Late-time tails in gravitational collapse of a self-interacting (massive) scalar-field and decay of a self-interacting scalar hair

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We study *analytically* the initial value problem for a *self-interacting* (massive) scalar field on a Reissner-Nordström spacetime. Following the no-hair theorem we examine the *dynamical* physical mechanism by which the *self-interacting* (SI) hair decays. We show analytically that the intermediate asymptotic behavior of SI perturbations is dominated by an oscillatory inverse power-law decaying tail. We confirm numerically this result. However, the numerical examination reveals that at late times the decay of SI hair is *slower* than any power law. [S0556-2821(98)03916-2]

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I. INTRODUCTION

The late-time evolution of various fields outside a collapsing star has important implications for two major aspects of black-hole physics: the no-hair theorem and the mass-inflation scenario. The *no-hair theorem* [1], introduced by Wheeler in the early 1970s, states that the *external* field of a black hole relaxes to a Kerr-Newman field characterized solely by the black-hole mass, charge, and angular momentum. Thus, it is of interest to explore the *dynamical* physical mechanism responsible for the relaxation of perturbations fields outside a black hole and to determine the decay *rates* of the various perturbations (which *differ* from one field to the other). The mechanism by which massless neutral fields are radiated away was first studied by Price [2]. The physical mechanism by which a massless charged scalar hair is radiated away was studied in [3,4]. In this paper we study the physical mechanism responsible for the decay of *self-interacting* (SI) *(massive)* scalar hair.

The asymptotic late-time tails along the outer horizon of a rotating or a charged black hole are used as the initial input for the ingoing perturbations which penetrate into the black-hole. These perturbations are the physical cause for the well-known phenomena of *mass-inflation* [5]. In this context, one should take into account the existence of *massive* tails outside the collapsing star. Here we study analytically the intermediate asymptotic behavior of such self-interacting (massive) perturbation fields. We study the late-time asymptotic behavior numerically and confirm the numerical results of Burko [6].

The plan of the paper is as follows. In Sec. II we describe the physical system and formulate the evolution equation considered. In Sec. III we formulate the problem in terms of the black-hole Green's function using the technique of spectral decomposition (this section is analogous to the one given in [4]). In Sec. IV we study the intermediate asymptotic evolution of SI scalar perturbations on a Reissner-Nordström background. We find an *oscillatory* inverse *power-law* behavior of the perturbations at a fixed radius and along the black-hole outer horizon. We find that the dumping exponents which describe the intermediate fall off of SI perturbations are *smaller* compared with the massless (neutral) dumping exponents. In Sec. V we verify our *analytical* results by numerical simulations. We conclude in Sec. VI with a brief summary of our results and their implications.

II. DESCRIPTION OF THE SYSTEM

We consider the evolution of self-interacting (SI) scalar perturbation fields outside a collapsing star. The external gravitational field of a spherically symmetric collapsing star of mass M and charge Q is given by the Reissner-Nordström metric

$$ds^2=-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2+\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2+r^2d\Omega^2. \tag{1}$$

Using the tortoise radial coordinate y, defined by $dy=dr/\lambda^2$ where $\lambda^2=1-2M/r+Q^2/r^2$, the metric becomes

$$ds^2=\lambda^2(-dt^2+dy^2)+r^2d\Omega^2. \tag{2}$$

The wave equation for the SI scalar field is

$$\phi_{;ab}g^{ab}-U'(|\phi|^2)\phi=0, \tag{3}$$

where $U(|\phi|^2)$ is the self-interaction potential and $U'(|\phi|^2)=dU(|\phi|^2)/d\phi^*$. Since we study the evolution of small perturbations we approximate $U'(|\phi|^2)$ by $m^2\phi$ (we assume that m is real), neglecting terms of higher order in ϕ. Resolving the field into spherical harmonics $\phi=\Sigma_{l,m}\psi^l_m(t,r)Y^m_l(\theta,\varphi)/r$ one obtains a wave equation for each multiple moment:

$$\psi_{,tt}-\psi_{,yy}+V\psi=0, \tag{4}$$

where

$$V=V_{M,Q,l,m}(r)=\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}-\frac{2Q^2}{r^4}+m^2\right]. \tag{5}$$

