Microfield dynamics of black holes

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The microcanonical treatment of black holes as opposed to the canonical formulation is reviewed and some major differences are displayed. We propose a microcanonical alternative to the thermodynamical expression for the number density and discuss its characteristics. In particular the decay rates are compared in the two different pictures and shown to predict significantly different fates for cosmological black holes. [S0556-2821(98)06716-2]

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I. INTRODUCTION

In spite of its many mathematical and physical inconsistencies and drawbacks, the treatment of black holes as thermodynamical systems has since its inception been the description preferred by most physicists investigating the nature of black holes. Not least among the drawbacks is the fact that the laws of quantum mechanics are violated, because the number density function of the emitted radiation as calculated using a thermal vacuum is characteristic of mixed states, while the incoming radiation may have been in pure states. Since black holes can in principle radiate away completely, the unitarity principle is violated.

In a series of papers [1–11] we have investigated an alternative description of black holes which is free of the problems encountered in the thermodynamical approach. In our approach black holes are considered to be extended quantum objects (p-branes). This point of view has recently been further supported by investigations in fundamental strings [12], where one finds that extended D-branes are a basic ingredient of the theory [13] and lead to black-hole-type solutions for which the area of the horizon is proved to measure the quantum degeneracy [14]. In the present work we consider a gas of p-brane black holes and show that the equilibrium configuration is decidedly non-thermal. We also define a new vacuum, the microcanonical or fixed-energy vacuum, and obtain within the context of mean field theory the wave functions for the radiation associated with such objects. Using the number density function for our vacuum, we calculate the black hole decay rate and compare it with that obtained from the thermodynamical description (see also [15,16] for an alternative derivation).

In Sec. II we present a brief summary of the thermodynamical description of processes involving black holes and discuss in detail the inconsistencies mentioned above. In Sec. III we discuss our interpretation of the WKB approximation as the quantum tunneling probability and review our results for the statistical mechanics of a gas of black holes. In Sec. IV we discuss the thermodynamical interpretation of black holes within the context of mean field theory and prove that the thermal vacuum is the false vacuum for a black hole system. We also present an alternative vacuum for such a system and the microcanonical number density which corresponds to this vacuum. In Sec. V we present the microcanonical wave functions for the in and out states and in Sec. VI we derive the black hole decay rate.

We use units with $c=G=\hbar=k_B=1$ unless differently stated.

II. THERMODYNAMICAL INTERPRETATION OF BLACK HOLES

Bekenstein's original observation [17] that the area of a black hole is in some way analogous to the thermodynamical concept of entropy was enlarged upon in Ref. [18] where the four laws of black hole thermodynamics were hypothesized. The mass difference of neighboring equilibrium states was shown to be related to the change in the black hole area A according to the relation (*Smarr formula*)

$$\Delta M=\kappa\Delta A+\varpi\Delta J+F\Delta Q, \tag{2.1}$$

where κ is the surface gravity and is related to the temperature by

$$T=\beta_H^{-1}=\frac{\kappa}{2\pi}. \tag{2.2}$$

J is the angular momentum of the black hole, Q its charge and ϖ, Φ play the role of potentials.

The partition function for the black hole is assumed to be determined as

$$Z(\beta)=\mathrm{Tr}\; e^{-\beta H}=e^{-S_H}. \tag{2.3}$$

The function S_H is the Hawking entropy which is given by

$$S_H=S_E-\beta_H\varpi J, \tag{2.4}$$

where S_E is the Euclidean action. The Hawking entropy is also related to the area of the black hole by

$$S_H=\frac{A}{4}. \tag{2.5}$$

Finally, in thermodynamical equilibrium the statistical mechanical density of states is given by

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$$\Omega = Z^{-1}(\beta) = e^{S_H}, \tag{2.6}$$

and the specific heat is

$$C_V = \frac{\partial E}{\partial T} = -\frac{\beta^2}{8\pi}. \tag{2.7}$$

The fact that the canonical specific heat, an intrinsically positive quantity, is negative in this interpretation is a clear signal that the thermodynamical analogy fails.

The thermodynamical interpretation of black holes has many such inconsistencies. A second problem can be best shown if we specialize the previous expressions, for instance, to the Schwarzschild black hole. It turns out that

$$S_H = S_E. \tag{2.8}$$

Further, since the radius of the horizon in this case is $2M$, one has

$$S_H = 4\pi M^2 \tag{2.9}$$

and

$$\beta_H = 8\pi M. \tag{2.10}$$

It then follows that the partition function as calculated from the microcanonical density of states,

$$Z(\beta) = \int_0^\infty dM\,\Omega(M)\,e^{-\beta M} = \int_0^\infty dM\,e^{4\pi M^2} e^{-\beta M} \to \infty, \tag{2.11}$$

is infinite for all temperatures and hence the canonical ensemble is not equivalent to the (more fundamental) microcanonical ensemble

$$Z(\beta) \neq \frac{1}{\Omega}, \tag{2.12}$$

as is required for thermodynamical equilibrium [19] (see also Sec. IV A for a more general analysis).

