus if one were to require the integral $\mathcal{I}$ to be positive for all affine parametrizations, one would simply recover the strong flare-out condition, while if we were to merely require that the integral $\mathcal{I}$ be positive for at least one choice of affine parameterization we would have the extremely weak constraint that $d\theta/du$ be positive for at least one point on the surface Σ. Either option though mathematically consistent is physically unreasonable, and the definition in terms of the sgn function is the best intermediate strength definition we have found. This comment also implies that constraints on weighted averages of the form $\int_{\Sigma_u} \sqrt{\gamma}\ f(x)\ (d\theta/du) d^2x$ are too subject to reparametrization effects to be useful.)

The conditions under which the average flare-out are appropriate arise for example for situations with multiple throat wormholes. Indeed, suppose we have a double throat wormhole where each of the two throats are flared-out in the strong sense. Then the spacetime between the throats contains an extremal hypersurface which is not minimal, but which can be minimal in the integrated, averaged sense. (See, e.g., [5,6].) Independently from this, averaged flare-out conditions of various types crop up in energy conditions averaged over the hypersurface [5,6].

Finally, it is also useful to define a weighted flare-out condition.

4. Definition: Averaged f-weighted flare-out condition

A two-surface satisfies the ''f-weighted flare-out'' condition if and only if it is extremal, $\theta=0$*, and*

$$\int_{\Sigma_u} \sqrt{\gamma} f(x)\ \mathrm{sgn}\left(\frac{d\theta}{du}\right) d^2x>0, \tag{37}$$

where f is a positive definite function defined on the two-surface.

Note that the strong flare-out condition implies both the simple flare-out condition and the averaged flare-out condition, but the simple flare-out condition does not necessarily imply the averaged flare-out condition (the integral might vanish). However, we see that if the averaged f-weighted flare-out condition is satisfied for all positive definite f, then it implies the simple flare-out condition, which follows from identifying $f(x)=\delta u(x)^2 \geqslant 0$ and using the minimality constraint (33).

5. Technical aside: degenerate throats

A class of wormholes for which we have to extend these definitions arises when the wormhole throat possesses an accidental degeneracy in the expansion of the null geodesics at the throat. The above discussion has been tacitly assuming that in the vicinity of the throat we can Taylor expand the expansion

$$\theta(x,u)=\theta(x,0)+u\left(\left.\frac{d\theta(x,u)}{du}\right|_{u=0}\right)+O(u^2), \tag{38}$$

with the constant term vanishing by the extremality constraint and the first derivative term being constrained by the flare-out conditions.

Now if the extremal two-surface has an accidental degeneracy with the first derivative term (and possibly higher-order terms) vanishing identically, then we would have to develop the above expansion further out to the first non-vanishing term. This would mean we would have to rephrase the flare-out in terms of these higher-order derivatives of the null geodesic expansion. In fact, the first non-vanishing term would appear at odd order in u:

$$\theta(x,u)=\frac{u^{2N-1}}{(2N)!}\left(\left.\frac{d^{2N-1}\theta(x,u)}{du^{2N-1}}\right|_{u=0}\right)+O(u^{2N}), \tag{39}$$

since the surface is by definition extremal. It must be odd in u otherwise the throat would be a point of inflection and not a true minimum of the area. Simply put, even-order surface deformations involve odd-order derivatives of the expansion. We can see this in another way by computing higher-order variations in the area. The condition that it be a minimum is

$$\delta^{2N}A(\Sigma_u)=\int_{\Sigma_u}\sqrt{\gamma}\,\frac{d^{2N-1}\theta}{du^{2N-1}}\big(\delta u(x)\big)^{2N}\,d^2x>0, \quad (40)$$

which leads to the flare-out condition being stated in terms of the $(2N-1)$-th derivative of the expansion. Note: for $N=1$, this reduces to the minimality constraint in (33). This motivates the following definition.

6. Definition: N-fold degenerate flare-out condition

A two-surface satisfies the N-fold degenerate flare out condition if and only if it is extremal, $\theta=0$, *the first* $(2N-2)$ *u-derivatives of* θ *vanish,* $(d^{2N-1}\theta(x,u)/du^{2N-1})\geqslant 0$ *everywhere on the surface and if finally, for at least some point x on the surface, the inequality is strict:*

$$\frac{d^{2N-1}\theta}{du^{2N-1}}>0. \quad (41)$$

Physically, at an N-fold degenerate point, the wormhole throat is seen to be extremal up to order $2N-1$ with respect to the derivatives of the expansion, i.e., the flare-out condition is delayed in the (outgoing) null direction with respect to throats in which the flare-out occurs at $N=1$, which (by the way we have set up the definition) corresponds to the strong flare-out condition.

These considerations bring us to the following surprising result already alluded to above: namely, there is no *a priori* reason for the two independent null variations $\delta u_+, \delta u_-$ to single out the *same* minimal hypersurface. That is, in general

$$\Sigma_{u+}\neq\Sigma_{u-}, \quad (42)$$

and we must conclude that generic time-dependent wormholes possess two throats. If these hypersurfaces are in causal contact then it will be possible to enter the wormhole via one throat and exit through the other. If the two throats are not in causal contact then the wormhole is not two-way traversable, and you have at best two one-way traversable wormholes with no way of getting back to where you started from.

