Hypothesis of path integral duality. II. Corrections to quantum field theoretic results

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In the path integral expression for a Feynman propagator of a spinless particle of mass m, the path integral amplitude for a path of proper length $\mathcal{R}(x,x'|g_{\mu\nu})$ connecting events x and x' in a spacetime described by the metric tensor $g_{\mu\nu}$ is $\exp\{-[m\,\mathcal{R}(x,x'|g_{\mu\nu})]\}$. In a recent paper, assuming the path integral amplitude to be invariant under the duality transformation $\mathcal{R}\to(L_P^2/\mathcal{R})$, Padmanabhan has evaluated the modified Feynman propagator in an arbitrary curved spacetime. He finds that the essential feature of this ''principle of path integral duality'' is that the Euclidean proper distance $(\Delta x)^2$ between two infinitesimally separated spacetime events is replaced by $[(\Delta x)^2+4L_P^2]$. In other words, under the duality principle the spacetime behaves as though it has a ''zero-point length'' L_P, a feature that is expected to arise in a quantum theory of gravity. In Schwinger's proper time description of the Feynman propagator, the weightage factor for a path with a proper time s is $\exp[-(m^2s)]$. Invoking Padmanabhan's ''principle of path integral duality'' corresponds to modifying the weightage factor $\exp[-(m^2s)]$ to $\exp\{-[m^2s+(L_P^2/s)]\}$. In this paper, we use this modified weightage factor in Schwinger's proper time formalism to evaluate the quantum gravitational corrections to some of the standard quantum field theoretic results in flat and curved spacetimes. In flat spacetime, we evaluate the corrections to (1) the Casimir effect, (2) the effective potential for a self-interacting scalar field theory, (3) the effective Lagrangian for a constant electromagnetic background and (4) the thermal effects in Rindler coordinates. In arbitrary curved spacetime, we evaluate the corrections to (1) the effective Lagrangian for the gravitational field and (2) the trace anomaly. In all these cases, we first briefly present the conventional result and then go on to evaluate the corrections with the modified weightage factor. We find that the extra factor $\exp[-(L_P^2/s)]$ acts as a regulator at the Planck scale thereby ''removing'' the divergences that otherwise appear in the theory. Finally, we discuss the wider implications of our analysis. [S0556-2821(98)02016-5]

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I. INTRODUCTION

This paper is a logical continuation of an earlier paper by Padmanabhan [1]. Here, we shall utilize the results obtained in [1] to evaluate the quantum gravitational corrections to some of the standard quantum field theoretic results in flat and curved spacetimes. In this section, we shall very briefly summarize the results that were presented in [1] and then proceed on to discuss the various applications.

In Schwinger's proper time formalism, the Euclidean space Feynman propagator is described by the integral [2]

$$G_{\rm F}(x,x'|g_{\mu\nu})=\int_0^\infty ds\ e^{-m^2 s}\ K(x,x';s|g_{\mu\nu}), \qquad (1)$$

where $K(x,x';s|g_{\mu\nu})$ is the probability amplitude for a particle to propagate from x to x' in a proper time interval s in a given background spacetime described by the metric tensor $g_{\mu\nu}$. The quantity $K(x,x;s|g_{\mu\nu})$ can usually be expressed as a path integral. In [1], Padmanabhan was able to introduce a fundamental length scale into the Feynman propagator above by invoking a ''principle of path integral duality.'' He had assumed that the fundamental length scale was the Planck length $L_P\equiv(G\hbar/c^3)^{1/2}$. [Actually, it can so happen that the fundamental length scale is not L_P but (ηL_P), where η is a factor of order unity. Therefore, in this paper, when we say that the fundamental length is L_P, we actually mean that it is of $O(L_P)$.] He found that the duality principle modifies the weightage given to a path of proper time s from $\exp[-(m^2s)]$ to $\exp\{-[m^2s+(L_P^2/s)]\}$. The resulting Euclidean Feynman propagator is then given by

$$G_{\rm F}^{\rm P}(x,x'|g_{\mu\nu})=\int_0^\infty ds\ e^{-m^2 s}\ e^{-(L_P^2/s)}\ K(x,x';s|g_{\mu\nu}). \qquad (2)$$

The presence of a fundamental length scale is a feature that is expected to arise in a quantum theory of gravity [3,4]. Hence, the modification of the weightage factor as mentioned above can be interpreted as being equivalent to introducing quantum gravitational corrections into standard field theory.

The result given above will be utilized when required. Further, we are interested in evaluating the one-loop effective Lagrangian for various systems with the presence of a fundamental length scale in the background (flat or curved) spacetime. The usual effective Lagrangian will be modified accordingly and this will determine the quantum gravitational corrections to standard results. We derive now the formula for the modified effective Lagrangian.