Furthermore, if quantum mechanical effects are taken into account, black holes can be shown to radiate [20,21]. In the thermodynamical approach the future-evolved *in* vacuum becomes temperature dependent (see Sec. V), and the radiation coming out of the black hole has a Planckian distribution

$$n_{\beta_H}(\omega) = \frac{1}{e^{\beta_H \omega} - 1}. \tag{2.13}$$

Since black holes can in principle radiate away completely, this result implies that information can be lost, because pure states can come into the black hole but only mixed states come out. The breakdown of the unitarity principle is one of the most serious drawbacks of the thermodynamical interpretation, since it requires the replacement of quantum mechanics with some new (unspecified) physics.

III. BLACK HOLES AS *p*-BRANES

The inconsistencies of the thermodynamical interpretation are an indication that the interpretation of e^{-S_H} as the canonical partition function is wrong. In the usual WKB approximation e^{-S_E} is the tunneling probability per unit volume for a particle to tunnel through a potential. In the present case we hypothesize that the probability to tunnel through the black hole's horizon is given by

$$P \simeq e^{-S_H}, \tag{3.1}$$

for any kind of black hole. The quantum degeneracy of states for the system is proportional to P^{-1} and is then given by

$$\sigma \simeq c\,e^{A/4}, \tag{3.2}$$

where the constant c is determined from quantum field theoretic corrections and can contain non-local effects. Recently an analogous and maybe deeper understanding of Eq. (3.2) has been obtained in string theory, where black hole solutions appear to be related to D-branes [12,13] and the relationship between area and entropy is recovered at least in the very special cases of tiny, extremal black holes [14] (the generalization to bigger, non-extremal black holes might just be a technical problem [22] or it might be a more substantial one [23]).

Explicit expressions can be obtained for the above quantities for some geometries.

A. *D*-dimensional Schwarzschild black hole

As a first example we can consider the Schwarzschild black hole, which in D dimensions has the Euclidean metric

$$ds^2 = e^{2\lambda} d\tau^2 + e^{-2\lambda} dr^2 + r^2 d\Omega_{D-2}^2, \tag{3.3}$$

where

$$e^{2\lambda} = 1 - \left(\frac{r_+}{r}\right)^{D-3}. \tag{3.4}$$

The area in D dimensions is

$$\frac{A}{4} = \frac{A_{D-2}}{16\pi} \beta_H r_+^{D-3}, \tag{3.5}$$

with

$$M = \frac{D-2}{16\pi} A_{D-2} r_+^{D-2}, \tag{3.6}$$

where A_{D-2} is the area of a unit $D-2$ sphere. Eliminating the horizon radius r_+ in favor of the mass, the area becomes

$$\frac{A}{4} = C(D) M^{(D-2)/(D-3)}, \tag{3.7}$$

where $C(D)$ is a mass-independent function:

$$C(D)=\frac{4^{(D-1)/(D-3)}\pi^{(D-2)/(D-3)}}{(D-3)(D-2)^{(D-2)/(D-3)}A_{D-2}^{1/(D-3)}}. \tag{3.8}$$

Substituting in for $A/4$ in the degeneracy of states expression we find

$$\sigma(M)\simeq ce^{C(D)M^{(D-2)/(D-3)}}. \tag{3.9}$$

Comparing this expression to those known for non-local field theories, we find that it corresponds to the degeneracy of states for an extended quantum object (p-brane) of dimension $p=(D-2)/(D-4)$. As has been demonstrated by several authors [25–27], an exponentially rising density of states is a clear signal of a non-local field theory. p-brane theories are the only known non-local theories in theoretical physics which can give rise to exponentially rising degeneracies.

B. KND black hole

In four dimensions the largest generalization of the Schwarzschild black hole is given by the Kerr-Newman family with the addition of a scalar field called the dilaton. The action of the Kerr-Newman dilaton (KND) black hole is found as an effective action in compactified string theories [24] and is given by

$$S=\frac{1}{16\pi}\int d^4x\sqrt{g}\left[R-\frac{1}{2}(\nabla\phi)^2-e^{-a\phi}F^2\right]+\Sigma, \tag{3.10}$$

where the first term on the right-hand side (RHS) is the volume contribution obtained by integrating on the whole region outside the outer horizon, R is the scalar curvature, ϕ is the dilaton field, a its coupling constant, F is the Maxwell field and Σ contains all surface terms.

In Ref. [10] the field equations derived from the action in Eq. (3.10) were expanded in the charge-to-mass ratio, Q/M, and a new perturbative static solution was found, which is of the form

$$ds^2=-\frac{\Delta\ \sin^2\ \theta}{\Psi}(d\varphi)^2+\Psi(dt-\omega d\varphi)^2+\rho^2\left[\frac{(dr)^2}{\Delta}+(d\theta)^2\right]. \tag{3.11}$$

The latter can be simplified upon substituting for the (bare) parameters M, Q and $J\equiv\alpha M$, the Arnowitt-Deser-Misner (ADM) mass, charge and angular momentum of the hole, and also by shifting the radial coordinate, $r\to r-a^2Q^2/6M$ (see Ref. [11] for the details). One finally obtains that the metric in Eq. (3.11) coincides (at order Q^2/M^2) with the Kerr-Newman solution (e.g., see [28]). This implies that the background dilaton field

$$\phi=-a\frac{r}{\rho^2}\frac{Q^2}{M}+\mathcal{O}(Q^4) \tag{3.12}$$

does not affect the singularity structure at order Q^2/M^2.

