Lightlike signals in general relativity and cosmology

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The modelling of light-like signals in general relativity taking the form of impulsive gravitational waves and light-like shells of matter is examined. Systematic deductions from the Bianchi identities are made. These are based upon Penrose's hierarchical classification of the geometry induced on the null hypersurface history of the signal by its embedding in the space-times to the future and to the past of it. The signals are not confined to propagate in a vacuum and thus their interaction with matter (a burst of radiation propagating through a cosmic fluid, for example) is also studied. Results are accompanied by illustrative examples using cosmological models, vacuum space-times, the de Sitter universe and Minkowskian space-time. [S0556-2821(98)06416-9]

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I. INTRODUCTION

In his classic work on impulsive gravitational waves (gravitational waves having a Dirac delta function profile) Penrose [1] introduced a hierarchical classification of intrinsic geometries that the null hypersurface history of the wave front inherits from the space-times it is embedded in. This classification is related to the physical characteristics of the light-like signal—a fact which emerges clearly from the analysis of light-like shells of matter and impulsive gravitational waves carried out by Barrabès and Israel [2]. They generalize the usual approach (see [3]) to the study of non-null singular hypersurfaces, based on the extrinsic curvature tensor of the hypersurface, to include the light-like case. The light-like signal can be an impulsive gravitational wave or a light-like shell of matter or both. The latter situation could be viewed as a model of a burst of gravitational radiation accompanied by a burst of neutrinos from a supernova [4]. In [1] and [2] the properties of the signal which can be obtained from the Bianchi identities are either implicit in the work or are only partially explicitly derived. In this paper we take a systematic approach to the deductions one can make from the Bianchi identities depending upon the type of induced geometry (in Penrose's sense) on the history of the signal. As one moves through the hierarchy of geometries more information becomes available from the Bianchi identities. As our study is not confined to vacuum space-times we in addition examine the interaction of a light-like signal with matter (a burst of radiation propagating through the cosmic fluid). We therefore analyze in some detail the influence of the light-like signal on a congruence of time-like world lines—the histories of galaxies in a cosmological model, for example. We show that for the signal to include a gravitational wave the shear of the fluid lines must jump across the history of the signal in space-time. For the signal to be a light-like shell of matter then a jump in the acceleration, vorticity or expansion of the histories of the fluid lines is necessary. Our deductions are presented (in Sec. IV below) in the form of a series of lemmas, each followed by illustrative examples involving cosmological models, vacuum space-times, the de Sitter universe and Minkowskian space-time.

The outline of the paper is as follows: In Sec. II we give an introduction to and brief summary of the Barrabès-Israel [2] formalism, the basic equations needed to study the behavior of a time-like congruence intersecting the history of the light-like signal in space-time and a description of the Penrose classification of induced geometries on the history of the signal. In Sec. III we systematically deduce the consequences of the twice-contracted Bianchi identities and the Bianchi identities which become available as one works through the hierarchy of induced geometries. Section IV contains the main conclusions of our work in the form of physical and geometrical applications of the results of Secs. II and III. The paper concludes with a discussion in Sec. V.

II. GEOMETRICAL PRELIMINARIES

The history of a light-like shell or an impulsive gravitational wave corresponds in the space-time manifold $\mathcal{M}$ to a singular null hypersurface across which the metric tensor $g_{\mu\nu}$ is only C^0: i.e., it is continuous but its first derivatives are discontinuous across the hypersurface. This can be used for example as a model for the emission of light-like matter and gravitational radiation due to a sudden change in properties such as the mass, the angular momentum and the mutipole moments of a gravitating body (an example of this is the production of bursts of neutrinos and gravitational radiation from a supernova). In this section we first make a brief review of a general formalism adapted to the case of singular null hypersurfaces which has been developed a few years ago [2], and of its application to the splitting of the light-like signal between a shell and a wave [4]. Then we analyze the discontinuities in the kinematical quantities (velocity, acceleration, expansion, shear and vorticity) of a congruence of time-like lines crossing a singular null hypersurface. Finally we present the Penrose classification [1] of intrinsic geom-

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etries that a singular null hypersurface inherits from its embedding in $\mathcal{M}$ and which will serve as a basis for the classification of the action of a null shell and/or a wave on a congruence of time-like lines.

We consider a space-time $\mathcal{M}$ which is divided into two halves $\mathcal{M}^+$ and $\mathcal{M}^-$ separated by a null hypersurface $\mathcal{N}$ and with a local coordinate system $\{x_+^\mu\}$ in $\mathcal{M}^+$ and $\{x_-^\mu\}$ in $\mathcal{M}^-$, Greek indices taking values 0,1,2,3, in terms of which the metric tensor components are $g^+_{\mu\nu}$ and $g^-_{\mu\nu}$ respectively. Let ξ^a, Latin indices taking values 1,2,3, be local intrinsic coordinates on $\mathcal{N}$. The parametric equations of the embeddings of $\mathcal{N}$ have the forms $x^\mu_\pm = f^\mu_\pm(\xi^a)$. The corresponding tangent basis vectors are $e_{(a)} = \partial/\partial\xi^a$, and their scalar products define the induced metric on $\mathcal{N}$:

$$g_{ab} = e_{(a)} \cdot e_{(b)} = g_{\mu\nu} e^\mu_{(a)} e^\nu_{(b)}\big|_\pm , \tag{2.1}$$

assumed the same on both faces of the hypersurface. Four dimensional scalar products are indicated by a centered dot as in Eq. (2.1) and evaluation of a quantity on the plus or minus side of $\mathcal{N}$ is indicated by a vertical stroke followed by a plus or minus (or both if the quantity is the same on both sides of $\mathcal{N}$).

Let n be normal to $\mathcal{N}$ with components $n^\mu_\pm$ viewed on the plus and minus sides. Thus

$$n \cdot n\big|_\pm = 0, \quad n \cdot e_{(a)}\big|_\pm = 0. \tag{2.2}$$

As $\mathcal{N}$ is a null hypersurface the normal n is tangent to it and can be decomposed along the tangent basis vectors as

$$n = n^a e_{(a)} . \tag{2.3}$$

Let us introduce a transversal N on $\mathcal{N}$ and require its projections onto the plus and minus sides of N to be equal, i.e.

$$[N \cdot N] = [N \cdot e_{(a)}] = 0, \tag{2.4}$$

where we use square brackets to denote the jump of any quantity F that is discontinuous across the hypersurface: $[F] = {}^+F - {}^-F$. In order to make sure that N is transverse to the hypersurface we require that it have a non-vanishing scalar product with the normal n:

$$N \cdot n = \eta^{-1}, \tag{2.5}$$

where η is some non-zero constant usually taken to be -1. Note that N is not uniquely defined by this equation as one may make a tangential displacement $N \mapsto N + \zeta^a(\xi^b) e_{(a)}$ with arbitrary functions ζ^a for a prescribed η. Next we define the transverse extrinsic curvature $\mathcal{K}_{ab}$ of the null hypersurface $\mathcal{N}$ as

$$\mathcal{K}_{ab} = -N \cdot \nabla_{e_{(b)}} e_{(a)} , \tag{2.6}$$

and its jump across the null hypersurface by

$$\gamma_{ab} = 2[\mathcal{K}_{ab}], \tag{2.7}$$

where ∇ denotes covariant differentiation with respect to the four-dimensional Levi-Cività connection. It can be shown [2] that the jumps γ_{ab} are independent of the choice of the transversal N and thus define a purely intrinsic property of the hypersurface.