Consider a D-dimensional spacetime described by the metric tensor $g_{\mu\nu}$. Assume that a quantized scalar field Φ of mass m satisfies the following equation of motion:

$$(\hat{H}+m^2)\Phi=0, \qquad (3)$$

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where $\hat{H}$ is a differential operator in the D-dimensional spacetime. Then, in Lorentzian space, the effective Lagrangian corresponding to the operator $\hat{H}$ is given by the integral [2]

$$\mathcal{L}_{\rm corr} = -\frac{i}{2}\int_0^\infty \frac{ds}{s}\, e^{-im^2 s}\; K(x,x;s|g_{\mu\nu}), \tag{4}$$

where

$$K(x,x;s|g_{\mu\nu}) \equiv \langle x|e^{-i\hat{H}s}|x\rangle. \tag{5}$$

(If the quantum field interacts either with itself or with an external classical field, then $\mathcal{L}_{\rm corr}$ will prove to be the correction to the Lagrangian describing the classical background.) The quantity $K(x,x;s|g_{\mu\nu})$ is the path integral kernel (in the coincidence limit) of a quantum mechanical system described by time evolution operator $\hat{H}$. The integration variable s acts as the time parameter for the quantum mechanical system.

We had mentioned earlier that invoking the ''principle of path integral duality'' corresponds to modifying the weightage factor $\exp[-(m^2 s)]$ to $\exp\{-[m^2 s + (L_P^2/s)]\}$. In Lorentzian space, this modified weightage factor is given by $\exp\{-[im^2 s - i(L_P^2/s)]\}$. Therefore, invoking the ''principle of path integral duality'' corresponds to modifying the effective Lagrangian given by Eq. (4) above to the following form:

$$\mathcal{L}_{\rm corr}^{\rm P} = -\frac{i}{2}\int_0^\infty \frac{ds}{s}\, e^{-im^2 s}\; e^{iL_P^2/s}\, K(x,x;s|g_{\mu\nu}), \tag{6}$$

where $K(x,x;s|g_{\mu\nu})$ is still given by Eq. (5). In this paper, we shall use this modified effective Lagrangian to evaluate the quantum gravitational corrections to standard quantum field theoretic results in flat and curved spacetimes.

The layout of the rest of the paper is as follows. In Secs. II–V, we evaluate the corrections to (1) the Casimir effect, (2) the effective potential for a self-interacting scalar field theory, (3) the effective Lagrangian for a constant electromagnetic background and (4) the thermal effects in the Rindler coordinates, in flat spacetime. In Secs. VI and VII, we evaluate the quantum gravitational corrections to (1) the gravitational Lagrangian and (2) the trace anomaly in an arbitrary curved spacetime. In all these sections, we shall first briefly discuss the conventional result and then go on to evaluate the corrections with the modified weightage factor. Finally, in Sec. VIII, we discuss the wider implications of our analysis.

II. CORRECTIONS TO THE CASIMIR EFFECT

The presence of a pair of conducting plates alters the vacuum structure of the quantum field and as a result there arises a non-zero force of attraction between the two conducting plates. This effect is called the Casimir effect. In the conventional derivation of the Casimir effect, the difference between the energy in the Minkowski vacuum and the Casimir vacuum is evaluated and a derivative of this energy difference gives the force of attraction between the Casimir plates (see, e.g., Ref. [5], pp. 138–141). To evaluate the Casimir force in such a fashion we need to know the normal modes of the quantum field with and without the plates. Therefore, if we are to evaluate the quantum gravitational corrections to the Casimir effect by the method described above, then we need to know how metric fluctuations will modify the modes of the quantum field. But, from the ''principle of path integral duality'' we only know how the quantum gravitational corrections can modify the effective Lagrangian. Therefore, in this section, we shall first present a derivation of the Casimir effect from the effective Lagrangian approach and then go on to evaluate the quantum gravitational corrections to this effect with the modified weightage factor. The system we shall consider here is a massless scalar field in flat spacetime. We shall evaluate the effective Lagrangian for two cases: (i) with vanishing boundary conditions on a pair of parallel plates and (ii) with periodic boundary conditions on the quantum field. We shall describe the first case in detail and just state the final results for the second.

A. Conventional result

Consider a pair of plates situated at $z=0$ and $z=a$. Let us assume that the scalar field Φ vanishes on these plates. For such a case, the operator $\hat{H}$ corresponds to that of a free particle with the condition that its eigenfunctions along the z-direction vanish at $z=0$ and $z=a$. Along the other $(D-1)$ perpendicular directions the operator $\hat{H}$ corresponds to that of a free particle without any boundary conditions. The complete quantum mechanical kernel can then be written as

$$K(x,x';s) = K_z(z,z';s)\times K_\perp(x_\perp, x'_\perp; s). \tag{7}$$

[We shall refer to the flat spacetime kernel $K(x,x';s|\eta_{\mu\nu})$ simply as $K(x,x';s)$.] In the limit $x_\perp \to x'_\perp$, $K_\perp(x_\perp, x'_\perp; s)$ is given by

$$K_\perp(x_\perp, x_\perp; s) = \left(\frac{i}{(4\pi i s)^{(D-1)/2}}\right). \tag{8}$$

The quantum mechanical kernel $K_z(z,z';s)$ along the z-direction corresponds to that of a particle in an infinite square well potential with walls at $z=0$ and $z=a$. The kernel for such a case is given by (see, e.g., Ref. [6], p. 46)

$$K_z(z,z';s) = \left(\frac{1}{(4\pi i s)^{1/2}}\right)\sum_{n=-\infty}^{\infty}\{\exp[i(z-z'+2na)^2/4s] - \exp[i(z+z'+2na)^2/4s]\} \tag{9}$$

which, in the coincidence limit (i.e. when $z=z'$), reduces to

$$K_z(z,z;s) = \left(\frac{1}{(4\pi i s)^{1/2}}\right)\left\{1 + 2\sum_{n=1}^{\infty} e^{(in^2a^2/s)} - \sum_{n=-\infty}^{\infty} e^{i(z+na)^2/s}\right\}. \tag{10}$$