III. FORMALISM

The time evolution of a SI scalar field described by Eq. (4) is given by

$$\psi(y,t)=\int [G(y,x;t)\psi_t(x,0)+G_t(y,x;t)\psi(x,0)]dx, \tag{6}$$

for $t>0$, where the (retarded) Green's function $G(y,x;t)$ is defined as

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}+V(r)\right]G(y,x;t)=\delta(t)\delta(y-x). \tag{7}$$

The causality condition gives us the initial condition $G(y,x;t)=0$ for $t\leqslant 0$. In order to find $G(y,x;t)$ we use the Fourier transform

$$\tilde{G}(y,x;w)=\int_{0^-}^{\infty}G(y,x;t)e^{iwt}dt. \tag{8}$$

The Fourier transform is analytic in the upper half w plane and it satisfies

$$\left(\frac{d^2}{dy^2}+w^2-V\right)\tilde{G}(y,x;w)=\delta(y-x). \tag{9}$$

$G(y,x;t)$ itself is given by the inversion formula

$$G(y,x;t)=\frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic}\tilde{G}(y,x;w)e^{-iwt}dw, \tag{10}$$

where c is some positive constant.

Next, we define two auxiliary functions $\tilde{\psi}_1(y,w)$ and $\tilde{\psi}_2(y,w)$ which are linearly independent solutions to the homogeneous equation

$$\left(\frac{d^2}{dy^2}+w^2-V\right)\tilde{\psi}_i(y,w)=0, \quad i=1,2. \tag{11}$$

Let the Wronskian be

$$W(w)=W(\tilde{\psi}_1,\tilde{\psi}_2)=\tilde{\psi}_1\tilde{\psi}_{2,y}-\tilde{\psi}_2\tilde{\psi}_{1,y}, \tag{12}$$

where $W(w)$ is y independent. Using the two solutions $\tilde{\psi}_1$ and $\tilde{\psi}_2$, the black-hole Green's function can be expressed as

$$\tilde{G}(y,x;w)=-\frac{1}{W(w)}\begin{cases}\tilde{\psi}_1(y,w)\tilde{\psi}_2(x,w), & y<x,\\ \tilde{\psi}_1(x,w)\tilde{\psi}_2(y,w), & y>x.\end{cases} \tag{13}$$

In order to calculate $G(y,x;t)$ using Eq. (10), one may close the contour of integration into the lower half of the complex frequency plane. Then, one finds three distinct contributions to $G(y,x;t)$ [7].

(1) *Prompt contribution*. This arises from the integral along the large semicircle. It is this part, denoted G^F, which propagates the *high*-frequency response. For large frequencies the Green's function approaches to the one of a *massless* scalar field on a flat spacetime background. This term contributes to the *short*-time response and can be shown to be effectively zero beyond a certain time. Thus, it is not relevant for the late-time behavior of the field.

(2) *Quasinormal modes*. These arise from the distinct singularities of $\tilde{G}(y,x;w)$ in the lower half of the complex w plane and is denoted by G^Q. These singularities occur at frequencies for which the Wronskian (12) vanishes. G^Q is just the sum of the residues at the poles of $\tilde{G}(y,x;w)$. Since each mode has $\mathrm{Im}\, w<0$, it decays *exponentially* with time.

(3) *Tail contribution*. As will be shown later the intermediate asymptotic tail is associated with the existence of a branch cut (in $\tilde{\psi}_2$) placed along the interval $-m\leqslant w\leqslant m$. This tail arises from the integral of $\tilde{G}(y,x;w)$ around the branch cut (denoted by G^C) which leads to an *oscillatory* inverse *power-law* behavior of the field. Since we are interested in the intermediate asymptotic behavior of a SI scalar field our goal is to evaluate $G^C(y,x;t)$.