As the metric tensor is only C^0 at the hypersurface the Riemann, Ricci and Weyl tensors in general each contain a singular Dirac δ-term. The stress-energy tensor which appears on the matter side of the Einstein field equations also contains a similar term which one interprets as the surface stress-energy tensor of a shell whose history in space-time is the null hypersurface $\mathcal{N}$. The existence of a surface stress-energy tensor is an intrinsic property of the hypersurface and it has intrinsic components S^{ab} which are given by [2]

$$16\pi\eta^{-1} S^{ab} = (g_*^{ac} n^b n^d + g_*^{bd} n^a n^c)\gamma_{cd} - g_*^{ab}\gamma^\dagger - n^a n^b (g_*^{cd}\gamma_{cd}), \tag{2.8}$$

where $\gamma^\dagger = \gamma_{ab} n^a n^b$. In this expression we have introduced a (non-unique) ''pseudo''-inverse metric [2] g_*^{ab} (as $\mathcal{N}$ is a null hypersurface the induced metric is degenerate and has no inverse) such that

$$g_*^{ab} g_{bc} = \delta^a_c - \eta n^a N_c , \tag{2.9}$$

where $N_c = N \cdot e_{(c)}$. It is important to note that the expression (2.8) for the surface stress-energy tensor is independent of the choice of the ''pseudo''-inverse metric.

Another important point is that only four of the six components of the jump γ_{ab} appear in S^{ab}, and correspond to $n^a\gamma_{ab}$ and $g_*^{ab}\gamma_{ab}$. This leaves two components which are decoupled from matter and describe an impulsive gravitational wave, as an analysis of the singular δ-part of the Weyl tensor reveals [4]. These two components are given by [4]

$$\hat{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2} g_*^{cd}\gamma_{cd} g_{ab} - 2\eta n^c \gamma_{c(a} N_{b)} + \eta^2 \gamma_{cd} n^c n^d N_a N_b , \tag{2.10}$$

and it is easy to see that $\hat{\gamma}_{ab} n^b = 0$ and $g_*^{ab}\hat{\gamma}_{ab} = 0$. The two non-vanishing components of $\hat{\gamma}_{ab}$ represent the two degrees of freedom of polarization of the wave.

A covariant counterpart to the above description exists and we now summarize the main results. The intrinsic quantities γ_{ab} and S^{ab} can be extended to four-tensors on the space-time manifold. In a local coordinate system $\{x^\mu\}$ covering both sides of $\mathcal{N}$ (for example it can be either $\{x^\mu_+\}$ or $\{x^\mu_-\}$) we define the tensors $\gamma_{\mu\nu}$ and $S^{\mu\nu}$ by the requirements.

$$\gamma_{\mu\nu} e^\mu_{(a)} e^\nu_{(b)} = \gamma_{ab} , \quad S^{ab} e^\mu_{(a)} e^\nu_{(b)} = S^{\mu\nu}. \tag{2.11}$$

It is easy to see that $S^{\mu\nu}$ satisfies $S^{\mu\nu} n_\nu = 0$ and is thus tangential to the hypersurface. It can also be shown that $\gamma_{\mu\nu}$ is directly related to the jump in the first derivatives of the metric tensor. One gets

$$[\partial_\alpha g_{\mu\nu}] = \eta n_\alpha \gamma_{\mu\nu}, \quad [N^\alpha \partial_\alpha g_{\mu\nu}] = \gamma_{\mu\nu}, \tag{2.12}$$

where we have used Eq. (2.5), and for the jumps of the Christoffel symbols,

$$[\Gamma^{\mu}_{\alpha\beta}]=\frac{\eta}{2}(\gamma^{\mu}_{\alpha}n_{\beta}+\gamma^{\mu}_{\beta}n_{\alpha}-n^{\mu}\gamma_{\alpha\beta}). \tag{2.13}$$

The covariant components $S_{\alpha\beta}$ of the surface stress-energy tensor of the null shell are given by

$$16\pi\eta^{-1}S_{\alpha\beta}=2\gamma_{(\alpha}n_{\beta)}-\gamma n_{\alpha}n_{\beta}-\gamma^{\dagger}g_{\alpha\beta}, \tag{2.14}$$

where

$$\gamma_{\alpha}=\gamma_{\alpha\beta}n^{\beta}, \quad \gamma=\gamma_{\alpha\beta}g^{\alpha\beta}, \tag{2.15}$$

and $\gamma^{\dagger}$ is defined above and can be rewritten as

$$\gamma^{\dagger}=\gamma_{\alpha\beta}n^{\alpha}n^{\beta}=\gamma_{\alpha}n^{\alpha}. \tag{2.16}$$

We consider now a congruence of time-like curves with unit tangent vector u^{α}, $u\cdot u=-1$, crossing the singular null hypersurface $\mathcal{N}$ (as previously we are using a coordinate system $\{x^{\mu}\}$ covering both sides of $\mathcal{N}$). This congruence can be arbitrarily chosen in each domain $\mathcal{M}^{\pm}$ and in the sequel we shall only consider the case where the tangent vector is continuous across $\mathcal{N}$. If u is tangent to matter world lines in $\mathcal{M}^{\pm}$, then choosing u continuous across $\mathcal{N}$ forbids $\mathcal{N}$ becoming the history of a shock wave propagating through the matter in the usual sense [5,6]. We choose u to be continuous across $\mathcal{N}$ for the following reasons: (a) this is the minimal requirement consistent with a delta function appearing in the Weyl tensor (as can be seen by applying the Ricci identities to u), (b) this allows finite jumps (and no delta function) to appear in the kinematical quantities associated with the integral curves of u (the acceleration, vorticity, shear and expansion) and (c) the jumps in the kinematical quantities are then simply related to the presence or otherwise of a light-like shell and/or a gravitational wave in the signal with history $\mathcal{N}$ (see Sec. IV below where these general results and physical examples are given).

Let us then consider a congruence of time-like curves with continuous 4-velocity u but with discontinuous first derivatives, having jumps across $\mathcal{N}$ described by the vector λ such that

$$[\partial_{\mu}u^{\alpha}]=\eta n_{\mu}\lambda^{\alpha} \quad \text{or} \quad [N^{\mu}\partial_{\mu}u^{\alpha}]=\lambda^{\alpha}, \tag{2.17}$$

where we have used Eq. (2.5). It follows that the jump in the 4-acceleration $a^{\alpha}=u^{\mu}\nabla_{\mu}u^{\alpha}$ is given by

$$\eta^{-1}[a^{\alpha}]=-s\lambda^{\alpha}-sU^{\alpha}-\frac{1}{2}(u\cdot U)n^{\alpha}, \tag{2.18}$$

where we have put $s=-u\cdot n>0$ and $U_{\alpha}=\gamma_{\alpha\beta}u^{\beta}$. Using $u\cdot a=0$ and the above equations one gets the following relation:

$$u\cdot U=-2u\cdot\lambda. \tag{2.19}$$

The expansion θ, shear $\sigma_{\alpha\beta}$, and vorticity $\omega_{\alpha\beta}$ of the time-like congruence are in general discontinuous across $\mathcal{N}$ and have jumps given by

$$\eta^{-1}[\theta]=-\frac{1}{2}s\gamma+\lambda\cdot n, \tag{2.20}$$

$$\eta^{-1}[\sigma_{\alpha\beta}]=-\frac{s}{2}\gamma_{\alpha\beta}+\lambda_{(\alpha}n_{\beta)}-sU_{(\alpha}u_{\beta)} -\frac{1}{2}(u\cdot U)n_{(\alpha}u_{\beta)}-s\lambda_{(\alpha}u_{\beta)}-\frac{\eta^{-1}}{3}[\theta]h_{\alpha\beta}, \tag{2.21}$$