Therefore, the complete kernel (in the coincidence limit) corresponding to the operator $\hat{H}$ with the conditions that its eigenfunctions vanish at $z=0$ and $z=a$ is given by

$$K(x,x;s)=\left(\frac{i}{(4\pi i s)^{D/2}}\right)\times\left\{1+2\sum_{n=1}^{\infty} e^{(in^2a^2/s)}-\sum_{n=-\infty}^{\infty} e^{i(z+na)^2/s}\right\}. \tag{11}$$

Substituting this kernel in Eq. (4) and setting $m=0$, we obtain that

$$\mathcal{L}_{\rm corr}=\left(\frac{1}{2(4\pi i)^{D/2}}\right)\int_0^{\infty}\frac{ds}{s^{(D/2)+1}}\times\left\{1+2\sum_{n=1}^{\infty} e^{in^2a^2/s}-\sum_{n=-\infty}^{\infty} e^{i(z+na)^2/s}\right\}. \tag{12}$$

In flat spacetime, in the absence of any boundary conditions on the quantum scalar field Φ, the quantum mechanical kernel for the operator $\hat{H}$ corresponds to that of a free particle. The kernel for such a case is given by

$$K^0(x,x;s)=\left(\frac{i}{(4\pi i s)^{D/2}}\right). \tag{13}$$

That is, there exists a non-zero $\mathcal{L}_{\rm corr}$ even for a free field in flat spacetime. Therefore, the flat spacetime contribution as given by

$$\mathcal{L}^0_{\rm corr}=\left(\frac{1}{2(4\pi i)^{D/2}}\right)\int_0^{\infty}\frac{ds}{s^{(D/2)+1}}\,e^{-im^2s} \tag{14}$$

has to be subtracted from all $\mathcal{L}_{\rm corr}$.

Subtracting the quantity $\mathcal{L}^0_{\rm corr}$ [given by Eq. (14) above with m set to zero] from the expression for $\mathcal{L}_{\rm corr}$ and then evaluating the resulting integral, we obtain that

$$\bar{\mathcal{L}}_{\rm corr}=(\mathcal{L}_{\rm corr}-\mathcal{L}^0_{\rm corr})=\left(\frac{\Gamma(D/2)}{2(4\pi a^2)^{D/2}}\right)\times\left\{2\sum_{n=1}^{\infty} n^{-D}-\sum_{n=-\infty}^{\infty}[n+(z/a)]^{-D}\right\}. \tag{15}$$

Now, consider the case $D=4$. The first series in the above expression for $\bar{\mathcal{L}}_{\rm corr}$ can be expressed in terms of the Riemann zeta function (see, e.g., Ref. [7], p. 334) and the second series can be summed using the following relation (cf. Ref. [8], Vol. I, p. 652):

$$\sum_{n=-\infty}^{\infty}(k+\alpha)^{-p}=\left(\frac{\pi(-1)^{p-1}}{(p-1)!}\right)\frac{d^{p-1}}{d\alpha^{p-1}}[\cot(\pi\alpha)], \tag{16}$$

with the result

$$\bar{\mathcal{L}}_{\rm corr}=\left(\frac{\pi^2}{1440a^4}\right)-\left(\frac{\pi^2}{96a^4\,\sin^4(\pi z/a)}\right)[3-2\sin^2(\pi z/a)]. \tag{17}$$

Let us now consider the case $D=2$. For such a case, from Eqs. (15) and (16), we obtain that

$$\bar{\mathcal{L}}_{\rm corr}=\left(\frac{\pi}{24a^2}\right)[1-3\,{\rm cosec}^2(\pi z/a)]. \tag{18}$$

Note that there for both cases $D=4$ and $D=2$, there arises a term that is dependent on z. [Compare these results with Eqs. (5.11) and (5.15) that appear in Ref. [9], p. 105.]

In the discussion above, we had assumed that the field vanishes at $z=0$ and $z=a$. Let us now consider the case wherein we impose periodic boundary conditions on the field along the z-direction; i.e., let us assume that $\Phi(z)=\Phi(z+a)$. For such a case, the kernel in D dimensions (in the coincidence limit) is found to be

$$K(x,x;s)=\left(\frac{i}{(4\pi i s)^{(D-1)/2}}\right)\left(\frac{1}{a}\right)\sum_{n=-\infty}^{\infty} e^{-4in^2\pi^2 s/a^2}. \tag{19}$$

The corresponding effective Lagrangian, after the flat spacetime contribution is subtracted out, is given by

$$\bar{\mathcal{L}}_{\rm corr}=\left(\frac{\Gamma(D/2)}{(\pi a^2)^{D/2}}\right)\sum_{n=1}^{\infty} n^{-D}=\left(\frac{\Gamma(D/2)}{(\pi a^2)^{D/2}}\right)\zeta(D), \tag{20}$$

where $\zeta(D)$ is the Riemann zeta-function (see, e.g., Ref. [7], p. 334). Let us now consider the case $D=4$. For such a case, we find that

$$\bar{\mathcal{L}}_{\rm corr}=\left(\frac{\Gamma(2)}{(\pi a^2)^2}\right)\zeta(4)=\left(\frac{\pi^2}{90\,a^4}\right). \tag{21}$$