B. Static limit

In a static spacetime, a wormhole throat is a closed two-dimensional spatial hypersurface of minimal area that, without loss of generality, can be located entirely within a single constant-time spatial slice [5,6]. Now, for any static spacetime, one can always decompose the spacetime metric in a block-diagonal form as

$$g_{ab}=-V_aV_b+{}^{(3)}g_{ab}, \quad (43)$$

where $V^a=\exp[\phi](\partial/\partial t)^a$ is a timelike vector field orthogonal to the constant-time spatial slices and ϕ is some function of the spatial coordinates only. In the vicinity of the throat we can always set up a system of Gaussian coordinates n so that

$${}^{(3)}g_{ab}=n_an_b+\gamma_{ab}, \quad (44)$$

where $n^a=(\partial/\partial n)^a$, $n^an_a=+1$, and γ_{ab} is the two-metric of the hypersurface. Putting these facts together implies that in the vicinity of any static throat we may write the spacetime metric as

$$g_{ab}=-V_aV_b+n_an_b+\gamma_{ab}. \quad (45)$$

But Eq. (16) holds in general, so comparing both metric representations yields the identity

$$-l_-^a l_+^b-l_+^a l_-^b=V^aV^b+n^an^b, \quad (46)$$

and the following (linear) transformation relates the two-metric decompositions and preserves the inner-product relations in Eq. (15):

$$l_-^a=\frac{1}{2}(V^a+n^a), \quad l_+^a=\frac{1}{2}(V^a-n^a). \quad (47)$$

Since the throat is static, γ_{ab} is time independent, hence when we come to vary the area (27) with respect to arbitrary perturbations in the two independent null directions we find that

$$\frac{\partial\gamma_{ab}}{\partial u_+}\delta u_+=\frac{1}{2}\left(\exp[\phi]\frac{\partial\gamma_{ab}}{\partial t}\delta t+\frac{\partial\gamma_{ab}}{\partial n}\delta n\right)=\frac{1}{2}\frac{\partial\gamma_{ab}}{\partial n}\delta n,$$
$$\frac{\partial\gamma_{ab}}{\partial u_-}\delta u_-=\frac{1}{2}\left(\exp[\phi]\frac{\partial\gamma_{ab}}{\partial t}\delta t-\frac{\partial\gamma_{ab}}{\partial n}\delta n\right)$$
$$=-\frac{1}{2}\frac{\partial\gamma_{ab}}{\partial n}\delta n. \quad (48)$$

Thus the variations are no longer independent, and reduce to taking a single surface variation in the spatial Gaussian direction. So, $\theta_+=0\Leftrightarrow\theta_-=0$ at the same hypersurface, proving that $\Sigma_{u_+}=\Sigma_{u_-}$ in the static limit, and so static wormholes have only one throat. An exhaustive analysis of the geometric structure of the generic static traversable wormhole may be found in [5,6].

With the definition of wormhole throat made precise we now turn to derive constraints that the stress energy tensor must obey on (or near) any wormhole throat. The constraints follow from combining the Raychaudhuri equation (25) with the flare-out conditions, and using the Einstein equation (11). It is clear that these constraints apply with equal validity at both the + and − throats, and in the following we cover both classes simultaneously and without risk of confusion by dropping the ± labels. We first treat the zero-torsion case.

C. Zero torsion

Since all throats are extremal hypersurfaces ($\theta=0$) the Raychaudhuri equation at the throat (25) reduces to

$$\frac{d\theta}{du}+\sigma^{ab}\sigma_{ab}=-8\pi T_{ab}\; l^al^b, \quad (49)$$

where we have used the Einstein equation (11) after setting the torsion terms to zero and the fact that the null geodesic congruences are hypersurface orthogonal, so that the twist $\omega_{ab}=0$ vanishes identically on the throat. We make no claim regarding the shear, except to point out that since σ_{ab} is purely spatial, its square $\sigma^{ab}\sigma_{ab}\geqslant 0$ is positive semi-definite everywhere (not just on the throat). Consider a marginally anti-trapped surface, i.e., a throat satisfying the simple flare-out condition. Then the stress energy tensor on the throat must satisfy

$$T_{ab}\ \ l^a l^b \leqslant 0. \tag{50}$$

The NEC is therefore either violated, or on the verge of being violated ($T_{ab}\ \ l^a l^b \equiv 0$), on the throat. Of course, whichever one of the two null geodesic congruences (l_+ or l_-) you are using to define the wormhole throat (anti-trapped surface), you must use the *same* null geodesic congruence for deducing null energy condition violations.