$$\eta^{-1}[\omega_{\alpha\beta}]=U_{[\alpha}n_{\beta]}+\lambda_{[\alpha}n_{\beta]}-sU_{[\alpha}u_{\beta]} -\frac{1}{2}(u\cdot U)n_{[\alpha}u_{\beta]}-s\lambda_{[\alpha}u_{\beta]}, \tag{2.22}$$

where the projection tensor $h_{\alpha\beta}=g_{\alpha\beta}+u_{\alpha}u_{\beta}$ has been introduced. We can alternatively express Eq. (2.22) in terms of the vorticity vector

$$\omega^{\alpha}=\frac{1}{2}\eta^{\alpha\beta\mu\nu}u_{\beta}\omega_{\mu\nu}, \tag{2.23}$$

where $\eta^{\alpha\beta\mu\nu}=(-g)^{-1/2}\epsilon_{\alpha\beta\mu\nu}$ with $g=\det(g_{\mu\nu})$ and $\epsilon_{\alpha\beta\mu\nu}$ is the four-dimensional Levi-Cività permutation symbol. The jump in the vorticity vector is given by

$$\eta^{-1}[\omega^{\alpha}]=\frac{1}{2}\eta^{\alpha\beta\mu\nu}u_{\beta}(U_{\mu}+\lambda_{\mu})n_{\nu}. \tag{2.24}$$

Finally we recall a classification introduced by Penrose [1] for the geometry that the null hypersurface $\mathcal{N}$ inherits from $\mathcal{M}^{+}$ and $\mathcal{M}^{-}$. As any null hypersurface $\mathcal{N}$ is generated by a uniquely defined two-parameter family of null geodesics one can consider a hierarchy of three types of intrinsic geometry in order of increasing structure.

Type I: the induced metrics match [our basic assumption (2.1)].

Type II: type I with parallel transport of the normal n along the null generators matching.

Type III: type II with parallel transport of any tangent vector to $\mathcal{N}$ along the null generators matching.

A type I geometry on $\mathcal{N}$ is the most general of the three types and is always assumed to be valid in our considerations. A type II geometry requires that $n^{\mu}\nabla^{+}_{\mu}n^{\alpha}=n^{\mu}\nabla^{-}_{\mu}n^{\alpha}$ or using Eq. (2.13) that $\gamma^{\dagger}=0$. If one defines the acceleration parameter κ by

$$n^{\mu}\nabla_{\mu}n^{\alpha}=\kappa n^{\alpha}, \tag{2.25}$$

one can show that $[\kappa]=\eta\gamma^{\dagger}/2$ so that a type II geometry implies $[\kappa]=0$. In particular this is realized when the null generators are affinely parametrized on both sides of $\mathcal{N}$. A type III geometry requires that $n^{\mu}\nabla^{+}_{\mu}v^{\alpha}=n^{\mu}\nabla^{-}_{\mu}v^{\alpha}$ for any vector v^{α} such that $n_{\alpha}v^{\alpha}=0$, and using Eq. (2.13) this implies that $\gamma_{\alpha}=An_{\alpha}$ where A is an arbitrary function on $\mathcal{N}$.

A physical interpretation in terms of the surface stress-energy tensor of a null shell, if it exists having history $\mathcal{N}$, can be given to the type II and III geometries above. As shown in [2] the surface pressure $p=-(\eta/16\pi)\gamma^{\dagger}$ and so a type II

induced geometry corresponds to a pressure-free null shell, i.e. a shell only admitting a surface energy density ηf and anisotropic surface stresses. For a type III geometry induced on $\mathcal{N}$ the surface stress-energy tensor reduces from Eq. (2.14) to $16\pi\eta^{-1}S_{\alpha\beta}=(2A-\gamma)n_\alpha n_\beta$, there are no surface stresses and the surface energy density is $\eta f=\eta(2A-\gamma)/16\pi$. If $A=\gamma/2$, then there is no shell and $\mathcal{N}$ is the history of an impulsive gravitational wave provided $\hat{\gamma}_{ab}\neq 0$.

III. CONSEQUENCES OF THE BIANCHI IDENTITIES

On the null hypersurface history of the light-like signal we have the normal n (which is tangential to $\mathcal{N}$ since $\mathcal{N}$ is null) and the time-like vector field u, for which $u\cdot n=-s<0$. In a local coordinate system $\{x^\mu\}$ covering both sides of $\mathcal{N}$, it is helpful to define on $\mathcal{N}$, but not tangent to $\mathcal{N}$, the null vector field

$$l^\mu=-\frac{1}{2s^2}\,n^\mu+\frac{1}{s}\,u^\mu, \tag{3.1}$$

with $l\cdot n=-1$. Now n,l can be supplemented by a complex null vector field m, with components m^μ, tangent to $\mathcal{N}$ and also orthogonal to l, and satisfying $m\cdot\bar{m}=+1$ (the overbar denoting complex conjugation). The null tetrad $\{n,l,m,\bar{m}$ defined on $\mathcal{N}$ will be useful for the purposes of displaying formulas below. If $\mathcal{N}$ has the equation $\Phi(x^\mu)=0$ and $n_\mu=\alpha^{-1}\Phi_{,\mu}$ for some function α defined on $\mathcal{N}$, then following from the results of Sec. II, the components of the Einstein tensor of the space-time $\mathcal{M}^+\cup\mathcal{M}^-$ have the form

$$G^{\mu\nu}=\hat{G}^{\mu\nu}\delta(\Phi)+\Theta(\Phi)^+G^{\mu\nu}+[1-\Theta(\Phi)]^-G^{\mu\nu}, \tag{3.2}$$

where δ is the Dirac delta function, Θ is the Heaviside step function with $\Phi>0$ in $\mathcal{M}^+$, $\Phi<0$ in $\mathcal{M}^-$ and

$$\Theta_{,\mu}=\alpha n_\mu\delta(\Phi). \tag{3.3}$$

In Eq. (3.3), ${}^\pm G^{\mu\nu}$ are the components of the Einstein tensors in $\mathcal{M}^\pm$ respectively and can be written as $-8\pi{}^\pm T^{\mu\nu}$ in terms of the respective energy-momentum-stress tensors. Also

$$\hat{G}^{\mu\nu}=-8\pi\alpha S^{\mu\nu}, \tag{3.4}$$

with $S^{\mu\nu}$ given by Eq. (2.14). Thus in particular

$$\hat{G}^{\mu\nu}n_\nu=0. \tag{3.5}$$

We now apply the twice-contracted Bianchi identities $\nabla_\nu G^{\mu\nu}\equiv 0$ to Eq. (3.2). On account of Eq. (3.5) the term in $\nabla_\nu G^{\mu\nu}$ involving the derivative of the delta function vanishes and we obtain

$$\nabla_\nu\hat{G}^{\mu\nu}+\alpha[G^{\mu\nu}n_\nu]=0. \tag{3.6}$$

Since $\hat{G}^{\mu\nu}$ in Eq. (3.2) is defined *on* $\mathcal{N}$, it only makes sense to calculate derivatives of $\hat{G}^{\mu\nu}$ tangential to $\mathcal{N}$. The number of such different tangential derivatives available to us depends upon the type of the induced geometry on $\mathcal{N}$. If the induced geometry is type I, then we obtain one meaningful equation from Eq. (3.6), namely,

$$-8\pi\eta^{-1}[T_{\mu\nu}n^\mu n^\nu]=\rho\gamma^\dagger. \tag{3.7}$$

Here square brackets as always denote the jump in the enclosed quantity across $\mathcal{N}$ and $\rho=m^\mu\bar{m}^\nu\nabla_\nu n_\mu$ is the expansion of the null geodesic integral curves of n ($[\rho]=0$ since ρ is intrinsic to $\mathcal{N}$). If the induced geometry is type II, then $\gamma^\dagger=0$. Hence the acceleration parameter κ introduced in Eq. (2.25) is continuous across $\mathcal{N}$. Now the meaningful equations emerging from the twice contracted Bianchi identities are Eq. (3.7) with $\gamma^\dagger=0$ and also

$$-16\pi\eta^{-1}[T_{\mu\nu}m^\mu n^\nu]=\gamma'_\mu m^\mu+3\rho\gamma_\mu m^\mu+\sigma\bar{m}^\mu\gamma_\mu. \tag{3.8}$$