For the case $D=2$, $\bar{\mathcal{L}}_{\rm corr}$ is given by

$$\bar{\mathcal{L}}_{\rm corr}=\left(\frac{\Gamma(1)}{(\pi a^2)}\right)\zeta(2)=\left(\frac{\pi}{6a^2}\right). \tag{22}$$

B. Results with the modified weightage factor

Let us now evaluate the effective Lagrangian with the modified weightage factor. Substituting the kernel (11) in Eq. (6) and setting $m=0$, we obtain that

$$\mathcal{L}^{\rm P}_{\rm corr}=\left(\frac{1}{2(4\pi i)^{D/2}}\right)\int_0^{\infty}\frac{ds}{s^{(D/2)+1}}\,e^{iL_P^2/s}\left\{1+2\sum_{n=1}^{\infty} e^{in^2a^2/s}-\sum_{n=-\infty}^{\infty} e^{i(z+na)^2/s}\right\}. \tag{23}$$

Just as in the case of $\mathcal{L}_{\rm corr}$, there exists a non-zero $\mathcal{L}^{\rm P}_{\rm corr}$ even for a free field in flat spacetime. This flat spacetime contribution has to be subtracted from all $\mathcal{L}^{\rm P}_{\rm corr}$. It is given by

$$\mathcal{L}_{\rm corr}^{\rm P0}=\left(\frac{1}{2(4\pi i)^{D/2}}\right)\int_0^\infty \frac{ds}{s^{(D/2)+1}}\, e^{-im^2 s}\, e^{iL_P^2/s}. \tag{24}$$

On subtracting this quantity (with m set to zero) from the expression for $\mathcal{L}_{\rm corr}^{\rm P}$, we get

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P}=(\mathcal{L}_{\rm corr}^{\rm P}-\mathcal{L}_{\rm corr}^{\rm P0})=\left(\frac{\Gamma(D/2)}{2(4\pi a^2)^{D/2}}\right)\times\left\{2\sum_{n=1}^{\infty}[n^2+(L_P^2/a^2)]^{-D/2}-\sum_{n=-\infty}^{\infty}\{[n+(z/a)]^2+(L_P^2/a^2)\}^{-D/2}\right\}. \tag{25}$$

The series in the above result cannot be written in closed form for the case $D=4$. So let us consider the case $D=2$. For such a case the series in $\bar{\mathcal{L}}_{\rm corr}^{\rm P}$ above can be expressed in a closed form. Making use of the two relations (cf. Ref. [8], Vol. I, p. 685)

$$\sum_{n=0}^{\infty}(n^2+\alpha^2)^{-1}=\left(\frac{1}{2\alpha^2}\right)+\left(\frac{\pi}{2\alpha}\right)\coth(\pi\alpha) \tag{26}$$

and

$$\sum_{n=-\infty}^{\infty}[(n+\alpha)^2+\beta^2]^{-1}=\left(\frac{\pi}{\beta}\right)\sinh(2\pi\beta)[\cosh(2\pi\beta)-\cos(2\pi\alpha)]^{-1}, \tag{27}$$

we find that $\bar{\mathcal{L}}_{\rm corr}^{\rm P}$ can be expressed as follows:

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P}=\left\{-\frac{1}{8\pi L_P^2}+\left(\frac{1}{8aL_P}\right)\coth(\pi L_P/a)+\left(\frac{1}{8aL_P}\right)\sinh(2\pi L_P/a)[\cos(2\pi z/a)-\cosh(2\pi L_P/a)]^{-1}\right\}. \tag{28}$$

In the limit of $L_P\to 0$, we find that

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P}\to\left\{\left(\frac{\pi}{24a^2}\right)[1-3\csc^2(\pi z/a)]-L_P^2\left(\frac{\pi^3}{360a^4}\right)[1+30\csc^2(\pi z/a)-45\csc^4(\pi z/a)]\right\}. \tag{29}$$

That is, the lowest order quantum gravitational corrections appear at $O(L_P^2/a^2)$.

The analogous results for the case of the periodic boundary conditions are given below. The modified effective Lagrangian for such a case is given by

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P}=\left(\frac{\Gamma(D/2)}{(\pi a^2)^{D/2}}\right)\sum_{n=1}^{\infty}(n^2+4L_P^2/a^2)^{-D/2}. \tag{30}$$

For the case $D=4$,

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P}=\left\{-\left(\frac{1}{32\pi^2L_P^4}\right)+\left(\frac{1}{32\pi aL_P^3}\right)\coth(2\pi L_P/a)+\left(\frac{1}{16a^2L_P^2}\right)\operatorname{cosech}^2(2\pi L_P/a)\right\} \tag{31}$$

and, in the limit $L_P\to 0$,

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P}\to\left\{\left(\frac{\pi^2}{90a^4}\right)-L_P^2\left(\frac{8\pi^4}{945a^6}\right)\right\}. \tag{32}$$