The surface term in Eq. (3.10) is given as

$$\Sigma=\frac{\beta_H}{2}M, \tag{3.13}$$

where the complexified time it has period β_H as given in Eq. (2.2). Also in this case there are two horizons:

$$r_\pm=M\pm\sqrt{M^2-\alpha^2-Q^2}, \tag{3.14}$$

where $\alpha=J/M$ and the minimum value admitted for the mass is $M_0=\sqrt{\alpha^2+Q^2}$, corresponding to the *extremal* case. The Euclidean action of the KND instanton is

$$S_E(M,\ J,\ Q;\ a)=\frac{A}{4}+\beta_H\varpi J, \tag{3.15}$$

and the area is given by

$$A=4\pi(r_+^2+\alpha^2)+\mathcal{O}(Q^4). \tag{3.16}$$

To order Q^4 the Euclidean action is

$$S_E(M,\ J,\ Q;\ a)=\frac{\beta_H}{2}\left(M-\frac{Q^2r_+}{r_+^2+\alpha^2}+\mathcal{O}(Q^4)\right). \tag{3.17}$$

The quantum degeneracy of states is then

$$\sigma_{KND}(M,\ J,\ Q;\ a)\sim e^{A/4}=e^{\pi(r_+^2+\alpha^2)}. \tag{3.18}$$

C. Statistical mechanics of KND black holes

Using the quantum degeneracy of states in Eq. (3.18) we can analyze the statistical mechanical properties of a gas of such black holes. The microcanonical density is defined as a function in the space of all possible configurations of $n>0$ black holes:

$$\Omega_{KND}(M,\ J,\ Q;\ a)=\sum_{n=1}^{\infty}\Omega_n(M,\ J,\ Q;\ a), \tag{3.19}$$

where Ω_n is determined from σ_{KND} through the relation

$$\Omega_n(M, J, Q; a) = \left[\frac{V}{(2\pi)^3}\right]^n \frac{1}{n!} \prod_{i=1}^{n} \int_{m_0}^{\infty} dm_i \int_{-m_i^2}^{+m_i^2} dj_i \int_{-\sqrt{m_i^2-\alpha_i^2}}^{+\sqrt{m_i^2-\alpha_i^2}} dq_i \sigma_{KND}(m_i, j_i, q_i; a) \int_{-\infty}^{+\infty} d^3 p_i \delta\left(M - \sum_{i=1}^{n} E_i\right) \times \delta\left(Q - \sum_{i=1}^{n} q_i\right) \delta\left(J - \sum_{i=1}^{n} j_i\right) \delta^3\left(\sum_{i=1}^{n} \vec{p}_i\right). \quad (3.20)$$

Here M, J and Q are respectively the total mass, angular momentum and charge of the gas: $m_0 \geqslant 0$ is the minimum allowed mass for each black hole in the gas. We are assuming in this relation that the black holes obey the particle-like dispersion relation

$$E_i^2 = p_i^2 + m_i^2. \quad (3.21)$$

The equilibrium configuration for such a gas is highly nonthermal. The most probable configuration turns out to be (see Ref. [29] for details) one massive black hole with

$$\text{mass} = M - (n-1) m_0$$

$$\text{charge} = Q - (n-1)\sqrt{1-\gamma^2} m_0$$

$$\text{angular momentum} = J - (n-1)\gamma m_0^2, \quad 0 \leqslant \gamma \leqslant 1. \quad (3.22)$$

The remaining $(n-1)$ black holes have

$$\text{mass} = m_0$$

$$\text{charge} = q_i = \sqrt{1-\gamma^2} m_0$$

$$\text{angular momentum} = j_i = \gamma m_0^2. \quad (3.23)$$

A numerical computation was carried out for the special case $n=2$, $\alpha=0$ (two Reissner-Nordström black holes). As shown in Fig. 1 (see also Fig. 2), the equipartition state is a saddle point, and the maxima correspond to one or the other of the black holes possessing (nearly) all of the mass and all of the charge. This suggests that

$$\gamma \sim 0. \quad (3.24)$$

Our picture of the gas is thus one in which one massive black hole carries all the charge and angular momentum, and is surrounded by $n-1$ lighter, Schwarzschild black holes. Then the density of states can be approximated by

$$\Omega(M,J,Q;a) \sim \left[\frac{cV}{(2\pi)^3}\right]^N \frac{1}{N!} e^{(N-1)A_{KND}(m_0, \gamma m_0^2, \sqrt{1-\gamma^2} m_0^2; a)/4} e^{A_{KND}(M-(N-1)m_0, J-(N-1)m_0^2\gamma, Q-\sqrt{1-\gamma}(N-1)m_0; a)/4}. \quad (3.25)$$

This is the statistical mechanical model of a black hole (the most massive one in the system) and its associated radiation (whose quanta are represented by the lighter black holes in the gas).

The most probable number N of black holes in the gas is found from the extremum condition $d\Omega_n/dn|_{n=N}=0$. The corresponding contribution to the sum in Eq. (3.19) can be used to approximate the full microcanonical density:

$$\Omega(M, J, Q; V, a) \simeq \Omega_N(M, J, Q; V, a). \quad (3.26)$$

The number N is given by $\Psi(N+1) \simeq \ln[cV/(2\pi)^3]$, where Ψ is the psi function [1].