Here $\gamma'_\mu=n^\nu\nabla_\nu\gamma_\mu$ and $\sigma=m^\mu m^\nu\nabla_\nu n_\mu$ is the complex shear of the generators of $\mathcal{N}$, which is intrinsic even for a type I geometry. Finally if the induced geometry is type III, then $\gamma_\mu=An_\mu$ for some function A defined on $\mathcal{N}$. Now the right hand sides of Eqs. (3.7) and (3.8) vanish and in addition we find that

$$\eta^{-1}[T_{\mu\nu}l^\mu n^\nu]=f'+(2\rho+\kappa)f, \tag{3.9}$$

where (as in Sec. II) $8\pi f=A-\frac{1}{2}\gamma$ and $f'=f_{,\mu}n^\mu$.

To obtain the Bianchi identities we use the tensor representing the left and right duals of the Riemann curvature tensor [7]:

$$G^{\mu\nu\rho\sigma}=\frac{1}{4}\,\eta^{\mu\nu\alpha\beta}\eta^{\rho\sigma\lambda\gamma}R_{\alpha\beta\lambda\gamma}, \tag{3.10}$$

where $R_{\alpha\beta\lambda\gamma}$ are the components of the Riemann curvature tensor and $\eta^{\alpha\beta\gamma\delta}=(-g)^{-1/2}\epsilon_{\alpha\beta\gamma\delta}$. Then the Bianchi identities read

$$\nabla_\sigma G^{\mu\nu\rho\sigma}\equiv 0. \tag{3.11}$$

The right hand side of Eq. (3.10) can be written in terms of the Riemann and Ricci tensors and the Ricci scalar as

$$-G^{\mu\nu\rho\sigma}=R^{\mu\nu\rho\sigma}-g^{\mu\rho}R^{\nu\sigma}-g^{\nu\sigma}R^{\mu\rho}+g^{\mu\sigma}R^{\nu\rho}+g^{\nu\rho}R^{\mu\sigma}+\frac{1}{2}\,R(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}). \tag{3.12}$$

It is then useful to substitute in Eq. (3.12) the Riemann tensor in terms of the Weyl tensor. For the space-time $\mathcal{M}^+\cup\mathcal{M}^-$ the tensor (3.10) has a decomposition similar to that of the Einstein tensor (3.2):

$$G^{\mu\nu\rho\sigma}=\hat{G}^{\mu\nu\rho\sigma}\delta(\Phi)+\Theta(\Phi)^+G^{\mu\nu\rho\sigma}+[1-\Theta(\Phi)]^-G^{\mu\nu\rho\sigma}, \tag{3.13}$$

with

$$\eta^{-1}\alpha^{-1}\hat{G}^{\mu\nu\rho\sigma} = -2n^{[\mu}\gamma^{\nu]\,[\rho}n^{\sigma]} - 2g^{\mu[\sigma}w^{\rho]\nu} + 2g^{\nu[\sigma}w^{\rho]\mu} - \gamma^{\dagger}g^{\mu[\rho}g^{\sigma]\nu}, \quad (3.14)$$

where $w^{\mu\nu} = \gamma^{(\mu}n^{\nu)} - \frac{1}{2}\gamma n^{\mu}n^{\nu}$. One readily sees that

$$\hat{G}^{\mu\nu\rho\sigma}n_{\sigma} = 0. \quad (3.15)$$

Thus when Eq. (3.13) is applied to Eq. (3.11) the term involving the derivative of the delta function vanishes and we obtain, from Eq. (3.11),

$$\nabla_{\sigma}\hat{G}^{\mu\nu\rho\sigma} + \alpha[G^{\mu\nu\rho\sigma}n_{\sigma}] = 0. \quad (3.16)$$

Now $\hat{G}^{\mu\nu\rho\sigma}$ is defined on $\mathcal{N}$ and so Eq. (3.16) only makes sense when it involves derivatives of $\hat{G}^{\mu\nu\rho\sigma}$ tangential to $\mathcal{N}$. This depends on the type of geometry induced on $\mathcal{N}$ by its embedding in $\mathcal{M}^{+}$ and in $\mathcal{M}^{-}$. If the geometry is type I, then we conclude, from Eq. (3.16),

$$-8\pi\eta^{-1}[T_{\mu\nu}n^{\mu}n^{\nu}] = \rho\gamma^{\dagger}, \quad (3.17)$$

$$2\eta^{-1}[\Psi_0] = -\sigma\gamma^{\dagger}, \quad (3.18)$$

$$2\eta^{-1}[\Psi_1] - 8\pi\eta^{-1}[T_{\mu\nu}m^{\mu}n^{\nu}] = \rho m^{\mu}\gamma_{\mu} - \sigma\overline{m}^{\mu}\gamma_{\mu}. \quad (3.19)$$

Here Eq. (3.17) coincides with Eq. (3.7) obtained from the twice-contracted Bianchi identities in the type I case. $[\Psi_0]$ and $[\Psi_1]$ are the jumps in the Newman-Penrose components of the Weyl tensors of $\mathcal{M}^{+}$ and $\mathcal{M}^{-}$ across $\mathcal{N}$ (we use a standard notation [8] for the components of the Weyl tensor, calculated on either side of $\mathcal{N}$, on the null tetrad $\{n, l, m, \overline{m}\}$). If the geometry induced on $\mathcal{N}$ is type II, then Eqs. (3.17)–(3.19) hold, with zeros on the right hand sides of Eqs. (3.17), (3.18), and we have, from Eq. (3.16),

$$2\eta^{-1}[\Psi_1] + 8\pi\eta^{-1}[T_{\mu\nu}m^{\mu}n^{\nu}] = -2\sigma\overline{m}^{\mu}\gamma_{\mu} - 2\rho m^{\mu}\gamma_{\mu} - m^{\mu}\gamma'_{\mu}. \quad (3.20)$$

We note that the difference between Eqs. (3.20) and (3.19) yields Eq. (3.8) obtained already from the twice-contracted Bianchi identities in the type II case. Finally if the geometry of $\mathcal{N}$ is type III, then Eqs. (3.17)–(3.20) hold with the right hand sides all vanishing and in addition we have

$$\eta^{-1}[\Psi_2] - \frac{2\pi}{3}\eta^{-1}[T] = -\frac{1}{2}\sigma\gamma_{\mu\nu}\overline{m}^{\mu}\overline{m}^{\nu} + 4\pi\{f' + (\rho+\kappa)f\}, \quad (3.21)$$

$$-4\pi\eta^{-1}[T_{\mu\nu}\overline{m}^{\mu}\overline{m}^{\nu}] = \frac{1}{2}\gamma'_{\mu\nu}\overline{m}^{\mu}\overline{m}^{\nu} + \frac{1}{2}(\rho+\kappa)\gamma_{\mu\nu}\overline{m}^{\mu}\overline{m}^{\nu} - 4\pi f\overline{\sigma}, \quad (3.22)$$

where $\gamma'_{\mu\nu} = n^{\sigma}\nabla_{\sigma}\gamma_{\mu\nu}$.