For the case $D=2$, $\bar{\mathcal{L}}_{\rm corr}^{\rm P}$ is given by

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P}=\left\{-\left(\frac{1}{8\pi L_P^2}\right)+\left(\frac{1}{4aL_P}\right)\coth(2\pi L_P/a)\right\} \tag{33}$$

and, in the limit of $L_P\to 0$, it reduces to

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P}\to\left\{\left(\frac{\pi}{6a^2}\right)-L_P^2\left(\frac{2\pi^3}{45a^4}\right)\right\}. \tag{34}$$

III. EFFECTIVE POTENTIAL OF A SELF-INTERACTING SCALAR FIELD THEORY

In this section, we shall consider a massive, self-interacting scalar field (in 4 dimensions, i.e. $D=4$) described by the action

$$\mathcal{S}[\Phi]=\int d^4x\;\mathcal{L}(\Phi)=\int d^4x\left\{\frac{1}{2}\partial^\mu\Phi\,\partial_\mu\Phi-\mathcal{V}(\Phi)\right\}=\int d^4x\left\{\frac{1}{2}\partial^\mu\Phi\,\partial_\mu\Phi-\frac{1}{2}m^2\Phi^2-\mathcal{V}_{\rm int}(\Phi)\right\}, \tag{35}$$

where m is the mass and $\mathcal{V}_{\rm int}(\Phi)$ represents the self-interaction of the scalar field. We are interested in studying the effects of small quantum fluctuations present in the system around some classical solution Φ_c. The classical solution will be assumed to be a constant or varying adiabatically so that its derivatives can be ignored. The effect of these fluctuations will be studied by expanding the action $\mathcal{S}[\Phi]$ about the classical solution Φ_c and integrating over these fluctuations to obtain an effective potential. This effective potential will contain corrections to the original potential $\mathcal{V}(\Phi_c)$.

Carrying out the calculation as mentioned above (for instance, see Ref. [10] for details), we find that the kernel $K(x,x:s)$ for such a case is given by

$$K(x,x;s)=\left(\frac{1}{16\pi^2 is^2}\right)\exp-[i\mathcal{V}''_{int}(\Phi_c)s], \tag{36}$$

where

$$\mathcal{V}''_{\rm int}(\Phi_c)\equiv\left(\frac{\partial^2\mathcal{V}_{\rm int}(\Phi)}{\partial\Phi^2}\right)_{\Phi=\Phi_c}.$$

We shall now use this kernel to evaluate the effective potential for the self-interacting scalar field. We shall first outline as to how the conventional result can be obtained and then go on to evaluate the quantum gravitational corrections with the modified weightage factor.

A. Conventional result

Substituting the kernel (36) in Eq. (4) we obtain that

$$\mathcal{L}_{\rm corr}=-\left(\frac{1}{32\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\,e^{-i\alpha s}, \tag{37}$$

where

$$\alpha=m^2+\mathcal{V}''_{\rm int}(\Phi_c).$$

This integral is quadratically divergent near $s=0$. In the conventional approach, this integral is evaluated (after performing a Euclidean rotation) by initially setting the lower limit to a small value, say, Λ and subsequently taking the limit $\Lambda\to 0$. In this limit, we need to retain only the leading terms. Performing partial integrations repeatedly, we find that

$$\mathcal{L}_{\rm corr}=\left(\frac{1}{64\pi^2}\right)\left\{\left[\frac{1}{\Lambda^2}-\frac{\alpha}{\Lambda}-\alpha^2\ln(\Lambda\mu)\right]-\alpha^2\ln\left(\frac{\alpha}{\mu}\right)-\gamma\alpha^2\right\}, \tag{38}$$

where μ is an arbitrary but finite parameter (it has been introduced to keep the argument of the logarithms dimensionless) and γ is the Euler-Mascheroni constant. The last two terms within the curly brackets, viz. $\alpha^2\ln(\alpha/\mu)$ and $\gamma\alpha^2$, do not depend on Λ and hence are finite in the limit $\Lambda\to 0$. The term inside the square brackets, however, diverges. There are linear, quadratic and logarithmically divergent terms. The quadratically divergent term is independent of α and being just an infinite constant it can be dropped while the other two divergent terms depend on α and hence cannot be ignored.

For an arbitrary $\mathcal{V}_{\rm int}$, no sense can be made out of the above expression for $\mathcal{L}_{\rm corr}$. Only those $\mathcal{V}_{\rm int}$ for which the divergent quantities $(\Lambda^{-1}\alpha)$ and $[\alpha^2\ln(\Lambda\mu)]$ have the same form as the original $\mathcal{V}(\Phi)$ can be considered. In such a case, the divergent terms can be absorbed into the constants that determine the form of $\mathcal{V}(\Phi)$ and the theory can then be suitably reinterpreted. Clearly, the above criteria will not be satisfied for a non-polynomial $\mathcal{V}(\Phi)$. In fact, only if $\mathcal{V}(\Phi)$ is a polynomial of quartic degree or less can the divergences be absorbed and the theory reinterpreted. As an example, consider the case

$$\mathcal{V}_{\rm int}(\Phi)=\frac{1}{4!}\lambda\Phi^4. \tag{39}$$

For such a case, it can be easily shown that the resulting effective potential for the scalar field is given by (for details, see Ref. [10])