IV. INTERMEDIATE ASYMPTOTIC BEHAVIOR OF A SELF-INTERACTING SCALAR FIELD

A. $M\ll r\ll M/(Mm)^2$ approximation

It is well known that the late-time behavior of *massless* fields is determined by the backscattering from asymptotically *far* regions [8,2,3]. Thus, the late-time tails of massless fields are dominated by the *low*-frequency contribution to the Green's function, for only low frequencies will be backscattered by the small spacetime curvature or by the small electromagnetic interaction at these asymptotic regions. On the other hand, it is also well known that *massive* tails exist even in a *flat* spacetime [9]. This phenomenon is related to the fact that different frequencies forming a massive wave packet have different phase velocities. As will be shown in this paper, the intermediate asymptotic behavior of a SI scalar field on a Reissner-Nordström background is dominated by *flat* spacetime effects. Namely, at intermediate times the backscattering from asymptotically *far* regions (which dominates the tails of massless fields) is *negligible* compared to the flat spacetime massive tails that appear here.

Let us assume that both the observer and the initial data are situated far away from the black hole. We expand the wave-equation (11) for the SI scalar field (in the field of the black hole) as a power series in M/r and Q/r and obtain (neglecting terms of order $O[(Mm/r)^2]$ and higher)

$$\left[\frac{d^2}{dr^2}+w^2-m^2+\frac{4Mw^2-2Mm^2}{r}-\frac{l(l+1)}{r^2}\right]\xi=0, \tag{14}$$

where $\xi=\lambda\tilde{\psi}$. The term proportional to M/r represents the Newtonian potential. If we further assume that both the observer and the initial data are situated in the region $r\ll M/(Mm)^2$ (and $M\ll r$), and we are interested in the intermediate asymptotic behavior of the field $[r\ll t\ll M/(Mm)^2]$, we can further approximate Eq. (14) by

$$\left[\frac{d^2}{dr^2}+w^2-m^2-\frac{l(l+1)}{r^2}\right]\xi=0. \tag{15}$$

Replacing Eq. (14) with Eq. (15) means that we *neglect* the backscattering of the field from asymptotically *far* regions. Thus, the intermediate asymptotic behavior of SI scalar perturbations on a black hole (or a star) background depends only on the field's parameters (namely, on the mass of the field) and it does *not* depend on the spacetime parameters. The validity of this conclusion is verified by numerical simulations (see Sec. V).

Let us now introduce a second auxiliary field $\tilde{\phi}$ defined by

$$\xi = r^{l+1} e^{i\sqrt{w^2-m^2}r}\tilde{\phi}(z), \tag{16}$$

where

$$z = -2iwr. \tag{17}$$

$\tilde{\phi}$(z) satisfies the confluent hypergeometric equation

$$\left[z\frac{d^2}{dz^2}+(2l+2-z)\frac{d}{dz}-(l+1)\right]\tilde{\phi}(z)=0. \tag{18}$$

The two basic solutions required in order to build the black-hole Green's function are (for $|w|\leqslant m$)

$$\tilde{\psi}_1 = Ar^{l+1}e^{-\varpi r}M(l+1,2l+2,2\varpi r) \tag{19}$$

and

$$\tilde{\psi}_2 = Br^{l+1}e^{-\varpi r}U(l+1,2l+2,2\varpi r), \tag{20}$$

where $\varpi=\sqrt{m^2-w^2}$. A and B are normalization constants. $M(a,b,z)$ and $U(a,b,z)$ are the two standard solutions to the confluent hypergeometric equation [10]. $U(a,b,z)$ is a many-valued function; i.e., there is a cut in $\tilde{\psi}_2$. Using Eqs. 13.6.6 and 13.6.24 of [10] one may write these solutions in a more familiar form

$$\tilde{\psi}_1 = \frac{1}{2}Ar^{1/2}\Gamma\left(l+\frac{3}{2}\right)\left(\frac{1}{2}\varpi\right)^{-(l+1/2)} I_{l+1/2}(\varpi r), \tag{21}$$

and

$$\tilde{\psi}_2 = \pi^{-1/2}Br^{1/2}(2\varpi)^{-(l+1/2)}K_{l+1/2}(\varpi r), \tag{22}$$

where $I_{l+\frac{1}{2}}$ and $K_{l+\frac{1}{2}}$ are modified Bessel functions.