We can now check whether the gas of black holes we have been describing satisfies the bootstrap condition [30]

$$\lim_{M\to\infty} \frac{\Omega(M, Q, J; V, a)}{\sigma_{KND}(M, Q, J; a)} = 1, \quad (3.27)$$

where σ_{KND} is the quantum degeneracy of a single black hole as given in Eq. (3.18). For the general case in Eq. (3.26), one obtains that Eq. (3.27) is satisfied if $m_0=0$ and

$$[e^{\Psi(N+1)}]^N \frac{1}{N!} = c. \quad (3.28)$$

As in the case of a gas of Schwarzschild black holes [1], this equation gives a relation between the constant c and the volume V. Correspondingly, one obtains the inverse microcanonical temperature

$$\beta = \frac{d \ln \Omega}{dE} \simeq \frac{d \ln \Omega_N}{dM} = \beta_H(M, J, Q; a). \quad (3.29)$$

In the limit $a=0$ one recovers the Kerr-Newman expression $\beta_H = 4\pi r_+$.

Our results show that the most probable state for a gas of Kerr-Newman dilaton black holes is very far from thermal equilibrium. Not only does one black hole acquire all of the mass as in the Schwarzschild case, but it also acquires all of

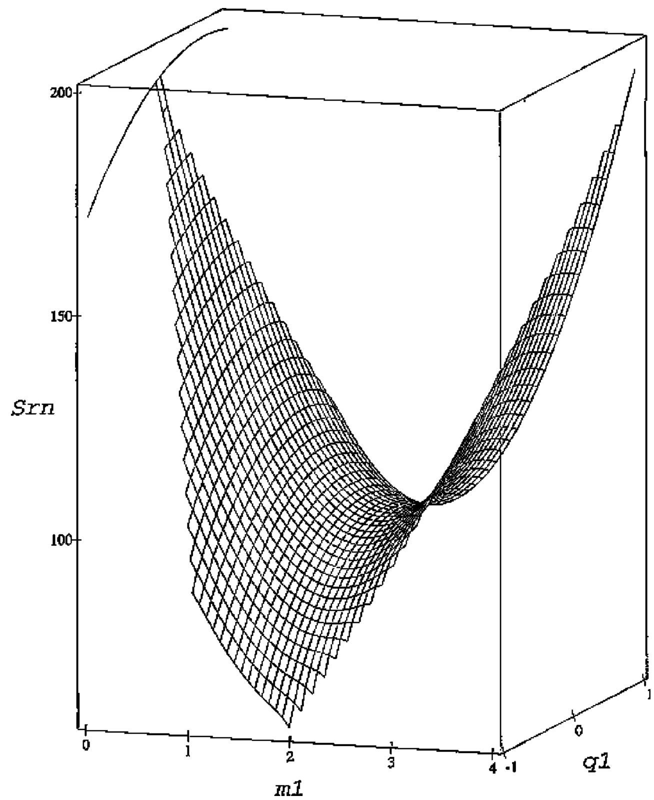

FIG. 1. The total Bekenstein-Hawking entropy S_{rn} for a system of two Reissner-Nordström black holes with total mass $M=4$ and total charge $Q=1$ as a function of m_1 and q_1.

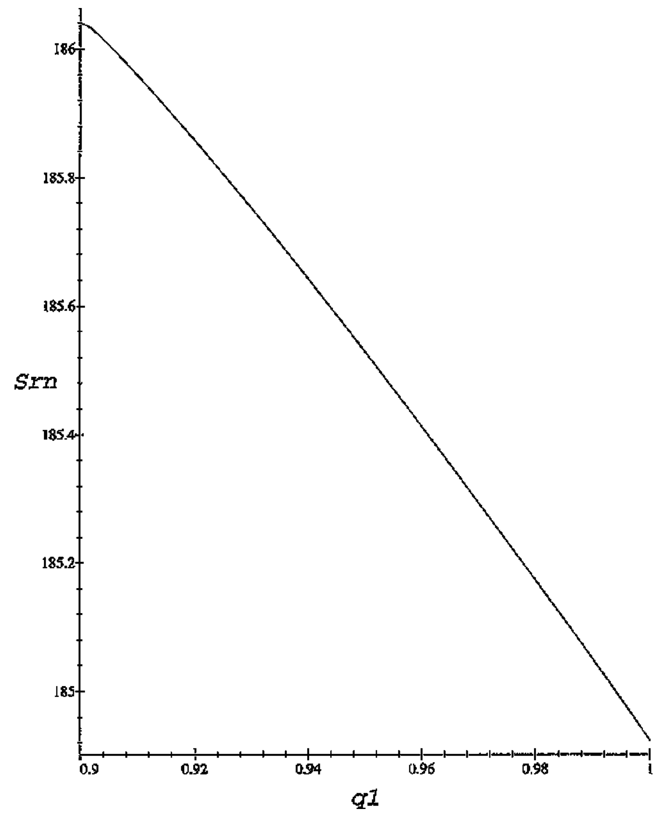

FIG. 2. When the lower limit for the mass of each black hole is $m_0=0.1$, the action S_{rn} in Fig. 1 has a maximum for $m_1=4-m_0$ and $q_1=1-m_0$, meaning that the second black hole is extremal ($m_2=q_2=m_0$).

the charge and all of the angular momentum of the whole gas. This is the reason the bootstrap condition is satisfied for the Kerr-Newman dilaton black holes at high energy.