$$\mathcal{V}_{\rm eff}=\mathcal{V}(\Phi_c)-\mathcal{L}_{\rm corr}=\frac{1}{2}m^2\Phi_c^2+\frac{1}{4!}\lambda\Phi_c^4-\mathcal{L}_{\rm corr}$$
$$=\frac{1}{2}m^2_{\rm corr}\Phi_c^2+\frac{1}{4!}\lambda_{\rm corr}\Phi_c^4+\mathcal{V}_{\rm finite}, \tag{40}$$

where

$$m^2_{\rm corr}=m^2+\frac{\lambda}{32\pi^2}\left(\frac{1}{2\Lambda}+m^2\ln(\Lambda\mu)\right),$$

$$\lambda_{\rm corr}=\lambda+\frac{3\lambda^2}{32\pi^2}\ln(\Lambda\mu) \tag{41}$$

and

$$\mathcal{V}_{\rm finite}=\frac{1}{64\pi^2}\left(m^2+\frac{\lambda}{2}\Phi_c^2\right)^2\left\{\ln\left[\frac{1}{\mu}\left(m^2+\frac{\lambda}{2}\Phi_c^2\right)\right]+\gamma\right\}. \tag{42}$$

The original potential $\mathcal{V}(\Phi)$ had two constants m and λ, which were the coefficients of Φ^2 and Φ^4. In $\mathcal{V}_{\rm eff}(\Phi_c)$, these are replaced by two other constants $m_{\rm corr}$ and $\lambda_{\rm corr}$ which are functions of m, λ and the parameter μ. These constants also contain divergent terms involving Λ. It is possible to interpret $\mathcal{V}(\Phi)$ suitably using renormalization group techniques. We shall not discuss these techniques here.

B. Results with the modified weightage factor

Let us now evaluate $\mathcal{V}_{\rm eff}$ with the modified weightage factor. Substituting the kernel (36) in Eq. (6), we obtain that

$$\mathcal{L}^{\rm P}_{\rm corr}=-\left(\frac{1}{32\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\,e^{iL_P^2/s}\,e^{-i\alpha s}, \tag{43}$$

where

$$\alpha=m^2+\mathcal{V}''_{\rm int}(\Phi_c).$$

This integral can be easily evaluated and the resulting $\mathcal{L}^{\rm P}_{\rm corr}$ can be expressed in a closed form as follows (see, for instance, Ref. [11], p. 340):

$$\mathcal{L}^{\rm P}_{\rm corr}=\left(\frac{\alpha}{16\pi^2 L_P{}^2}\right)\; K_2(2L_P\sqrt{\alpha}), \qquad (44)$$

where $K_2(2L_P\sqrt{\alpha})$ is the modified Bessel function. Since

$$\mathcal{L}^{\rm P0}_{\rm corr}=-\left(\frac{1}{32\pi^2}\right)\int_0^\infty \frac{ds}{s^3}\; e^{-im^2 s}\; e^{iL_P^2/s} =\left(\frac{m^2}{16\pi^2 L_P^2}\right)\; K_2(2L_P m), \qquad (45)$$

on subtracting this quantity from $\mathcal{L}^{\rm P}_{\rm corr}$, we obtain that

$$\overline{\mathcal{L}}^{\rm P}_{\rm corr}=(\mathcal{L}^{\rm P}_{\rm corr}-\mathcal{L}^{\rm P0}_{\rm corr}) =\left(\frac{1}{16\pi^2 L_P^2}\right)\{\alpha\; K_2(2L_P\sqrt{\alpha})-m^2\; K_2(2L_P m)\}, \qquad (46)$$

and therefore,

$$\mathcal{V}^{\rm P}_{\rm eff}=\left\{\left(\frac{1}{2}m^2\Phi_c^2+\mathcal{V}_{int}(\Phi_c)\right) -\left(\frac{1}{16\pi^2 L_P^2}\right)[\alpha\; K_2(2L_P\sqrt{\alpha})-m^2\; K_2(2L_P m)]\right\}. \qquad (47)$$

This expression for the effective potential is applicable for arbitrary $\mathcal{V}_{int}$. This is in contrast to the conventional approach where the divergences appearing in $\mathcal{L}_{\rm corr}$ forced the potential to be a polynomial of quartic degree or less. Further, there is no need to introduce an arbitrary parameter μ as was required in the conventional approach. The need for such a parameter arose because of the cutoff Λ that was introduced by hand in order to isolate the divergent terms appearing in $\mathcal{L}_{\rm corr}$. The introduction of a fundamental length L_P in the theory dispenses with such a need.