As a final preliminary we note that if the coefficient of the delta function in the Weyl tensor $\hat{C}^{\mu\nu\rho\sigma}$ is calculated using $\hat{G}^{\mu\nu\rho\sigma}$ given by Eq. (3.14) and $S^{\mu\nu}$ given by Eq. (2.14) and then if its components are calculated in the Newman-Penrose form $\hat{\Psi}_A$ (say) for $A = 0,1,2,3,4$, we obtain

$$\hat{\Psi}_0 = 0, \quad \hat{\Psi}_1 = 0, \quad \hat{\Psi}_2 = -\frac{1}{6}\,\eta\,\gamma^{\dagger},$$

$$\hat{\Psi}_3 = -\frac{1}{2}\,\eta\gamma_{\mu}\overline{m}^{\mu}, \quad \hat{\Psi}_4 = -\frac{1}{2}\,\eta\gamma_{\mu\nu}\overline{m}^{\mu}\overline{m}^{\nu}. \quad (3.23)$$

This shows (cf. [1,2]) that the delta function in the Weyl tensor is in general Petrov type II. If the induced geometry on $\mathcal{N}$ is type II, then $\gamma^{\dagger} = 0$ and $\hat{\Psi}_A$ is Petrov type III, whereas if the induced geometry on $\mathcal{N}$ is type III, then $\gamma^{\dagger} = 0$ and $\gamma_{\mu}\overline{m}^{\mu} = 0$ and $\hat{\Psi}_A$ is Petrov type N. The signal with history $\mathcal{N}$ contains a gravitational wave if $\hat{\Psi}_4 \neq 0$.

IV. PHYSICAL AND GEOMETRICAL APPLICATIONS

We now draw physical and geometrical conclusions from the results outlined in Secs. II and III, in the form of a series of lemmas with illustrative examples.

Combining the jumps (2.18), (2.20)–(2.22) across $\mathcal{N}$, in the kinematical quantities associated with a time-like congruence intersecting $\mathcal{N}$, with the Newman-Penrose components $\hat{\Psi}_A$ [given by Eq. (3.23)] of the coefficients of the δ-function in the Weyl tensor, we obtain the following, with straightforward algebra:

Lemma 1:

(1) $[\sigma_{\mu\nu}m^{\mu}m^{\nu}] \neq 0 \Leftrightarrow \hat{\Psi}_4 \neq 0$;

(2) *if* $[\sigma_{\mu\nu}] = 0$, *then* $\hat{\Psi}_4 = 0$ *and*

(a) $[a^{\mu}m_{\mu}] \neq 0 \Leftrightarrow \hat{\Psi}_3 \neq 0$,

(b) $[\omega^{\mu}] \neq 0 \Leftrightarrow \hat{\Psi}_3 \neq 0$;

(3) *if* $[\sigma_{\mu\nu}] = 0$ *and* $[a^{\mu}] = 0$, *then* $\hat{\Psi}_3 = \hat{\Psi}_4 = 0$ *and* $[\theta] \neq 0 \Leftrightarrow \hat{\Psi}_2 \neq 0$.

We note from Eqs. (2.18) and (2.24) that $[a^{\mu}] = 0 \Rightarrow [\omega^{\mu}] = 0$. The converse is not true because again from Eqs. (2.18) and (2.24) we find that if $[\omega^{\mu}] = 0$, then $[a^{\mu}] = s^{-2}n_{\lambda}[a^{\lambda}](n^{\mu} - su^{\mu})$ and we can only conclude from this that in general $m_{\mu}[a^{\mu}] = 0$. This explains the appearance of all components of $[\omega^{\mu}]$ in part (2b) of the lemma and of only one complex component of $[a^{\mu}]$ in part (2a) of the lemma.

We are particularly interested in lemma 1 when the time-like congruence of integral curves of u are the world lines of the cosmic fluid in a cosmological model. The first part of the lemma says that if the signal with history $\mathcal{N}$ includes a gravitational wave, then its effect on the cosmic fluid is to cause a jump across $\mathcal{N}$ in a complex component of the shear of the congruence and, if the passage of the signal through the fluid does not result in a jump in the fluid shear, then the signal cannot contain a gravitational wave. In this latter case the signal is a light-like shell of matter with a Petrov type II delta function in the Weyl tensor if the vorticity of the fluid

jumps across $\mathcal{N}$ or if a complex component of the fluid 4-acceleration jumps across $\mathcal{N}$. If only the expansion of the fluid jumps across $\mathcal{N}$, then part (3) of the lemma shows that the delta function in the Weyl tensor is Petrov type III.

There is an interesting analogy between lemma 1 and the usual decomposition of perturbations of cosmological models into scalar, vector and tensor parts with the tensor perturbations describing propagating gravitational waves and the other perturbations describing inhomogeneity in the matter distribution (see [9] and [10], for example). In lemma 1 the analogue of the tensor perturbations is the jump in the shear of the time-like congruence which by part (1) is necessary for the signal with history $\mathcal{N}$ to include a gravitational impulse wave. The analogues of the vector and scalar perturbations are the jumps in the 4-acceleration and vorticity on the one hand and in the expansion on the other hand, leading, by parts (2) and (3), to the possibility of the signal being a light-like shell of matter.

To illustrate lemma 1 with an example of a signal consisting of a gravitational impulsive wave and a light-like shell propagating through the Einstein–de Sitter universe (say) we must choose a cosmological model left behind by the signal (the space-time $\mathcal{M}^+$ to the future of the null hypersurface $\mathcal{N}$) which has the properties: (a) its fluid 4-velocity joins continuously to that of the Einstein–de Sitter universe on $\mathcal{N}$ and (b) its fluid 4-velocity has shear. Thus the line element of $\mathcal{M}^-$ is that of the Einstein–de Sitter universe which, in coordinates $x^\mu_- = (t, r, \phi, z)$, reads

$$ds^2 = -dt^2 + t^{4/3\beta}(dr^2 + r^2 d\phi^2 + dz^2), \tag{4.1}$$

where β is a constant. Here the t lines are the world lines of the particles of a perfect fluid with isotropic pressure p and proper density μ satisfying the equation of state $p = (\beta - 1)\mu$. A simple example of a space-time $\mathcal{M}^+$ satisfying the requirements (a) and (b) above is the anisotropic Bianchi type I space-time [11] with line element, in coordinates $x^\mu_+ = (t_+, r_+, \phi_+, z_+)$,

$$ds_+^2 = -dt_+^2 + A_+^2(dr_+^2 + r_+^2 d\phi_+^2) + B_+^2 dz_+^2, \tag{4.2}$$

where

$$A_+ = t_+^{(3\beta-2)/6\beta}, \quad B_+ = t_+^{2/3\beta}. \tag{4.3}$$

The t_+ lines are the world lines of a perfect fluid with isotropic pressure p_+ and proper density μ_+ satisfying $p_+ = \mu_+$. As a boundary between $\mathcal{M}^-$ and $\mathcal{M}^+$ take the null hypersurface $\mathcal{N}$ to be given by

$$r_+ = T_+(t_+), \quad \frac{dT_+}{dt_+} = \frac{1}{A_+}, \tag{4.4}$$

in the plus coordinates and by

$$r = T(t), \quad \frac{dT}{dt} = t^{-2/3\beta}, \tag{4.5}$$

in the minus coordinates. As intrinsic coordinates on $\mathcal{N}$ we can use $\xi^a = (r, \phi, z)$. The induced line elements on $\mathcal{N}$ from $\mathcal{M}^+$ and $\mathcal{M}^-$ match [as required by Eq. (2.1)] if

$$t_+ = t, \quad r_+ = \frac{6\beta}{(3\beta+2)}\left[\frac{(3\beta-2)}{3\beta} r\right]^{(3\beta+2)/(6\beta-4)},$$

$$\phi_+ = \left(\frac{3\beta+2}{6\beta-4}\right)\phi, \quad z_+ = z. \tag{4.6}$$