IV. MEAN FIELD THEORY

To study particle production and propagation in black hole geometries we now turn to a second semiclassical approximation. In the mean field approximation fields are quantized on a classical black hole background. Black holes have non-trivial topologies which causally separate two regions of space. For this reason the number of degrees of freedom is doubled, and two Fock spaces are required to describe quantum processes occurring in the vicinity of a black hole. Calculations of quantities associated with such processes can be carried out in ways analogous to calculations in thermofield dynamics (TFD) [31], but with an overall fixed energy [microfield dynamics (MFD) [32]].

A. Canonical formulation

In the context of mean field theory the thermal vacuum for quantum fields scattered off of black holes (that is the initial vacuum $|0_{in};t=0\rangle$ propagated to large later times) can be written as

$$|0_{in};t\rightarrow+\infty\rangle=\frac{1}{Z^{1/2}(\beta_H)}\sum_{n=0}^{\infty}e^{-\beta_H n\omega/2}|n\rangle\otimes|\tilde{n}\rangle\equiv|\mathrm{out}\rangle, \tag{4.1}$$

where the partition function $Z(\beta)$ is

$$Z=\sum_{n=0}^{\infty}e^{-\beta n\omega} \tag{4.2}$$

and the states $|\tilde{n}\rangle$ are a complete orthonormal basis for the region of space causally disconnected from an external observer. Operators corresponding to physically measurable quantities are defined on the basis set $|n\rangle$ for states outside the horizon. The ensemble average (expectation value) of a physical observable $\hat{O}$ in the out region is

$$\langle\mathrm{out}|\hat{O}|\mathrm{out}\rangle=\frac{1}{Z(\beta)}\sum_n e^{-n\beta\omega}\langle n|\hat{O}|n\rangle. \tag{4.3}$$

As usual the temperature is determined by the surface gravity according to Eq. (2.2).

For example if the operator $\hat{O}$ is the number operator

$$\hat{O}=a^{\dagger}a, \tag{4.4}$$

for particles of rest mass m, the ensemble average given in Eq. (4.3) is the particle number density

$$n_{\beta_H}(m,k)=\frac{1}{e^{\beta_H\omega_k(m)}-1}. \tag{4.5}$$

This expression can be immediately used for Schwarzschild black holes. Its generalization to particles (black holes) with spin and charge is straightforward and amounts to summing also over J and Q (which we avoid writing explicitly in this section to keep the notation simpler).

To describe particle interactions one needs the particle propagator, which can be determined from Eq. (4.3) by means of the time-ordered product $\langle \mathrm{out}|T\phi(x_1)\phi(x_2)|\mathrm{out}\rangle$. The Fourier transform of this expression is

$$\Delta_\beta = \frac{1}{k^2 - m^2 + i\epsilon} - 2\pi i\,\delta(k^2 - m^2)\,n_\beta(m,k), \qquad (4.6)$$

with n_β given by Eq. (4.5). These expressions are valid if black holes are described by a local field theory. However, as discussed in Sec. II, the particle number distribution given in Eq. (4.5) implies loss of coherence. The in state is a pure state

$$|0_{in};\ t=0\rangle = |0\rangle \otimes |\tilde{0}\rangle, \qquad (4.7)$$

but the number density obtained from the outgoing states is a thermal distribution.

In the microcanonical approach black holes are quantum excitations of p-branes, and hence non-local effects must be taken into account. This is accomplished by summing over all possible masses (angular momenta and charges):

$$n_{\beta_H}(k) = \int_0^\infty dm\,\sigma(m)\,n_{\beta_H}(m,k). \qquad (4.8)$$

Inclusion of non-local effects changes the thermal vacuum to

$$|\mathrm{out}\rangle = \frac{1}{Z^{1/2}(\beta_H)}\left[\prod_{m,k}\ \sum_{n_{k,m}=0}^{\infty}\right] \times \prod_{m,k} e^{-\beta_H n_{k,m}\omega_{k,m}/2}|n_{k,m}\rangle \otimes |\tilde{n}_{k,m}\rangle, \qquad (4.9)$$

in which the m and k labels are shown explicitly to emphasize the dependence of the states on the mass and momentum. The quantity in square brackets represents the product of the sums over the discrete values of the momentum and mass. The canonical partition function extracted from this expression is

$$Z(\beta_H) = \exp\left(-\frac{V}{(2\pi)^{D-1}} \int_{-\infty}^{+\infty} d^{D-1}\vec{k} \int_0^\infty dm\,\sigma(m) \times \ln[1 - e^{-\beta_H \omega_k(m)}]\right), \qquad (4.10)$$

where the discrete mass and momentum indices have been changed to continuous values. A system in thermodynamical equilibrium must satisfy the condition

$$\int_0^\infty \Omega(E)\,e^{-\beta E_H} dE = \exp\left(-\frac{V}{(2\pi)^{D-1}} \int_{-\infty}^{+\infty} d^{D-1}\vec{k} \int_0^\infty dm\,\sigma(m) \times \ln[1 - e^{-\beta_H \omega_k(m)}]\right). \qquad (4.11)$$

This is Hagedorn's self-consistency condition [30]. It is well known that only strings $(p=1)$ satisfy this condition,

$$\sigma(m) \sim e^{bm} \quad (m \to \infty), \qquad (4.12)$$

for $\beta_H > b =$ Hagedorn's inverse temperature. But black holes are not strings, as has been inferred from the quantum mechanical density of states for Schwarzschild black holes in Sec. III A,

$$\sigma(M) \sim e^{CM^{(D-2)/(D-3)}}. \qquad (4.13)$$

Therefore black holes do not satisfy Hagedorn's condition $[(D-2)/(D-3) > 1$ for $D > 3]$. The black hole system is not in thermal equilibrium because it does not fulfill the self-consistency condition required for thermal equilibrium. We are thus led to the conclusion that the thermal vacuum is the false vacuum for a black hole.