Let us now specialize to the case of the quartic interaction as given by Eq. (39). For such a case, the corrections to the parameters of the theory can be obtained from $\mathcal{V}^{\rm P}_{\rm eff}$ as follows:

$$(m^{\rm P}_{\rm corr})^2=\left(\frac{\partial^2\mathcal{V}^{\rm P}_{\rm eff}}{\partial\Phi_c^2}\right)_{\Phi_c=0}=\left\{m^2-\left(\frac{\lambda}{16\pi^2 L_P^2}\right)\; K_2(2L_P m)-\left(\frac{m\lambda}{16\pi^2 L_P}\right)\; K_2'(2L_P m)\right\} \qquad (48)$$

and

$$\lambda^{\rm P}_{\rm corr}=\left(\frac{\partial^4\mathcal{V}^{\rm P}_{\rm eff}}{\partial\Phi_c^4}\right)_{\Phi_c=0}=\left\{\lambda-\left(\frac{9\lambda^2}{32\pi^2 m L_P}\right)\; K_2'(2L_P m)-\left(\frac{3\lambda^2}{16\pi^2}\right)\; K_2''(2L_P m)\right\}, \qquad (49)$$

where K_2' and K_2'' denote first and second derivatives of the modified Bessel function K_2 with respect to the argument, respectively. In the limit $L_P\to 0$, we find that

$$(m^{\rm P}_{\rm corr})^2=m^2+\left(\frac{\lambda(2\gamma-1)}{32\pi^2}\right)m^2+\left(\frac{\lambda}{32\pi^2}\right)\left[\frac{1}{L_P^2}+m^2\ln(L_P^2 m^2)\right]+\left(\frac{\lambda m^4 L_P^2}{128\pi^2}\right)[\ln\,(L_P^2 m^2)+(4\gamma+3)],$$

$$\lambda^{\rm P}_{\rm corr}=\lambda+\left(\frac{3\gamma}{16\pi^2}\right)\lambda^2+\left(\frac{3\lambda^2}{32\pi^2}\right)\ln(L_P^2 m^2)+\left(\frac{3\lambda^2 m^2 L_P^2}{32\pi^2}\right)[\ln(L_P^2 m^2)+2(\gamma-1)],$$

$$\begin{aligned}\mathcal{V}^{P}_{\rm finite}=&\frac{1}{2}m^2\left[\frac{\lambda}{32\pi^2}\;\ln\left(1+\frac{\lambda}{2m^2}\Phi_c^2\right)-\frac{\lambda}{64\pi^2}\right]\Phi_c^2\\ &+\frac{1}{4!}\left[\frac{3\lambda^2}{32\pi^2}\;\ln\left(1+\frac{\lambda}{2m^2}\Phi_c^2\right)-\frac{9\lambda^2}{64\pi^2}-\frac{11\lambda^2 m^2}{192\pi^2}L_P^2-\frac{3\lambda^2 m^2}{32\pi^2}L_P^2\;\ln(L_P^2 m^2)\right]\Phi_c^4+\left(\frac{m^4}{64\pi^2}\right)\;\ln\left(1+\frac{\lambda}{2m^2}\Phi_c^2\right)\\ &+\left(\frac{L_P^2}{192\pi^2}\right)\left(m^2+\frac{1}{2}\lambda\Phi_c^2\right)^3\;\ln\left(1+\frac{\lambda}{2m^2}\Phi_c^2\right)-\left(\frac{m^6}{192\pi^2}\right)L_P^2\;\ln(L_P^2 m^2),\end{aligned} \qquad (50)$$

where γ is the Euler-Mascheroni constant. By comparing Eqs. (42) and (50), it can be easily seen that the divergences in $m_{\rm corr}^{\rm P}$ and $\lambda_{\rm corr}^{\rm P}$ are of the same form as in $m_{\rm corr}$ and $\lambda_{\rm corr}$. Quadratic and logarithmic divergences arise in these expressions. The finite terms appearing in $(m_{\rm corr}^{\rm P})^2$ and $\lambda_{\rm corr}^{\rm P}$ which are independent of L_P are different from those appearing in $\mathcal{V}_{\rm finite}^{\rm P}$ in Eq. (50). This is because the form of the cutoff used is different. It is also clear from the above expression that for $m\neq 0$, $\mathcal{V}_{\rm finite}^{P}$ is finite in the limit $L_P\to 0$.

IV. CORRECTIONS TO THE EFFECTIVE LAGRANGIAN FOR ELECTROMAGNETIC FIELD

The system we shall consider in this section consists of a complex scalar field Φ interacting with the electromagnetic field represented by the vector potential A^μ. It is described by the following action (see, e.g., Ref. [12], p. 98):

$$\begin{aligned}\mathcal{S}[\Phi,A^\mu]&=\int d^4x\ \mathcal{L}(\Phi,A^\mu)\\&=\int d^4x\ \Big\{(\partial_\mu\Phi+iqA_\mu\Phi)(\partial^\mu\Phi^*-iqA^\mu\Phi^*)\\&\qquad -m^2\Phi\Phi^*-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\Big\},\end{aligned}\tag{51}$$

where q and m are the charge and the mass associated with a single quantum of the complex scalar field, the asterisk denotes complex conjugation and

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu .\tag{52}$$

We shall assume that the electromagnetic field behaves classically; hence A^μ is just a c-number, while we shall assume the complex scalar field to be a quantum field so that Φ is an operator valued distribution. Varying the action (51) with respect to the complex scalar field Φ, we obtain the following Klein-Gordon equation:

$$(\hat{H}+m^2)\Phi\equiv[(\partial_\mu+iqA_\mu)(\partial^\mu+iqA^\mu)+m^2]\Phi=0.\tag{53}$$