We must first check that the 4-velocities of the fluid particles with histories in $\mathcal{M}^+$ and $\mathcal{M}^-$ are continuous across $\mathcal{N}$. This has to be done with care as we now have two local coordinate systems $\{x^\mu_+\}$ and $\{x^\mu_-\}$ on either side of $\mathcal{N}$, overlapping on $\mathcal{N}$ according to Eqs. (4.6). Let ${}^+v^\mu = (1,0,0,0)$ and ${}^-u^\mu = (1,0,0,0)$. Then ${}^+v^\mu$, ${}^-u^\mu$ are the fluid 4-velocities in $\mathcal{M}^+$ and $\mathcal{M}^-$ respectively. Let ${}^+u^\mu$ be the same vector as ${}^-u^\mu$ but calculated on the plus side of $\mathcal{N}$. We then compare (on $\mathcal{N}$) ${}^+v^\mu$ with ${}^+u^\mu$ and, if they are equal, then the fluid 4-velocity is continuous across $\mathcal{N}$. To do this we utilize the tangent basis vectors $e_{(a)} = \partial/\xi^a$ [with $\xi^a = (r, \phi, z)$ in this case] introduced at the beginning of Sec. II. Then ${}^+u^\mu$ is the same vector as ${}^-u^\mu$ if

$$[u \cdot u] = [u \cdot e_{(a)}] = 0. \tag{4.7}$$

These are the same conditions (2.4) that a transversal N on $\mathcal{N}$ has to satisfy and indeed u can be used as a transversal if desired. The four conditions (4.7) determine ${}^+u^\mu$ uniquely and for the example we are considering we obtain ${}^+u^\mu = (1,0,0,0)$. Hence ${}^+u^\mu = {}^+v^\mu$ and the fluid 4-velocity is continuous across $\mathcal{N}$. Now using the theory outlined in Sec. II above we find that $\gamma_{\mu\nu} = 0$ except for [quoting the non-vanishing components of $\gamma_{\mu\nu}$ in the coordinate system $\{x^\mu_-\}$ (say)]

$$\gamma_{11} = \frac{(\beta-2)}{\beta} t^{(2-\beta)/2\beta}, \quad \gamma_{22} = \frac{9\beta\,(\beta-2)}{(3\beta-2)^2} t^{(9\beta-2)/6\beta}. \tag{4.8}$$

With $n^\mu_- = (t^{(-3\beta+2)/6\beta}, t^{-(3\beta+2)/6\beta}, 0, 0)$ and $m^\mu_- = 2^{-1/2} t^{-2/3\beta}(0, 0, ir^{-1}, 1)$ we find that

$$\gamma_\mu = \gamma_{\mu\nu} n^\nu_- = \delta^1_\mu \frac{(\beta-2)}{\beta} t^{(2-3\beta)/3\beta}, \tag{4.9}$$

$$\gamma^\dagger = \gamma_\mu n^\mu_- = \frac{(\beta-2)}{\beta} t^{(2-9\beta)/6\beta}, \tag{4.10}$$

$$\gamma_{\mu\nu} \overline{m}^\mu \overline{m}^\nu = -\frac{(\beta-2)}{2\beta} t^{-(3\beta+2)/6\beta}, \tag{4.11}$$

where in Eq. (4.11) we have written r in terms of t following from Eqs. (4.5). Comparison now with Eqs. (3.23) shows that $\hat{\Psi}_3 = 0$ but $\hat{\Psi}_2 \neq 0$ and $\hat{\Psi}_4 \neq 0$. In addition we find that the vector field λ on $\mathcal{N}$ introduced in Eqs. (2.17) vanishes, as does $U_\alpha = \gamma_{\alpha\beta}{}^- u^\beta$. We see from Eqs. (4.9)–(4.11) that the

geometry induced on $\mathcal{N}$ is a type I geometry and that $\mathcal{N}$ is the history of both an impulsive gravitational wave and a light-like shell.

In the general case of a type I induced geometry on $\mathcal{N}$ we notice that the equations following from the Bianchi identities (3.17)–(3.19) are all algebraic relations between some components of $\gamma_{\mu\nu}$ and some of the jumps in the energy-momentum-stress tensors and the Weyl tensors of $\mathcal{M}^+$ and $\mathcal{M}^-$ across $\mathcal{N}$. The further consequences of the Bianchi identities when the induced geometry is type II or III [Eqs. (3.8), (3.9), (3.20)–(3.22)] can all be viewed as propagation equations for components of $\gamma_{\mu\nu}$ along the generators of $\mathcal{N}$ (derivatives along the generators being indicated by a prime). This is consistent because the type I geometry by itself excludes the possibility of a unique parameter being assigned to the geodesic generators of $\mathcal{N}$ on both the plus and minus sides and hence unique propagation equations along these generators of quantities defined on $\mathcal{N}$ cannot exist.

We emphasize the algebraic nature of the Bianchi identities in the case of a type I geometry by stating the following:
Lemma 2:
If the geometry induced on $\mathcal{N}$ is type I, then
(a) if $\rho\neq 0$ and/or $\sigma\neq 0$, $\hat{\Psi}_2$ satisfies

$$\rho\hat{\Psi}_2=\frac{4\pi}{3}[T_{\mu\nu}n^\mu n^\nu], \tag{4.12}$$

$$\sigma\hat{\Psi}_2=\frac{1}{3}[\Psi_0]; \tag{4.13}$$

(b) if $\rho^2\neq|\sigma|^2\neq 0$, $\hat{\Psi}_3$ is given by

$$[\Psi_1]-4\pi[T_{\mu\nu}m^\mu n^\nu]=-\hat{\Psi}_3^*\rho+\hat{\Psi}_3\sigma \tag{4.14}$$

and its complex conjugate (here $\hat{\Psi}_3^*$ is the complex conjugate of $\hat{\Psi}_3$).

We note again that ρ,σ are intrinsic to $\mathcal{N}$ ($[\rho]=0=[\sigma]$) for a type I geometry. For the cosmological example given above the expansion ρ and shear σ of the generators of $\mathcal{N}$ are given by

$$\rho=\frac{1}{r}\left(\frac{3\beta+2}{3\beta-2}\right) \quad \text{and} \quad \sigma=\frac{1}{\sqrt{2}r}, \tag{4.15}$$

while $[T_{\mu\nu}n^\mu n^\mu]=s^2[\mu+p]$, with s defined after Eq. (2.18), and since on $\mathcal{N}$ the continuous 4-velocity u is orthogonal to the complex null vector m tangent to $\mathcal{N}$, $[T_{\mu\nu}m^\mu n^\nu]=0$ and one can readily verify that the algebraic equations in lemma 2 are satisfied.

The richest induced geometry is of course type III and in this case we can, with additional assumptions, deduce from the Bianchi identities some interesting conclusions which we summarize in the following:
Lemma 3:
If the geometry on $\mathcal{N}$ is type III and if $\mathcal{M}^\pm$ are vacuum space-times, then $[\Psi_0]=[\Psi_1]=0$,

$$[\Psi_2]=\sigma\hat{\Psi}_4-4\pi\eta\rho f, \tag{4.16}$$

and thus if $[\Psi_2]=0$ and $\hat{\Psi}_4\neq 0$, then
(1) $\sigma=0$ and $\rho\neq 0\Rightarrow f=0$,
(2) $\sigma=0$ and $\rho=0\Rightarrow f\neq 0$ is possible,
(3) $\sigma\neq 0\Rightarrow\rho\neq 0$ and $f\neq 0$,
where the surface stress-energy tensor of the light-like shell now has the form $S_{\alpha\beta}=\eta f n_\alpha n_\beta$.