Using Eq. (10), one finds that the branch cut contribution to the Green's function is given by

$$G^C(y,x;t) = \frac{1}{2\pi}\int_{-m}^{m}\tilde{\psi}_1(x,\varpi) \times\left[\frac{\tilde{\psi}_2(y,\varpi e^{\pi i})}{W(\varpi e^{\pi i})}-\frac{\tilde{\psi}_2(y,\varpi)}{W(\varpi)}\right]e^{-iwt}dw. \tag{23}$$

For simplicity we assume that the initial data have a considerable support only for r values which are smaller than the observer's location. This, of course, does not change the *late*-time behavior.

Using Eqs. 9.6.30 and 9.6.31 of [10], one finds

$$\tilde{\psi}_1(r,\varpi e^{\pi i}) = \tilde{\psi}_1(r,\varpi) \tag{24}$$

and

$$\tilde{\psi}_2(r,\varpi e^{\pi i}) = -\tilde{\psi}_2(r,\varpi)+\frac{B}{A}\frac{\pi^{1/2}(-1)^{l+1}2^{-2l}}{\Gamma(l+\frac{3}{2})}\tilde{\psi}_1(r,\varpi). \tag{25}$$

Using Eqs. (24) and (25) it is easy to see that

$$W(\varpi e^{\pi i}) = -W(\varpi), \tag{26}$$

from which we obtain the relation

$$\frac{\tilde{\psi}_2(r,\varpi e^{\pi i})}{W(\varpi e^{\pi i})}-\frac{\tilde{\psi}_2(r,\varpi)}{W(\varpi)} = \frac{B}{A}\frac{\pi^{1/2}(-1)^{l}2^{-2l}}{\Gamma(l+\frac{3}{2})}\frac{\tilde{\psi}_1(r,\varpi)}{W(\varpi)}. \tag{27}$$

Since $W(\varpi)$ is r independent, we may use the $\varpi r\to 0$ asymptotic expansions of the Bessel functions (given by Eqs. 9.6.7 and 9.6.9 in [10]) in order to evaluate it. One finds

$$W(\varpi) = -\frac{1}{4}AB\pi^{-1/2}(2l+1)\Gamma\left(l+\frac{1}{2}\right)\varpi^{-(2l+1)}. \tag{28}$$

Finally, substituting Eqs. (27) and (28) into Eq. (23) we obtain

$$G^C(y,x;t) = \frac{(-1)^{l+1}2^{2-2l}}{A^2(2l+1)\Gamma(l+\frac{1}{2})\Gamma(l+\frac{3}{2})} \times\int_0^m \tilde{\psi}_1(y,\varpi)\tilde{\psi}_1(x,\varpi)\varpi^{2l+1}e^{-iwt}dw. \tag{29}$$

B. Intermediate asymptotic behavior at a fixed radius

It is easy to verify that in the large t limit the effective contribution to the integral in Eq. (29) arises from $|w| = O(m-1/t)$ or equivalently $\varpi = O(\sqrt{m/t})$. This is due to the rapidly oscillating term e^{-iwt} which leads to a mutual cancellation between the positive and the negative parts of the integrand. In order to obtain the intermediate asymptotic behavior of the field at a fixed radius (where $x,y \ll t$), we may use the $\varpi r \ll 1$ limit of $\tilde{\psi}_1(r,\varpi)$. Using Eq. 9.6.7 from [10] one finds

$$\tilde{\psi}_1(r,\varpi) \simeq \frac{1}{2} A r^{l+1}. \tag{30}$$

We obtain

$$G^C(y,x;t) = \frac{(-1)^{l+1}\pi^{1/2} m^{l+1}}{2^l(2l+1)\Gamma(l+\frac{1}{2})}(xy)^{l+1} t^{-(l+1)} J_{l+1}(mt), \tag{31}$$

which in the $t \gg m^{-1}$ limit becomes

$$G^C(y,x;t) = \sqrt{\frac{2}{\pi}}\frac{(-1)^{l+1}}{(2l+1)!!} m^{l+1/2}(xy)^{l+1} t^{-(l+3/2)} \times \cos\left[mt - \left(\frac{1}{2}l + \frac{3}{4}\right)\pi\right]. \tag{32}$$

Thus, the intermediate asymptotic behavior of the SI field at a fixed radius is dominated by an *oscillatory* inverse *power-law* tail.