For throats satisfying the strong flare-out condition, we have instead the stronger statement that for all points on the throat,

$$T_{ab} l^a l^b \leqslant 0, \quad \text{and } \exists x \in \Sigma_u \quad \text{such that} \quad T_{ab}\ \ l^a l^b < 0, \tag{51}$$

so that the NEC is indeed violated for at least *some* points lying on the throat. By continuity, if $T_{ab}\ \ l^a l^b<0$ at x, then it is strictly negative within a finite open neighborhood of x: $B_\epsilon(x)$. For throats that are strongly anti-trapped surfaces, we derive the most stringent constraint stating that

$$T_{ab}\ \ l^a l^b < 0 \quad \forall x \in \Sigma_u, \tag{52}$$

so that the NEC is violated *everywhere* on the throat.

Weaker, integrated energy conditions are obtained for throats satisfying the averaged flare-out conditions. For a throat that is flared-out on the average, integrating the Raychaudhuri equation (49) over the throat implies

$$\int_{\Sigma_u} \sqrt{\gamma}\ \mathrm{sgn}\ (T_{ab}\ \ l^a l^b) d^2x < 0, \tag{53}$$

indicating that the NEC, when averaged over the throat, is strictly violated (Warning: this has nothing to do with the violation of the averaged null energy condition, or ANEC. In the ANEC, the averaging is defined to take place along inextendible null geodesics. See, in particular, [9].) By the same token, throats satisfying the f-weighted averaged flare-out condition imply that

$$\int_{\Sigma_u} \sqrt{\gamma} f(x)\ \mathrm{sgn}\ (T_{ab}\ \ l^a l^b)\ d^2x < 0, \tag{54}$$

indicating that the sign of the NEC, when weighted with the positive definite function $f(x)$ is strictly violated on the average over the throat.

What can we say about the energy conditions in the region surrounding the throat? This requires knowledge of the expansion, shear, and twist in the neighborhood of the throat. Luckily, we can dispense with the twist immediately. Indeed, the (torsion-free) twist equation (26) is a simple, first-order linear differential equation:

$$\frac{d\omega_{ba}}{du} = -\theta\omega_{ba} - 2\sigma^c_{[a}\omega_{b]c}, \tag{55}$$

whose exact solution (if somewhat formal in appearance) is

$$\omega_{ab}(u) = \exp\left(-\int_0^u \theta(s)ds\right)\ \mathcal{U}_a{}^c(u)\ \ \mathcal{U}_b{}^d(u)\omega_{cd}(0), \tag{56}$$

where the quantity $\mathcal{U}(u)$ denotes the path-ordered exponential

$$\mathcal{U}_a{}^c(u) = \mathcal{P}\ \exp\left(-\int_0^u \sigma\ \ ds\right)_a{}^c. \tag{57}$$

So, an initially hypersurface orthogonal congruence remains twist-free everywhere, both on and off the throat: $\omega_{ba}(0)=0 \Rightarrow \omega_{ba}(u)=0$. Then the equation

$$\frac{d\theta}{du} + \frac{1}{2}\theta^2 + \sigma^{ab}\sigma_{ab} = -8\pi T_{ab}\ \ l^a l^b, \tag{58}$$

is seen to be valid for all u. Coming back to simply-flared throats, we have two pieces of information regarding the expansion: namely that $\theta(0)=0$ and $[d\theta(u)/du]_{u=0}\geqslant 0$, so that if we expand θ in a neighborhood of the throat as in Eq. (38), then we have that

$$\frac{d\theta(u)}{du} = \left.\frac{d\theta(u)}{du}\right|_{u=0} + O(u), \tag{59}$$

so over each point x on the throat, there exists a finite range in affine parameter $u\in(0,u_x^*)$ for which $d\theta(u)/du \geqslant 0$. Since both θ^2 and $\sigma^{ab}\sigma_{ab}$ are positive semi-definite, we conclude that the stress energy is either violating, or on the verge of violating, the NEC along the partial null curve $\{x\}\times(0,u_x^*)$ based at x. If the throat is of the strongly flared variety, then we see that the NEC is definitely violated at least over some finite regions surrounding the throat: $\cup_x\{x\}\times(0,u_x^*)$, and including the base points x. For strongly anti-trapped surfaces, the NEC is violated everywhere in a finite region surrounding the entire throat, and including the throat itself.

Finally, if the throat is N-fold degenerate (and $N>1$), then there exist points x on the throat for which $(d^{2N-1}\theta(x,u)/du^{2N-1})|_{u=0}>0$. This implies that the first derivative

$$\frac{d\theta(x,u)}{du} = \frac{(2N-1)u^{2N-2}}{(2N)!}\left.\frac{d^{2N-1}\theta(x,u)}{du^{2N-1}}\right|_{u=0} + O(u^{2N-1}), \tag{60}$$

is positive along a partial null curve $\{x\}\times(0,u_x^*)$ based at x and it follows by Eq. (49) that the NEC is violated along the finite ‘‘bristles’’ $\cup_x\{x\}\times(0,u_x^*)$.

TABLE I. Summary of the flare-out conditions for wormhole throats; all quantities are evaluated on the throat. The flare-out conditions are understood to apply to both throats, and we drop the $\pm$ label.