B. Microcanonical formulation

The true vacuum for a black hole can be obtained by first writing the thermal vacuum in terms of the density matrix $\hat{\rho}$ for a system in thermal equilibrium:

$$|0(\beta)\rangle = \hat{\rho}(\beta;\mathbf{H})|\mathscr{I}\rangle, \qquad (4.14)$$

where

$$\hat{\rho}(\beta,\mathbf{H}) = \frac{\rho(\beta,\mathbf{H})}{\langle \mathscr{I}|\rho(\beta,\mathbf{H})|\mathscr{I}\rangle}$$

$$\rho(\beta;\mathbf{H}) = e^{-\beta\mathbf{H}}$$

$$|\mathscr{I}\rangle = \left[\prod_{k,m}\ \sum_{n_{k,m}}\right] \prod_{k,m} |n_{k,m}\rangle \otimes |\tilde{n}_{k,m}\rangle. \qquad (4.15)$$

The trace of an observable operator is given by

$$\mathrm{Tr}\ \hat{O} = \langle \mathscr{I}|\hat{O}|\mathscr{I}\rangle. \qquad (4.16)$$

For example the free field propagator can be determined from

$$\Delta_\beta^{ab} = -i\langle \mathscr{I}|T\phi^a(x_1)\phi^b(x_2)\hat{\rho}|\mathscr{I}\rangle. \qquad (4.17)$$

The superscripts on ϕ refer to the member of the thermal doublet [31],

$$\phi^a = \begin{pmatrix} \phi \\ \tilde{\phi}^\dagger \end{pmatrix}, \qquad (4.18)$$

being considered. The Fourier transform of $\Delta_\beta^{11}(x_1,x_2)$ (the physical component) is equal to Δ_β given in Eq. (4.6).

If instead of treating black holes as objects in thermal equilibrium at fixed temperature T and corresponding vacuum $|\beta_H\rangle$ we treat the black holes as having fixed energy E, we can formally define the microcanonical vacuum as

$$|E\rangle=\frac{1}{\Omega(E)}\int_0^E \Omega(E-E')L^{-1}_{E-E'}[|\beta_H\rangle]dE', \quad (4.19)$$

where L^{-1} is the inverse Laplace transform. Our analysis of the WKB approximation for black holes in Secs. II and III shows that the assumption that black holes are at fixed energy is physically more reasonable than assuming that they are at fixed temperature. Using this basis physical correlation functions are expressed as

$$G_E^{a_1,\dots,a_N}(1,2,\dots,N)=\langle\mathscr{T}|T\phi^{a_1}(1),\dots,\phi^{a_N}(N)|E\rangle. \quad (4.20)$$

Interaction effects can be taken into account by means of the microcanonical propagator

$$\Delta_E^{11}(k)=\frac{1}{k^2-m^2+i\epsilon}-2\pi i\,\delta(k^2-m^2)n_E(m,k), \quad (4.21)$$

where $n_E(m,k)$ is the microcanonical number density,

$$n_E(m,k)=\sum_{l=1}^{\infty}\frac{\Omega(E-l\omega_k(m))}{\Omega(E)}\theta(E-l\omega_k), \quad (4.22)$$

which is our candidate alternative to Eq. (4.5) for the distribution of particles emitted by a black hole.

V. WAVE FUNCTIONS

The analysis carried out so far is global in nature. In fact, although we were able to show consistent equilibrium configurations for gases of black holes and number densities for the emitted radiation in such configurations, the geometry of spacetime never appears explicitly in the final expressions. Of course, one is also interested in the local properties of spacetime, and this is most intriguing in the present case because the above results include implicitly any back-reactions of the radiation on the metric.

Thus we need a probe to test the spacetime which corresponds to the microcanonical vacuum described in the previous section. We then turn to the study of the propagation of waves and show that the wave functions in the microcanonical vacuum can be obtained by making a formal replacement in the wave functions obtained for the thermal vacuum.