C. Intermediate asymptotic behavior along the black-hole outer horizon

Next, we consider the behavior of the SI scalar field at the black-hole outer horizon r_+. While Eqs. (21) and (22) are approximate solutions to the wave equation (11) in the $M \ll r \ll M/(Mm)^2$ region, they do not represent the solution near the horizon. As $y \to -\infty$ the wave equation (11) can be approximated by the equation

$$\tilde{\psi}_{,yy} + w^2 \tilde{\psi} = 0. \tag{33}$$

Thus, we chose

$$\tilde{\psi}_1(y,w) = C(w) e^{-iwy}, \tag{34}$$

and we use the form (30) for $\tilde{\psi}_1(x,w)$. In order to match the $y \ll -M$ solution to the $y \gg M$ solution we assume that the two solutions have the same temporal dependence. This assumption has been proven to be very successful for massless neutral [11] and charged [3,4] perturbations. In other words we assume that $C(w)$ is w independent. In this case one should replace the roles of x and y in Eqs. (23) and (29). Using Eq. (29), we obtain

$$G^C(y,x;t) = \Gamma_0 \sqrt{\frac{2}{\pi}}\frac{(-1)^{l+1}}{(2l+1)!!} m^{l+1/2} y^{l+1} v^{-(l+3/2)} \times \cos\left[mv - \left(\frac{1}{2}l + \frac{3}{4}\right)\pi\right], \tag{35}$$

where Γ_0 is a constant.

Thus, the intermediate asymptotic behavior of the SI field along the black-hole outer horizon is dominated by an *oscillatory* inverse *power-law* tail.

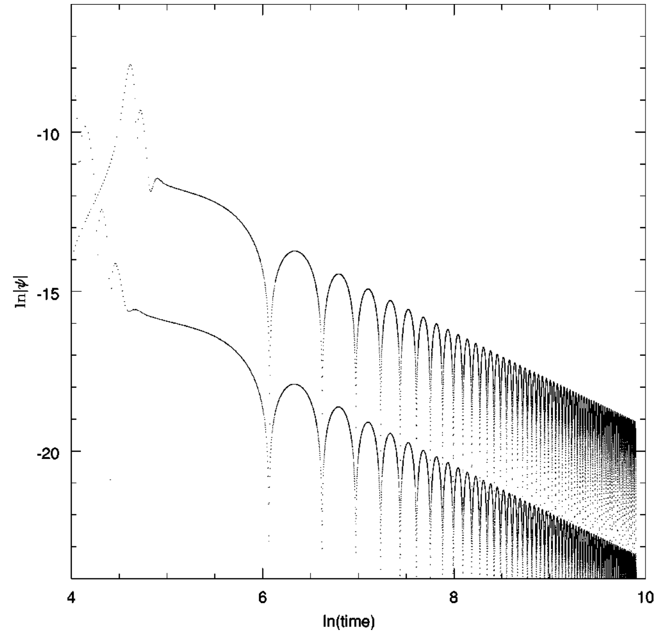

FIG. 1. Evolution of the SI field $|\psi|$ on a Reissner-Nordström background, for $l=0, M=0.5, Q=0.45$ and $m=0.01$. The initial data are a Gaussian distribution with $v_0=50$ and $\sigma=2$. The field at a fixed radius ($y=50$) is shown as a function of t. Along the black-hole outer horizon the field is shown as a function of v. The oscillatory *power-law* falloff is manifest at intermediate times. The power-law exponents are -1.48 at a fixed radius (top curve) and -1.47 along the black-hole outer horizon (bottom curve). These values are to be compared with the *analytically* predicted value of -1.5. The period of the oscillations is $T=\pi/m$ to within 0.1%, in agreement with the predicted value.