We first note that part (1) of lemma 3 explains the ''miracle'' whereby the Penrose spherical impulsive wave propagating through flat space-time [1] *automatically* satisfies the vacuum field equations. The history $\mathcal{N}$ of the signal in this case is a future null cone which is a shear-free ($\sigma=0$) expanding ($\rho\neq 0$) null hypersurface. The induced geometry is type III and thus by lemma 3(1) the surface stress-energy tensor $S_{\alpha\beta}$ must vanish ($f=0$). An example of part (1) of lemma 3 in which $\mathcal{M}^\pm$ are not flat is provided by taking $\mathcal{M}^\pm$ to be two Petrov type III Robinson-Trautman [12] vacuum space-times with line elements of the form

$$ds_\pm^2=-2r_\pm^2p_\pm^{-2}d\zeta_\pm d\bar{\zeta}_\pm+2dudr_\pm+K_\pm du^2, \tag{4.17}$$

with $p_\pm=p_\pm(\zeta_\pm,\bar{\zeta}_\pm)$ and

$$K_\pm=\Delta_\pm\log p_\pm, \quad \Delta_\pm K_\pm=0, \tag{4.18}$$

where $\Delta_\pm=2p_\pm^2\partial^2/\partial\zeta_\pm\partial\bar{\zeta}_\pm$. These two space-times are joined together on the shear-free, expanding null hypersurface $\mathcal{N}$ with the equation $u=0$, with Eq. (2.1) satisfied if

$$\zeta_+=h(\zeta_-) \quad \text{and} \quad r_+=F(\zeta_-,\bar{\zeta}_-)r_-, \tag{4.19}$$

where h is an analytic function of ζ_- and $F(\zeta_-,\bar{\zeta}_-)=p_+/(|h'|p_-)$. In coordinates labelled $x^\mu=(\zeta_-,\bar{\zeta}_-,r_-,u)$ we find that $\gamma_{\mu\nu}=0$ except for γ_{11} and $\gamma_{22}=\bar{\gamma}_{11}$ with

$$\gamma_{11}=-2r\frac{F'}{F}\frac{\partial}{\partial\zeta}\log(F'p_-^2), \tag{4.20}$$

where $F'=\partial F/\partial\zeta_-$. Thus the induced geometry is type III, there is no surface stress-energy tensor on $\mathcal{N}$ and, since $\hat{\Psi}_4=\frac{1}{2}\gamma_{11}r_-^{-2}p_-^2\neq 0$, $\mathcal{N}$ is the history of an impulsive gravitational wave.

Part (2) of lemma 3 shows that if $\mathcal{N}$ is a null hyper*plane* (with generators having vanishing shear and expansion) and if the matching of $\mathcal{M}^+$ and $\mathcal{M}^-$ on $\mathcal{N}$ satisfying Eq. (2.1) is such that the induced geometry is type III, then $\mathcal{N}$ can be the history of a plane impulsive gravitational wave and/or a plane light-like shell of matter. For example take $\mathcal{M}^+$ to be a pp-wave space-time with line element

$$ds_+^2=dx_+^2+dy_+^2+2dudv_++H(x_+,y_+,u)du^2, \tag{4.21}$$

with $H_{x_+x_+}+H_{y_+y_+}=0$ (subscripts here denoting partial derivatives). Take $\mathcal{M}^-$ to be flat space-time with line element

$$ds_-^2=dx^2+dy^2+2dudv. \tag{4.22}$$

Now match $\mathcal{M}^+$ to $\mathcal{M}^-$ on the null hyperplane $\mathcal{N}$ ($u=0$) with [1]

$$x_+ = x, \quad y_+ = y, \quad v_+ = v + h(x,y), \tag{4.23}$$

to ensure that Eq. (2.1) is satisfied. Using the theory of Secs. II and III above with $x^\mu_- = (\xi^a, u)$ with $\xi^a = (x,y,v)$ we find that $\gamma_{\mu 4} = 0$ and otherwise $\gamma_{ab} = -h_{,ab}$. Thus with $n^\mu_- = \delta^\mu_3$ we have $\gamma_\mu = \gamma_{\mu 3} = 0$ and so the geometry induced on $\mathcal{N}$ is type III. We also find that

$$8\pi f = -\frac{1}{2}\,\gamma = h_{xx} + h_{yy} \tag{4.24}$$

and

$$\hat{\Psi}_4 = -\frac{1}{2}\,(h_{xx} - h_{yy}) + i h_{xy}. \tag{4.25}$$

This shows explicitly that a light-like shell and a plane impulsive wave can co-exist, each with history $\mathcal{N}$.

A simple example of part (3) of lemma 3 is a cylindrical fronted light-like signal with history $\mathcal{N}$ in flat space-time. Thus $\mathcal{M}^\pm$ have line elements

$$ds^2_\pm = (u + v_\pm)^2 d\phi^2_\pm + dz^2_\pm + 2du dv_\pm. \tag{4.26}$$

Now $\mathcal{N}\,(u=0)$ is a null hypersurface generated by shearing null geodesics $(\sigma \neq 0)$. We match the induced metrics on $\mathcal{N}$ with

$$\phi_+ = q(\phi), \quad z_+ = z, \quad v_+ = v/q', \tag{4.27}$$

with $q' = dq/d\phi$. In coordinates $x^\mu_- = (\phi, z, v, u)$ we find that $\gamma_{\mu\nu} = 0$ except for

$$\gamma_{11} = 2v\left\{\frac{q''}{q'} - \frac{3}{2}\left(\frac{q''}{q'}\right)^2 + q'^2 - 1\right\}. \tag{4.28}$$

Thus with $n^\mu_- = \delta^\mu_3$ we see that $\gamma_\mu = 0$ and the induced geometry on $\mathcal{N}$ is type III. The shear σ and expansion ρ of the null geodesic generators of $\mathcal{N}$ satisfy

$$\rho = \sigma = \frac{1}{2v}, \tag{4.29}$$

while

$$4\pi f = \hat{\Psi}_4 = -\frac{1}{4v^2}\,\gamma_{11}. \tag{4.30}$$

Thus in general $f \neq 0$ and a shell and impulsive wave co-exist. We see that no signal exists with history $\mathcal{N}$ if and only if $\gamma_{11} = 0$. It is interesting to note that if no signal exists on $\mathcal{N}$ and the isometric transformations preserving this state form a *group*, then they are given by Eqs. (4.27) with $q(\phi) = \phi + c$, and $c = \text{const}$. There also exist other disconnected isometric transformations of $\mathcal{N}$, of the form (4.27), when no signal exists on $\mathcal{N}$, but these transformations do not form a group.

A corresponding lemma to lemma 3 which has applications to light-like signals propagating through a cosmic fluid is the following:

Lemma 4:
If the geometry on $\mathcal{N}$ is type III and if $\mathcal{M}^\pm$ are perfect fluid space-times with u continuous across $\mathcal{N}$, then $[\Psi_0] = [\Psi_1] = [\mu + p] = 0$ *and*

$$[\Psi_2] - \frac{4\pi}{3}[\mu] = \sigma\hat{\Psi}_4 - 4\pi\eta\rho f, \tag{4.31}$$

and thus if $[\Psi_2] = (4\pi/3)[\mu]$ *and* $\hat{\Psi}_4 \neq 0$, *then the deductions are the same as (1)–(3) of lemma 3.*

The special case of the de Sitter universe is obtained by putting $-8\pi p = 8\pi\mu = \Lambda$, where Λ is the cosmological constant. We will confine our observations on lemma 4 to the de Sitter case, leaving further applications of lemma 4 to another occasion.