Flare-out condition	Expansion	Constraints on the throat
Simple	$\theta=0$	$\frac{d\theta}{du}\geqslant 0$
Strong	$\theta=0$	$\frac{d\theta}{du}\geqslant 0$, and $\exists x\in\Sigma_u \;\; \frac{d\theta}{du}>0$
Strongly anti-trapped	$\theta=0$	$\forall x\in\Sigma_u, \;\; \frac{d\theta}{du}>0$
Averaged	$\theta=0$	$\int_{\Sigma_u}\sqrt{\gamma}\,\text{sgn}\,(\frac{d\theta}{du})d^2x>0$
f-averaged	$\theta=0$	$\int_{\Sigma_u}\sqrt{\gamma}f(x)\,\text{sgn}\,(\frac{d\theta}{du})d^2x>0$, for an $f(x)\geqslant 0$
N-fold degenerate	$\theta=0$	$\frac{d^m\theta}{du^m}=0$, for $m=1,\cdots,2N-2$ and $\frac{d^{2N-1}\theta}{du^{2N-1}}\geqslant 0$

D. Non-zero H-torsion

Torsion, although contributing additional terms to the Einstein (11) and Raychaudhuri equations (25) does not necessarily alleviate the problem of the violation of the NEC on or near wormhole throats. This state-of-affairs holds at both throats so without loss of generality, take the $(+)$ throats and consider the term linear in H that appears in Eq. (25). This can be simplified as follows:

$$l^c_+ \;\; H^d_{cb} \;\; B^{+b}{}_d = l^c_+ \;\; H^d_{cb}(v^{+b}_d - l^b_+\gamma^e_d l^c_- \nabla_e l_{+c}),$$
$$= l^c_+ \;\; H^d_{cb} \;\; \omega^{+b}_d \tag{61}$$

since the mixed spatial-null components of $B^{+b}_{\;\;d}$ are orthogonal to H^d_{cb}, and by virtue of the latter's antisymmetry, projects out the twist from the purely spatial tensor v^{+b}_d. Now consider an initially hypersurface orthogonal null congruence, then at the throat of the wormhole we have

$$\frac{d\theta_+}{du_+} + \sigma^{+ab}\sigma^+{}_{ab} = -8\pi T_{ab} \;\; l^a_+ l^b_+ - 2H_{ade}H^{de}_b \;\; l^a_+ l^b_+, \tag{62}$$

after using the expression for the Ricci tensor in Eq. (13).

We could now run through the list of flare-out conditions (see Table I) as before and we would obtain, as expected, constraints on the combination of stress energy and torsion appearing on the right-hand side of Eq. (62). Thus, for a simply-flared throat, or marginally anti-trapped surface, we must have

$$4\pi T_{ab} \;\; l^a_+ l^b_+ + H_{ade}H^{de}_b \;\; l^a_+ l^b_+ \leqslant 0, \tag{63}$$

at the throat and one might propose sweeping the violations into the torsion sector. We will find that this is not possible. For illustrative purposes, suppose we consider the ansatz

$$H_{abc} = \frac{1}{\sqrt{-g}}\epsilon_{abce}w^e(x), \tag{64}$$

for any vector field w^e. Then the combination

$$H_{ade}H^{de}_b \;\; l^a_+ l^b_+ = +2(w^a l_{+a})^2 \geqslant 0, \tag{65}$$

is positive definite for all w. Such a torsion-field aggravates the violation of the NEC and all of the above constraints on the stress tensor derived at and near the throat in the zero-torsion case apply as well to throats in the presence of this class of non-zero torsion. Actually, with a little more work, it is possible to relax the assumption of total antisymmetry and demonstrate that *all* torsion leads to enhanced violation of the NEC. To see why this comes about first consider the general decomposition of an arbitrary antisymmetric rank-two tensor $A_{ab}=-A_{ba}$ in terms of null vectors and spatial projector. We find that we can write

$$A_{ab} = a l_{-[a}l_{+b]} + \gamma^c_{[a}\gamma^d_{b]}A_{cd} + 2l_{-[a}l^c_+\gamma^d_{b]}A_{dc}$$
$$+ 2l_{+[a}l^c_-\gamma^d_{b]}A_{dc}, \tag{66}$$

where the coefficient $a=-2l^c_- l^d_+ A_{cd}$. Now evaluate this for $A_{de}=l^a_+ H_{ade}$. One finds that $a=-2l^a_- l^b_+ l^c_+ H_{cab}=0$. The third term above also vanishes since $l_{-[a}\gamma^d_{b]}l^c_+ l^e_+ H_{edc}=0$, which leaves us with

$$A_{ab} = \tilde{A}_{ab} + 2l_{+[a}l^c_-\gamma^d_{b]}A_{cd}, \tag{67}$$

where $\tilde{A}_{ab}=\gamma^c_{[a}\gamma^d_{b]}A_{dc}$ is a purely spatial tensor. Now, the square of Eq. (67) involves only the purely spatial components:

$$A_{de}A^{de} \equiv l^a_+ H_{ade} l^b_+ H^{de}_b = \tilde{A}_{de}\tilde{A}^{de} \geqslant 0, \tag{68}$$

and this is precisely the combination appearing in Eq. (63). Thus, the torsion terms cannot be made to absorb any energy violations. On the contrary, torsion tends to focus null geodesics. While the ''normal'' stress energy must continue to violate (or be on the verge of violating) the NEC on the throat, the presence of any non-zero torsion does act to lessen the violation off the throat. This is simply because torsion acts as a source of the twist, and even if the twist vanishes on the throat, nonvanishing twist is eventually generated in the neighborhood surrounding the throat, as can be appreciated by examining Eq. (26), and twist comes in with the just the right sign in the Raychaudhuri equation. Of course, without further input, we have no way of knowing if this happens in the region near the throat or far away from the throat. If it occurs near the throat, then the energy violations in that region might be (partially) absorbed into the twist, but the violation persists nonetheless.

V. CONFORMALLY EXPANDING MORRIS-THORNE WORMHOLE

We shall illustrate these basic concepts and constructs with the following explicit example. Consider the time-dependent spherically symmetric inter-universe wormhole described by a pair of coordinate patches in which the metric takes the form

$$ds^2=\Omega^2(t)\left(-dt^2+\frac{dr_{1,2}^2}{1-b(r_{1,2})/r_{1,2}}+r_{1,2}^2[d\theta^2+\sin^2\theta\; d\phi^2]\right). \tag{69}$$

This metric is conformally related to a zero-tidal force inter-universe Morris-Thorne wormhole by a simple time-dependent but space-independent conformal factor [20–22]. (Other versions of time-dependent wormholes are discussed in [23–25].) Each coordinate system used to exhibit the metric given above covers only half the wormhole spacetime, and there are two radial coordinates, r_1 and r_2, each of which runs only from r_0 to infinity, where r_0 is obtained by solving the implicit equation $b(r_0)=r_0$. See [1,3]. The two radial coordinates cover two distinct universes and overlap only at $r_1=r_0=r_2$ which defines the *center* of the wormhole (we will find that the center coincides with the throat only in the static limit). For simplicity this wormhole is taken to be symmetric under interchange of the two asymptotically flat regions but this is not essential to the analysis.

It should be clear that we look for throats within *each* coordinate patch separately. We will see below that for suitable energy conditions, the above metric corresponds to a wormhole with two time-dependent throats, each throat residing in one of the two universes joined by the wormhole.

A. First coordinate patch

The throats, when and if they exist, will be located on spheres of (instantaneous) radii $\Omega(t)r_1$ (where $r_1\geqslant r_0$) possessing the spatial metric (written in block-diagonal form)

$$\gamma_{1ab}=\Omega^2\;\; r_1^2\;\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \begin{bmatrix}1 & 0\\ 0 & \sin^2\theta\end{bmatrix}\end{pmatrix}. \tag{70}$$

We can easily find a set of two independent null vectors orthogonal to the spheres in this patch; they are given by

$$l_\pm^a=\frac{1}{\sqrt{2}\Omega}\left(1,\pm\left(1-\frac{b(r_1)}{r_1}\right)^{1/2},0,0\right), \tag{71}$$

and it is easy to verify that all the inner-product relations (15) are satisfied and that the metric (69) in this patch can be decomposed in terms of $l_{+a}, l_{-a}, \gamma_{1ab}$ just as in Eq. (16). The expansions of these null geodesics are calculated in a straightforward manner:

$$\theta_\pm=\gamma_1^{ab}\nabla_a l_{\pm b}=\sqrt{2}\frac{\dot\Omega}{\Omega^2}\pm\frac{\sqrt{2}}{r_1\Omega}\left(1-\frac{b(r_1)}{r_1}\right)^{1/2}, \tag{72}$$

where the overdot stands for the derivative with respect to (conformal) time t. The derivatives, taken with respect to the affine parameter, used for testing for flare-out are $(d\theta_+/du_+=l_+^t\partial\theta_+/\partial t+l_+^r\partial\theta_+/\partial r)$, etc.,

$$\frac{d\theta_\pm}{du_\pm}=\frac{1}{\Omega^2}\left[\left(\frac{\ddot\Omega}{\Omega}-2\frac{\dot\Omega^2}{\Omega^2}\right)\mp\frac{\dot\Omega}{r_1\Omega}\left(1-\frac{b(r_1)}{r_1}\right)^{1/2}-\frac{1}{r_1^2}\left(1-\frac{b(r_1)}{r_1}\right)+\frac{1}{2r_1^2}\left(-b'(r_1)+\frac{b(r_1)}{r_1}\right)\right]. \tag{73}$$

Now we can search for throats in this patch. First we locate the extremal hypersurfaces; these coincide with the zeroes of the expansions:

$$\theta_\pm=0\Leftrightarrow\frac{1}{r_1}\left(1-\frac{b(r_1)}{r_1}\right)^{1/2}=\mp\frac{\dot\Omega}{\Omega}, \tag{74}$$

which defines the time-dependent throat radius $r_1^*(t)$ implicitly. We note that the factor involving the square root is always positive semi-definite, hence we find that (in the r_1 coordinate patch) it is only θ_- that can vanish for an expanding ($\dot\Omega>0$) background, while it is θ_+ that can vanish for a collapsing ($\dot\Omega<0$) background. There is, therefore, always only one extremal hypersurface in the first patch.