V. NUMERICAL RESULTS

It is straightforward to integrate Eq. (4) using the method described in [11]. The late-time evolution of a SI scalar field is independent of the form of the initial data used. The results presented here are for a Gaussian pulse on $u=0$

$$\psi(u=0,v) = A \exp\{-[(v-v_0)/\sigma]^2\}, \tag{36}$$

where the amplitude A is physically irrelevant due to the linearity of Eq. (4). It should be noted that the evolution equation (4) is invariant under the rescaling

$$r \to ar, \quad t \to at, \quad M \to aM, \quad Q \to aQ, \quad m \to m/a, \tag{37}$$

where a is some positive constant. The black-hole mass and charge are set equal to $M=0.5$ and $Q=0.45$, respectively. We have chosen an initial field profile with $m=0.01$, $v_0=50$, and $\sigma=2$. We studied the behavior of the field ψ at a fixed radius and along the black-hole outer horizon (approximated by the null surface $u=u_{max}$, where u_{max} is the largest value of u on the grid). The numerical results for the $l=0$ mode are shown in Fig. 1. Initially, the evolution is dominated by the prompt contribution and quasinormal ringing. However, at intermediate times a definite oscillatory *power-law* falloff is manifest. The power-law exponents are

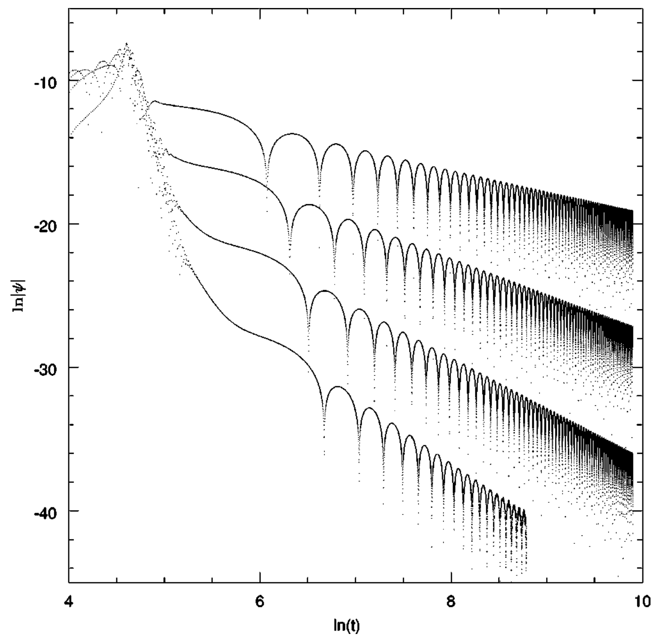

FIG. 2. The amplitude of the field $|\psi(y=50,t)|$ for different multipoles $l=0$, 1, 2, and 3 (from top to bottom). The power-law exponents are $-1.49,-2.50,-3.50$, and -4.51, respectively, in excellent agreement with the *analytically* predicted values of $-1.5,-2.5,-3.5$, and -4.5. The oscillations period is $T=\pi/m$ to within 0.1%, in agreement with the predicted value. The initial data are those of Fig. 1.

-1.48 at a fixed radius (top curve) and -1.47 along the black-hole outer horizon (bottom curve). These values are to be compared with the *analytically* predicted value of -1.5; see Eqs. (32) and (35). The period of the oscillations (for $|\psi|$) equals $T=\pi/m$ to within 0.1%, again in agreement with the predicted value. Figure 2. depicts the dependence of the intermediate asymptotic tails (at a fixed radius $y=50$) on the multipole index l. The numerical values of the power-law exponents, describing the fall off of the field at intermediate times (after a period of quasinormal ringing), are -1.49, $-2.50,-3.50$, and -4.51 for $l=0$, 1, 2, and 3, respectively. These numerical values are in excellent agreement with the *analytically* predicted values of $-1.5,-2.5,-3.5$ and -4.5. Again, the period of the oscillations is $T=\pi/m$ to within 0.1%, in agreement with the predicted value.

The analytical derivations and their numerical confirmations presented so far are restricted to the *intermediate* asymptotic regime. The *late*-time evolution of the field (for the $l=0$ mode) is shown in Fig. 3 (for the clarity of the figure we display only the maxima of the oscillations). The initial data are those of Fig. 1. Shown is the behavior of the field $|\psi|$ along the asymptotic regions of timelike infinity i_+ (top curve) and along the black-hole outer horizon (bottom curve). It is clear that the field's amplitude *decays* with time, in agreement with the no-hair theorem. The decay rate is *slower* than any power law. Again, we find that the field $|\psi|$ oscillates with a period of $T=\pi/m$ to within 0.2%. We have found that the larger the field's mass is, the sooner it leaves the intermediate asymptotic phase of an inverse power-law decay.