A. Thermal vacuum

In flat 4-dimensional spacetime with spherical coordinates t,r,θ,ϕ an incoming spherical wave collapsing on a point is given in null coordinates asymptotically by

$$\psi_{in}=\frac{Y_{lm}(\theta,\phi)}{\sqrt{8\pi^2\omega}}\frac{e^{-i\omega v}}{r} \quad v=t+r_*, \quad (5.1)$$

where r_* is the so-called *tortoise* coordinate. An outgoing wave has the form

$$\psi_{out}=\frac{Y_{lm}(\theta,\phi)}{\sqrt{8\pi^2\omega}}\frac{e^{-i\omega u}}{r} \quad u=t-r_*. \quad (5.2)$$

If we now consider waves propagating on a black hole background, e.g. a Schwarzschild black hole, and do not take into account back-reactions, then the incoming wave becomes (see for example [33])

$$\psi_{in}=\begin{cases}\dfrac{Y_{lm}(\theta,\phi)}{\sqrt{8\pi^2\omega}}\dfrac{e^{i(\omega/\kappa)\ln(v_0-v)}}{r}, & v<v_0,\\[2mm] 0, & v>v_0.\end{cases} \quad (5.3)$$

This wave obeys the wave equation in a background with surface gravity κ. The in states for the two vacua are related by the Bogoliubov transformation

$$\left.\begin{matrix}\alpha_{\omega\omega'}\\ \beta_{\omega\omega'}\end{matrix}\right\}=\frac{1}{2\pi}\int_{-\infty}^{v_0}dv\left(\frac{\omega'}{\omega}\right)^{1/2}e^{\pm i\omega' v}e^{i(\omega/\kappa)\ln[(v_0-v)/c]}, \quad (5.4)$$

where c is a constant. The two coefficients α and β are related by the Wronskian condition

$$\sum_{\omega'}[|\alpha_{\omega\omega'}|^2-|\beta_{\omega\omega'}|^2]=1. \quad (5.5)$$

In Eq. (5.5) the variable ω' has been rendered a discrete variable by box normalization of the wave functions. Also backscattering of the fields from the spacetime curvature has been ignored. The integrals in Eq. (5.4) can be evaluated explicitly, and one finds that

$$|\alpha_{\omega\omega'}|^2=e^{2\pi\omega/\kappa}|\beta_{\omega\omega'}|^2. \quad (5.6)$$

Substituting this relation into Eq. (5.5) one obtains the Planckian distribution

$$n_{\beta_H}(\omega)=\sum_{\omega'}|\beta_{\omega\omega'}|^2=\frac{1}{e^{\beta_H\omega}-1}. \quad (5.7)$$

B. Microcanonical vacuum

The relationship between α and β in Eq. (5.6) arises because the logarithmic term in Eq. (5.4) introduces a branch cut, and the integration around this branch cut causes the factor multiplying this term (times 2π) to appear in the exponential multiplying β. Thus if we simply make the formal replacement

$$\frac{2\pi\omega}{\kappa}\to\ln[1+n_E^{-1}(\omega)], \quad (5.8)$$

where $n_E(\omega)$ is the microcanonical number density as expressed in Eq. (4.22), the relevant waves are of the form

$$\psi_{out}=\frac{Y_{lm}(\theta,\phi)}{\sqrt{8\pi^2\omega}}\frac{e^{-i\omega u}}{r}, \tag{5.9}$$

and

$$\psi_{in}=\begin{cases}\dfrac{Y_{lm}(\theta,\phi)}{\sqrt{8\pi^2\omega}}\dfrac{e^{(i/2\pi)\ln[1+n_E^{-1}(\omega)]\ln(v_0-v)}}{r}, & v<v_0,\\ 0, & v>v_0.\end{cases} \tag{5.10}$$

The relation between α and β now becomes

$$|\alpha_{\omega\omega'}|^2=e^{\ln[1+n_E^{-1}(\omega)]}|\beta_{\omega\omega'}|^2, \tag{5.11}$$

which gives, for the sum over ω',

$$\sum_{\omega'}|\beta_{\omega\omega'}|^2=n_E(\omega). \tag{5.12}$$

Of course the wave in Eq. (5.10) does not satisfy the same wave equation as the wave in Eq. (5.3), but it will satisfy a wave equation in a background whose metric includes back-reaction and non-local effects.

VI. BLACK HOLE DECAY RATES

The rate of decay of any kind of black hole can be calculated from the number density $n(\omega)$ representing the available states which can be fed into by the Hawking radiation. Thus the number density (5.7) obtained from the thermodynamical picture should apply also to black holes with very small mass. But for black holes with mass of the order of the Planck mass or less we will show that the thermodynamical picture is no longer a good approximation of the true microcanonical description and one expects the two approaches to lead to significantly different predictions for the survival of cosmological black holes (with small mass) in the present universe.

For simplicity we consider only 4-dimensional Schwarzschild black holes. In this case the microcanonical density is given by (we restore the fundamental constants)

$$\Omega(M)=e^{4\pi M^2/M_p^2}, \tag{6.1}$$

with the mass M of the black hole times c^2 being the fixed energy E and $M_p=\sqrt{\hbar c/G}$ the Planck mass. Substituting the expression for Ω into Eq. (4.22), we find, for the number density,

$$n_M(\omega)=\sum_{l=1}^{Mc^2/\hbar\omega}e^{(-8\pi lM/M_p^2c^2)\hbar\omega+(4\pi l^2/M_p^2c^4)\hbar^2\omega^2}. \tag{6.2}$$

Then we notice that in the classical limit $\hbar\to 0$ (at fixed c and G) the Planck mass vanishes and the sum over l runs up to ∞ for fixed M and ω. Therefore one recognizes that the $\hbar\to 0$ limit is equivalent to taking $M/M_p\to\infty$ at fixed ω and neglecting the [positive and $\mathcal{O}(\hbar)$] ω^2 term in the exponent with respect to the [negative and $\mathcal{O}(\hbar^0)$] ω term. In this case we recover the thermal number density (5.7).