Irrespective of expansion or collapse, the flare out evaluated on that extremal hypersurface works out to be

$$\left.\frac{d\theta_\pm}{du_\pm}\right|_{\theta_\pm=0}=\frac{1}{\Omega^2}\left(\left[\frac{\ddot\Omega}{\Omega}-2\frac{\dot\Omega^2}{\Omega^2}\right]+\frac{1}{2r_1^*(t)^2}\left[-b'(r_1^*(t))+\frac{b(r_1^*(t))}{r_1^*(t)}\right]\right). \tag{75}$$

The flare out of the hypersurface is a function of time. Note that the second grouped term on the right-hand side is always greater than or equal to zero while the first grouped term can, in principle, have any sign, depending on the nature of the background expansion (or contraction). This observation was proposed in [21,22] as a means of temporarily suspending the energy condition violations for dynamic wormholes. However, the Einstein tensor associated with the above metric (69) can be easily worked out [21,22] and taking its projection along the radial null direction yields the combination

$$G_{\hat{t}\hat{t}}+G_{\hat{r}\hat{r}}=8\pi(\rho_1-\tau_1)$$
$$=\Omega^{-2}\left(-\frac{b(r_1)}{r_1^3}+\frac{b'(r_1)}{r_1^2}-2\frac{\ddot{\Omega}}{\Omega}+4\frac{\dot{\Omega}^2}{\Omega^2}\right),\tag{76}$$

where ρ_1 and τ_1 denote the energy density and radial tension as seen by an observer in the proper reference frame. Evaluate this at $r_1=r_1^*(t)$ and compare it to Eq. (75) to conclude that any conformal factor that is chosen so as to suspend the violation of the NEC, will at the same time eradicate the flare-out condition:

$$(\rho_1-\tau_1)\geqslant 0\Leftrightarrow\frac{d\theta_\pm}{du_\pm}\Big|_{r_1^*(t)}\leqslant 0,\tag{77}$$

and the hypersurface at $r_1^*(t)$ will *not be flared-out.* In other words, the extremal hypersurface will be a throat of the simply flared-out variety if and only if the NEC is violated or on the verge of being violated there.

This is completely compatible with the topological censorship theorem [9]. If one picks an ingoing radial null geodesic along which the NEC is always satisfied, then by the above argument the expansion can never flare out, one is forced to continue moving inward, and so one cannot pass through a wormhole throat.

B. Second coordinate patch

Many of the results from the first coordinate patch can be carried over to the second coordinate patch with a few key flips in signs. The throats in this second patch, when and if they exist, will be located on spheres of (instantaneous) radii $\Omega(t)r_2$ (with $r_2\geqslant r_0$) possessing the spatial metric (written in block-diagonal form)

$$\gamma_{2ab}=\Omega^2\ \ r_2^2\ \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \begin{bmatrix}1 & 0\\ 0 & \sin^2\theta\end{bmatrix}\end{pmatrix}.\tag{78}$$

We can easily find a set of two independent null vectors orthogonal to the spheres in this patch; they are given by

$$l_\pm^a=\frac{1}{\sqrt{2}\Omega}\left[1,\mp\left(1-\frac{b(r_2)}{r_2}\right)^{1/2},0,0\right].\tag{79}$$

It is easy to verify that the key sign flip above guarantees that the vector fields $l_\pm$ defined as patch one connect smoothly with their definitions on patch two. Furthermore all the inner-product relations (15) are satisfied and the metric (69) in this patch can still be decomposed in terms of l_{+a}, l_{-a}, and γ_{2ab} just as in Eq. (16). Their respective expansions are calculated in a straightforward manner:

$$\theta_\pm=\gamma_2^{ab}\nabla_a l_{\pm b}=\sqrt{2}\frac{\dot{\Omega}}{\Omega^2}\mp\frac{\sqrt{2}}{r_2\Omega}\left(1-\frac{b(r_2)}{r_2}\right)^{1/2}.\tag{80}$$

The search for throats in this second patch proceeds just as above. For the location of the extremal hypersurfaces we now have

$$\theta_\pm=0\Leftrightarrow\frac{1}{r_2}\left(1-\frac{b(r_2)}{r_2}\right)^{1/2}=\pm\frac{\dot{\Omega}}{\Omega},\tag{81}$$

which now defines the throat radius $r_2^*(t)$ implicitly. We again note that the left-hand side is always positive semidefinite, hence we find that it is now θ_+ that vanishes for an expanding background while it is θ_- that vanishes for a collapsing background (in this patch.). Therefore, there is again exactly one extremal hypersurface in this patch. Note that because of the crucial sign flip, whichever of the two expansions it is that vanishes in coordinate patch one, it is the *other* expansion that will now vanish in patch two.