To illustrate Eq. (4.31) of lemma 4 we let $\mathcal{M}^\pm$ both be de Sitter universes (with different cosmological constants $\Lambda_\pm$) having line elements

$$ds^2_\pm = \frac{2v^2_\pm d\zeta_\pm d\bar{\zeta}_\pm + 2du_\pm dv_\pm}{\left(1 + \frac{1}{6}\Lambda_\pm u_\pm v_\pm\right)^2}. \tag{4.32}$$

Here $\mathcal{N}\ (u_+ = u_- = 0)$ is a future null cone generated by expanding $(\rho \neq 0)$ shear-free $(\sigma = 0)$ null geodesics. We match $\mathcal{M}^\pm$ on $\mathcal{N}$ with a Penrose [1] warp

$$\zeta_+ = h(\zeta_-), \quad v_+ = \frac{v_-}{|h'|}, \tag{4.33}$$

where h is an analytic function of ζ_- and $h' = dh/d\zeta_-$. Now the induced geometry on $\mathcal{N}$ is type III. In general $\hat{\Psi}_4 = -\chi/2v \neq 0$ with

$$\chi = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 \tag{4.34}$$

and

$$8\pi\rho f = \frac{1}{3}[\Lambda]. \tag{4.35}$$

This is the form taken by Eq. (4.31) for this example since now ${}^\pm\Psi_A = 0$ for $A = 0,1,2,3,4$, $8\pi[\mu] = [\Lambda]$, $\eta = +1$ and $\sigma = 0$. Thus, if $[\Lambda] = 0$, then since $\rho = 1/v \neq 0$ we must have $f = 0$ and so $\mathcal{N}$ is the history of an impulsive gravitational wave [13].

Finally as an illustration of conclusion (2) of lemma 4 we consider $\mathcal{M}^+$ to be a Schwarzschild space-time with line element in Kruskal form

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{64m^3}{r}\,e^{(1 - r/2m)} dU dV, \tag{4.36}$$

with $r = r(UV)$ given by

$$\left(\frac{r}{2m} - 1\right)e^{(r/2m - 1)} = 2UV, \tag{4.37}$$

and we take $\mathcal{M}^-$ to be de Sitter space-time (with $\Lambda>0$) with line element

$$ds^2=r^2(d\theta^2+\sin^2\theta d\phi^2)-2\frac{(1+\lambda r)^2}{\lambda^2}dUdV, \tag{4.38}$$

where $\lambda^2=\Lambda/3$ and $r=r(UV)$ is given by

$$\frac{1-\lambda r}{1+\lambda r}=2UV. \tag{4.39}$$

These match (cf. [2]) on the horizon $\mathcal{N}(U=0)$ if $2m\lambda=1$. We have rescaled one of the null coordinates in Eqs. (4.36) and (4.37) to make the metric tensors given via the line elements (4.36) and (4.38) continuous across $\mathcal{N}$. The horizon $U=0$ is a null hyperplane generated by shear-free ($\sigma=0$), expansion-free ($\rho=0$) null geodesics. In the continuous coordinates (U,V,θ,ϕ) above we find using Eqs. (2.12) that $\gamma_{\mu\nu}=0$ except $\gamma_{22}=\gamma_{11}\sin^2\theta=-3V\sin^2\theta/4m^2$ and $f=3V/32\pi m^2\neq 0$. If the situation above is reversed and $\mathcal{M}^+$ is de Sitter space-time (with $\Lambda>0$) and $\mathcal{M}^-$ is Schwarzschild space-time, then $\gamma_{\mu\nu}=0$ except for $\gamma_{22}=\gamma_{11}\sin^2\theta=3V\sin^2\theta/4m^2$ and $f=-3V/32\pi m^2$. In either case the induced geometry is type III. The equation $[\Psi_2]=(4\pi/3)[\mu]$ becomes $1/4m^2=\lambda^2$. There is no gravitational wave present ($\hat{\Psi}_4=0$) and $\mathcal{N}$ is the history of a light-like shell.

V. DISCUSSION

The lemmas that we have established above fall into two different categories. Lemma 1 concerns the interaction between a null shell and/or a wave and any time-like congruence with a continuous tangent vector at the intersection with $\mathcal{N}$. It shows the close relationship existing between the presence of a wave ($\hat{\Psi}_4\neq 0$) and the shear of the time-like congruence ($[\sigma_{\mu\nu}m^\mu m^\nu]\neq 0$). There is a complementary result due to Penrose [1] to the effect that for a null geodesic congruence crossing $\mathcal{N}$ with continuous tangent, a jump in the complex shear is necessary for $\mathcal{N}$ to be the history of an impulsive gravitational wave. On the other hand lemmas 2–4 relate the properties of the null hypersurface (embodied in $\rho,\sigma,\hat{\Psi}_A$) to the outside medium (described by ${}^{\pm}T^{\mu\nu}$) and the outside geometry (described by ${}^{\pm}\Psi_A$). The different jumps $[T_{\mu\nu}n^\mu n^\mu],[T_{\mu\nu}m^\mu n^\mu],[T_{\mu\nu}l^\mu l^\mu]$ tell how the fluid lines and fluid properties (energy density, pressure) are modified by the presence of the light-like signal. For instance Eq. (3.6) can be written as

$$\nabla_\nu(\alpha S^{\mu\nu})=-\alpha[T^{\mu\nu}n_\nu], \tag{5.1}$$

and represents an equation of conservation for energy and momentum [2]. For a pure impulsive gravitational wave it reduces to $[T^{\mu\nu}n_\nu]=0$, which also holds for a shock wave [5,6].

Finally a couple of technical points which have arisen above merit discussion. At the beginning of Sec. III we note that in a local coordinate system $\{x^\mu\}$ covering both sides of $\mathcal{N}$ the equation of $\mathcal{N}$ is $\Phi(x^\mu)=0$ (say) and so as normal we can take $n_\mu=\alpha^{-1}\Phi_{,\mu}$ where α is some function defined on $\mathcal{N}$. In the passage from type I geometry to type II geometry the acceleration parameter κ becomes continuous across $\mathcal{N}$. In this case we are entitled to put $\alpha=1$ and so make κ vanish on $\mathcal{N}$. However, situations can arise in applications with a type II or type III induced geometry on $\mathcal{N}$ in which the most natural parameter to use along the generators of $\mathcal{N}$ is not an affine parameter. Then, although $[\kappa]=0$, we have $\kappa\neq 0$ and for this reason we have retained κ in Eqs. (3.9), (3.21) and (3.22) [κ does not appear in Eq. (3.8) or (3.20) even when non-zero].

In introducing a time-like congruence crossing the history $\mathcal{N}$ of the light-like signal in Sec. II we chose to examine the case in which the unit time-like tangent vector field u^μ is continuous across $\mathcal{N}$ but may have a jump in its derivative described via a vector field λ^μ introduced in Eqs. (2.17). As we pointed out in Sec. II this assumption forbids $\mathcal{N}$ being the history of a shock wave in the usual sense (a shock wave in a gas with macroscopic 4-velocity u^μ, for example [5,6]). In this latter case the tangent to the congruence would itself jump across $\mathcal{N}$. This complicates the study of the interaction of the time-like congruence with the light-like signal by introducing delta functions (singular on $\mathcal{N}$) into the kinematical variables associated with the congruence and is a topic for further study.

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