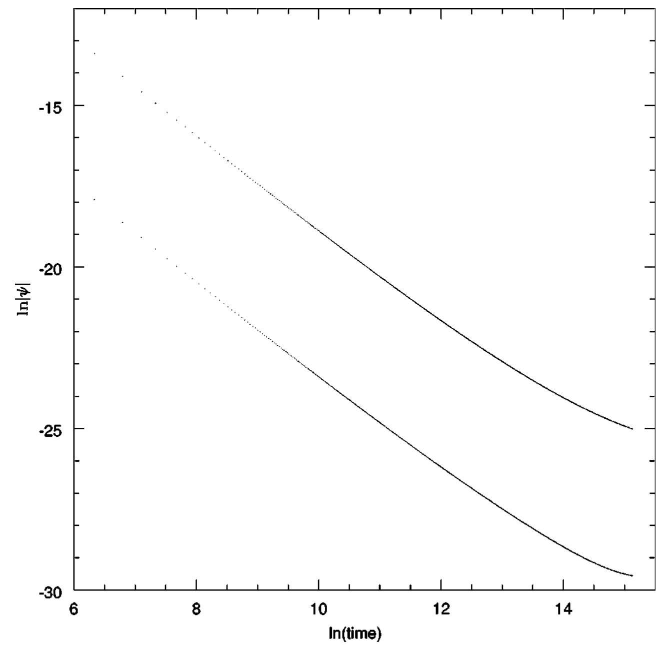

FIG. 3. Late-time evolution of the SI field $|\psi|$ on a Reissner-Nordström background. The field at future timelike infinity ($y=50$) is shown as a function of t (top curve). Along the black-hole outer horizon the field is shown as a function of v (bottom curve). For the clarity of the figure we display only the maxima of the oscillations (the field oscillates with a period of $T=\pi/m$ to within 0.2%). At *late* times the field's amplitude decays *slower* than any power law. The initial data are those of Fig. 1.

VI. SUMMARY AND PHYSICAL IMPLICATIONS

We have studied analytically the intermediate asymptotic evolution of a SI scalar field on a Reissner-Nordström background. Following the *no-hair theorem* we have focused attention on the physical mechanism by which a SI hair decays. The main results and their physical implications are the following.

Oscillatory inverse *power-law* tails develop at intermediate times at a fixed radius and along the black-hole outer horizon [as long as the initial data have a considerable support only in the region $M \ll r \ll M/(Mm)^2$; actually, our analytical derivations hold even in the case $M \ll r \ll Ml(l+1)/(Mm)^2$ for $l>0$). The dumping exponents, describing the fall-off of a SI field at intermediate times, are *smaller* compared with those of *massless* neutral perturbations [2,11]. For $l>(2/\sqrt{3})|eQ|-\frac{1}{2}$ (where e is the field's charge) these dumping exponents are also smaller than those of *massless* charged perturbations [4]. While the asymptotic behavior of *massless* perturbations is dominated by backscattering from asymptotically far regions [and thus depends on the *spacetime* parameters M (for neutral perturbations) and Q (for charged perturbations)], the intermediate asymptotic behavior of a SI field depends on the *field's* parameters (namely, on the field's mass m). In other words, dealing with SI perturbations, one may *neglect* the backscattering from asymptotically far regions at intermediate times.

Using the numerical scheme we have shown that at *late* times the inverse power law decay is replaced by another pattern of decay, which is *slower* than any power-law. This late-time behavior deserves a further analytic study. The late-time behavior of SI perturbations implies that a black hole which forms from a gravitational collapse of SI fields becomes ''bald'' *slower* than one which forms during a gravitational collapse of massless fields. Since SI perturbation fields decay at late times slower than any power law, they are expected to cause a *mass-inflation* singularity during a gravitational collapse which leads to the formation of a black hole. Moreover, SI fields are expected to *dominate* the mass-inflation phenomena during a gravitational collapse (this is caused by the *slower* decay of SI perturbations compared with massless ones).

ACKNOWLEDGMENTS

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