Because of the assumed symmetry between the two patches the rest of the analysis follows through without difficulty and we can again see that any conformal factor Ω that is chosen so as to suspend the violation of the NEC, will at the same *time* eradicate the flare-out condition at this second throat:

$$(\rho_2-\tau_2)\geqslant 0\Leftrightarrow\frac{d\theta_\pm}{dv_\pm}\Big|_{r_2^*(t)}\leqslant 0.\tag{82}$$

Once again, this extremal hypersurface will be a throat of the simply flared-out variety if and only if the NEC is violated or on the verge of being violated there.

[As indicated previously, the assumption that the wormhole is symmetric under the interchange of the two asymptotically flat regions is not essential to the analysis. To generalize this point one just needs to choose two un-equal shape functions $b_1(r_1)$ and $b_2(r_2)$ that need be linked only by the fact that they simultaneously satisfy $b_1(r_0)=r_0=b_2(r_0)$. It is now a simple exercise to go through the preceding formulas making minor changes as appropriate.]

C. Static limit

In the static limit, we have $\dot{\Omega}=0$ and the simultaneous vanishing of the expansions now occurs at the unique point where the two coordinate patches overlap: $b(r_0)=r_0$, this value being none other than the center of the wormhole: therefore, the static wormhole has only one throat, and the throat coincides with the center of the wormhole. We thus recover the zero-tidal force Morris-Thorne wormhole. Reality of the expansions further restrains the b function to satisfy $b(r)\leqslant r$ so that $b'(r_0)\leqslant 1$. The flare-outs of this unique throat with respect to either coordinate patch are

$$\frac{d\theta_\pm}{du_\pm}\Big|_{r_0}=\frac{1}{r_0^2}(-b'(r_0)+1)\geqslant 0,\tag{83}$$

so that the sphere of constant radius r_0 is a throat satisfying the simple flare-out condition and is therefore a marginally anti-trapped surface. It follows immediately from the above theorems, and in complete agreement with the standard analyses, that the NEC is either violated, or on the verge of being violated, at the throat. Note of course, that if these inequalities are strictly positive at any point on the throat, then these derivatives are strictly positive everywhere on the the throat (by spherical symmetry) and the throat satisfies the strong flare-out condition everywhere and is therefore a strongly anti-trapped surface. The NEC is strictly violated in this case.

D. Summary

This worked example shows how important it is to distinguish the ''center'' of the wormhole, defined by looking at the spatial behavior of a fixed time slice, from the throat of the wormhole, defined by the flare-out condition applied to null geodesics that are actually trying to traverse the wormhole.

If the null geodesics ever succeed in getting through the traversable wormhole, into the ''other universe,'' then they must at some stage have passed a region where their expansion satisfied the flare-out condition, and this region is what we define to be the throat of the wormhole. By the analysis of this paper, the NEC must be violated at or near this throat. The ''center'' of the wormhole is the wrong place to look for NEC violations, except in the static limit where the two throats coalesce trapping the center between them.

VI. GENERAL TIME-DEPENDENT SPHERICALLY SYMMETRIC TRAVERSABLE WORMHOLE

The most general metric describing a time-dependent spherically symmetric spacetime can (with appropriate choice of an atlas of coordinate patches) be written as

$$ds^2 = -e^{2\psi}\left(1-\frac{2m}{r}\right)dv^2 + 2e^{\psi}dv\ dr + r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{84}$$

Here $\psi(v,r)$ and $m(v,r)$ are two independent functions of the radial coordinate r and an advanced time-parameter v ($v \approx t+r$ at large r) [26]. This metric can also be adapted to describe an inter-universe wormhole. As in the previous example, the coordinate system employed covers only half the wormhole spacetime and so two patches will be required and the radial coordinate $r \in [r_0, \infty)$, where $r_0(v)$ is again the center of the wormhole. We should then introduce four independent functions: $\psi_{1,2}$ and $m_{1,2}$ where the labels refer to the two coordinate patches. These functions must satisfy a smoothness condition at $r = r_0(v)$ if there is to be no δ function material concentrated on the throat (the extrinsic curvatures should match across the center of the wormhole, see [3]).

In the interest of brevity and notational economy, we will focus on one of the two coordinate patches only. So consider one of the universes joined by the wormhole. A throat, when it exists, will be a sphere of radius $r \geqslant r_0$ with the spatial metric given by Eq. (70) with $\Omega = 1$. The two independent sets of null vectors orthogonal to the sphere are found to be given by

$$l_+^a = \left[1, \frac{1}{2}e^{\psi}\left(1-\frac{2m}{r}\right), 0, 0\right], \quad l_-^a = (0, -e^{-\psi}, 0, 0). \tag{85}$$

The expansions of the associated two sets of null rays are

$$\theta_+ = 2\gamma^{\theta\theta}\nabla_\theta l_{+\theta} = \frac{1}{2}e^{\psi}\left(1-\frac{2m}{r}\right), \tag{86}$$

and

$$\theta_- = 2\gamma^{\theta\theta}\nabla_\theta l_{-\theta} = -\frac{2}{r}e^{-\psi}, \tag{87}$$

respectively. Provided $\psi(r,v)$ is non-singular (a good idea if there are to be no horizons), the only expansion which can have zeros is θ_+ and $\theta_+ = 0 \Leftrightarrow 2m(r,v) = r$, so that $r = r(v)$ gives the time-dependent radius of the extremal sphere.