The above approximation works only for $\hbar\omega\ll Mc^2/l$, which shows that high frequency Hawking photons (with $\omega\sim Mc^2/\hbar$) are responsible for the failure of the canonical description and that corrections to the thermal spectrum become effective when the total energy emitted in a given mode ($\sim l\hbar\omega$) is of the same order of the mass of the black hole (Mc^2). Therefore one expects that the microcanonical description begins to depart appreciably from the thermal description for relatively small black holes, that is for black holes which have already emitted a large fraction of their mass, and fails completely for black holes of Planck mass or less. This is consistent with the naive idea that, when the amount of energy emitted by a black hole becomes comparable to M, the (vacuum) geometry assumed to describe the initial space-time becomes a less reliable approximation.

Further, the ω^2 term in Eq. (6.2) is a reflection of the fact that black holes have negative microcanonical specific heat. To see this let us set $c=G=\hbar=1$ again and rewrite Eq. (4.22) as

$$n_M=\sum_{l=1}^{\infty}\frac{\Omega(M-l\omega_k(m))}{\Omega(M)}\theta(M-l\omega_k)$$
$$=\sum_{l=1}^{M/\omega}\exp[S_E(M-l\omega)-S_E(M)]. \tag{6.3}$$

Taylor-expanding the last expression we find

$$n_M=\exp\left(-l\omega\frac{\partial S}{\partial M}+\frac{l^2\omega^2}{2}\frac{\partial^2 S}{\partial M^2}+\dots\right). \tag{6.4}$$

The first term in the exponential is proportional to β_H while the second term is proportional to β_H^2/C_V. We recall here that our results rely on the use of the microcanonical ensemble as the only self-consistent approach for systems with degeneracies given by expressions of the form in Eq. (3.2), which, in turn, are expected to apply to p-branes [1,2,25–27]. The same expressions (6.3) and (6.4) have been obtained recently in Ref. [15] and further generalized in [16] starting from an effective D-string model for black hole microstates in the microcanonical ensemble.

Using the expressions for n_{β_H} in Eq. (5.7) and n_M in Eq. (6.2), we can calculate the decay rates for radiating black holes predicted by the two theories. The luminosity is given by

$$L=\frac{1}{2\pi}\int_0^\infty d\omega\,\omega^3\Gamma(\omega)n_i(\omega),\quad i=\beta,\ M. \tag{6.5}$$

The factor Γ takes into account backscattering from the spacetime curvature and in this capacity would replace the 1 on the right hand side of Eq. (5.5). Carrying out the integration for the thermal number density with $\Gamma \sim 1$ and multiplying by the horizon area to get the rate at which mass is lost, we find

$$\left.\frac{dM}{dt}\right|_{\beta} \simeq -\frac{\hbar c^4}{G^2 M^2}. \qquad (6.6)$$

For $M \sim M_p$ we approximate the sum in the microcanonical number density (6.2) by retaining only the dominant $l=1$ term and the integration over ω in Eq. (6.5) can then be carried out explicitly. In this approximation we obtain

$$\left.\frac{dM}{dt}\right|_{M} \simeq -\frac{cG}{\hbar^2} M^4. \qquad (6.7)$$

This rate is in agreement with what one would expect from a model of quantum black holes as extended objects (p-branes). The more massive the black hole is, i.e. the higher the excited state of the p-brane, the more rapid the decay. The constant in front of the M^4 term is very large, so that the decay rate is very high until masses much less than the Planck mass are reached. In this region of mass Eq. (6.7) is no longer adequate because the luminosity L in Eq. (6.5) would have to be obtained using the full sum in Eq. (6.2).

The thermodynamical theory predicts that the decay rate blows up as $M \to 0$ and that it takes a finite time for the black hole to completely evaporate,

$$M \sim (M_0^3 - 3\delta t)^{1/3}, \qquad (6.8)$$

where $\delta = \hbar c^4 / G^2$. Thus one does not expect to find primordial black holes in the present universe. However, as we have explained above, for late stages of the evaporation process the decay rate is better approximated by the expression in Eq. (6.7) which, on the other hand, predicts that primordial, microscopic black holes could still be around today. In fact the corresponding decay rate goes to zero as a power of M and for $M \sim M_p$,

$$M \sim \frac{M_0}{(1+3\alpha M_0^3 t)^{1/3}}, \qquad (6.9)$$

where $\alpha = cG/\hbar^2$, or even slower for $M \ll M_p$.

VII. CONCLUSION

For black hole masses near the Planck mass the microcanonical approach is clearly preferable to the thermodynamical approach in the semiclassical quantization processes described above. It is free of the inconsistencies present in the thermodynamical approach, and its predictions seem to be more physically reasonable, e.g. a finite black hole decay rate throughout the life of the black hole.

The use of a fixed energy basis for the Hilbert space of the theory instead of the usual thermal state implies that black holes are particle states. In our interpretation of black holes as quantum objects the associated quantum degeneracy of states obtained from the inverse of the tunneling probability points to the identification of black holes with the excitation modes of p-branes. The self-consistent treatment of black holes as quantum extended objects implies that black holes are elementary particles.

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