The flare-out evaluated at this hypersurface is readily calculated to be $[d/du_+ = l_+^a \nabla_a = l_+^v(\partial/\partial v) + l_+^r(\partial/\partial r)]$

$$\left.\frac{d\theta_+}{du_+}\right|_{\theta_+=0} = -\frac{2}{r^2(v)}e^{\psi}\left.\left(\frac{\partial m(r,v)}{\partial v}\right)\right|_{r(v)}. \tag{88}$$

The Einstein equations are easy to work out in this metric. At this throat of the wormhole, the null-null component yields

$$\left.\left(\frac{\partial m(r,v)}{\partial v}\right)\right|_{\theta_+=0} = 4\pi r^2 T_{vv}, \tag{89}$$

so that

$$\left.\frac{d\theta_+}{du_+}\right|_{\theta_+=0} \geqslant 0 \Leftrightarrow T_{vv}|_{\theta_+=0} \leqslant 0. \tag{90}$$

Once again, this throat will be simply flared if and only if the null energy condition is violated, or on the verge of being violated, at the throat. If the violations are suspended at the throat, the hypersurface will not satisfy any flare-out condition, and so ceases to be a throat. (For instance, this is what occurs in Refs. [21–25].) An entirely similar analysis can be carried out for the other coordinate patch. Again, there are are total of two time dependent throats and again, they coalesce into a single throat located at r_0 in the static limit.

VII. DISCUSSION

We have presented a local geometric definition of a wormhole throat that generalizes the notion of ''flare-out'' to an arbitrary time-dependent wormhole and is free from technical assumptions about global properties. Flare-out is manifested in the properties of light rays (null geodesics) that traverse a wormhole: bundles of light rays that enter the wormhole at one mouth and exit from the other must have cross-sectional area that first decreases, reaching a true minimum at the throat, and then increases. These properties can be quantified precisely in terms of the expansion $\theta_\pm$ of the (future-directed) null geodesics together with its derivative $d\theta_{\pm}/du_{\pm}$, where all quantities are evaluated at the two-dimensional spatial hypersurface comprising the throat. Strictly speaking, this flaring-out behavior of the outgoing null geodesics (l_{+}) defines one throat: the ''outgoing'' throat. But one can also ask for the flaring-out property to be manifested in the propagation of the set of ingoing null geodesics (l_{-}) as they traverse the wormhole, and this leads one to define a second, or ''ingoing'' throat. In general, these two throats need not be identical, but for the static limit they do coalesce and are indistinguishable.

The flaring-out property implies that all wormhole throats are in fact *anti-trapped* surfaces, an identification that was anticipated some time ago by Page [8]. With this definition and using the Raychaudhuri equation, we are able to place rigorous constraints on the Ricci tensor and the stress-energy tensor at the throat(s) of the wormhole as well as in the regions near the throat(s). We find, as expected, that wormhole throats generically violate the null energy condition and we have provided several theorems regarding this matter.

The nature of the energy-condition violations associated with wormhole throats has led numerous authors to try to find ways of evading or minimizing the violations. Most attempts to do so focus on alternative gravity theories in which one may be able to force the extra degrees of freedom to absorb the energy-condition violations (some of these scenarios are discussed in [6], see also [27,28]). But the energy condition violations are still always present, as sweeping the energy condition violations into a particular sector surely does not make the problem go away. As a striking case in point, we have treated in detail the case of gravity plus torsion. If we identify the torsion with that appearing naturally in the spectrum of closed strings, then we find it actually worsens the violations of the NEC at the throats. More recently it has been realized that time-dependence lets one move the energy condition violating regions around in *time* [21–25]. Temporary suspension of the violation of the NEC at a time-dependent throat also leads to a simultaneous obliteration of the flare-out property of the throat itself, so this strategy ends up destroying the throat and nothing is to be gained. (See also [6].) In arriving at this conclusion it is crucial to note that we have defined flare-out in terms of the expansion properties of light rays at the throat and *not* in terms of ''shape'' functions or embedding diagrams. While the latter can certainly be used without risk for detecting flare-out in static wormholes, they are at best misleading if applied to dynamic wormholes. This is simply because the embedding of a wormhole spacetime requires selecting and lifting out a particular time slice and embedding this instantaneous spatial three geometry in a flat Euclidean $\mathbf{R}^3$. For a static wormhole, any constant time slice will suffice, and if the embedded surface is flared-out in the spatial direction orthogonal to the throat, then it is flared-out in spacetime as well. But if the wormhole is dynamic, flare-out in the spatial direction does not imply flare-out in the *null* directions orthogonal to the throat.

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