A. Conventional result

In what follows, we shall evaluate the effective Lagrangian for a constant electromagnetic background. A constant electromagnetic background can be described by the vector potential $A^\mu=(-Ez,-By,0,0)$, where E and B are constants. The electric and the magnetic fields that this vector potential gives rise to are given by $\mathbf{E}=E\hat{z}$ and $\mathbf{B}=B\hat{z}$, where $\hat{z}$ is the unit vector along the positive x-axis. The operator $\hat{H}$ corresponding to the vector potential above is then given by

$$\hat{H}\equiv(\partial_t^2-\nabla^2-2iqEz\partial_t-2iqBy\partial_x-q^2E^2+q^2B^2).\tag{54}$$

The kernel $K(x,x;s)$ corresponding to such an operator is given by (see, e.g., Ref. [10])

$$K(x,x;s)=\left\{\left(\frac{1}{16\pi^2 is^2}\right)\left(\frac{qEs}{\sinh(qEs)}\right)\left(\frac{qBs}{\sin(qBs)}\right)\right\}.\tag{55}$$

Substituting this kernel in the expression for $\mathcal{L}_{\rm corr}$ in Eq. (4), we get

$$\begin{aligned}\mathcal{L}_{\rm corr}=-\left(\frac{1}{16\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\,e^{-i(m^2-i\epsilon)s}\left(\frac{qas}{\sinh(qas)}\right)\\ \times\left(\frac{qbs}{\sin(qbs)}\right),\end{aligned}\tag{56}$$

where a and b are related to the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ by the relations $(a^2-b^2)=(\mathbf{E}^2-\mathbf{B}^2)$ and $(ab)=(\mathbf{E}\cdot\mathbf{B})$.

We can now interpret the real part of $\mathcal{L}_{\rm corr}$ as the correction to the Lagrangian describing the classical electromagnetic background given by

$$\mathcal{L}_{\rm em}=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}=\frac{1}{2}(\mathbf{E}^2-\mathbf{B}^2)=\frac{1}{2}(a^2-b^2).\tag{57}$$

The real part of $\mathcal{L}_{\rm corr}$ can be regularized by subtracting the flat space contribution which is obtained by setting both a and b to zero. Such a regularization then leads us to the following result:

$$\begin{aligned}\mathrm{Re}\ \bar{\mathcal{L}}_{\rm corr}=-\left(\frac{1}{16\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\cos(m^2s)\\ \times\left\{\left(\frac{qas}{\sinh(qas)}\right)\left(\frac{qbs}{\sin(qbs)}\right)-1\right\}.\end{aligned}\tag{58}$$

Near $s=0$, the expression in the curly brackets above goes as $[-q^2s^2(a^2-b^2)/6]$. Hence, Re $\bar{\mathcal{L}}_{\rm corr}$ is still logarithmically divergent near $s=0$. But this divergence is proportional to the original Lagrangian $\mathcal{L}_{\rm em}$, and because of this feature, we can absorb this divergence by redefining the field strengths and charge. Or, in other words, we can renormalize the field strengths and charge by absorbing the logarithmic divergence into them in the following fashion. We write

$$\mathcal{L}_{\rm eff}=(\mathcal{L}_{\rm em}+\mathrm{Re}\ \bar{\mathcal{L}}_{\rm corr})=(\mathcal{L}_{\rm em}+\mathcal{L}_{\rm div})+(\mathrm{Re}\ \bar{\mathcal{L}}_{\rm corr}-\mathcal{L}_{\rm div}),\tag{59}$$

where we have defined $\mathcal{L}_{\rm div}$ as follows:

$$\begin{aligned}\mathcal{L}_{\rm div}&=-\left(\frac{1}{16\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\cos(m^2s)\left\{-\frac{1}{6}q^2s^2(a^2-b^2)\right\}\\&=\left(\frac{Z}{2}\right)(a^2-b^2)=\left(\frac{Z}{2}\right)(\mathbf{E}^2-\mathbf{B}^2)=Z\ \mathcal{L}_{\rm em},\end{aligned}\tag{60}$$

where Z is a logarithmically divergent quantity described by the integral

$$Z=\left(\frac{q^2}{48\pi^2}\right)\int_0^\infty\frac{ds}{s}\cos(m^2 s). \tag{61}$$

Therefore, we can write

$$\mathcal{L}_{\rm eff}=(\mathcal{L}_{\rm em}+\mathcal{L}_{\rm div})+({\rm Re}\ \bar{\mathcal{L}}_{\rm corr}-\mathcal{L}_{\rm div})=(1+Z)\ \mathcal{L}_{\rm em}+\mathcal{L}_{\rm finite}, \tag{62}$$

where $\mathcal{L}_{\rm finite}=({\rm Re}\ \bar{\mathcal{L}}_{\rm corr}-\mathcal{L}_{\rm div})$ is a finite quantity described by the integral

$$\mathcal{L}_{\rm finite}=-\left(\frac{1}{16\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\cos(m^2 s)\left\{\left(\frac{qas}{\sinh(qas)}\right)\times\left(\frac{qbs}{\sin(qbs)}\right)-1+\frac{1}{6}q^2s^2(a^2-b^2)\right\}. \tag{63}$$

All the divergences now appear in Z. Redefining the field strengths and charges as

$$\mathbf{E}_{\rm phy}=(1+Z)^{1/2}\ \mathbf{E},\quad \mathbf{B}_{\rm phy}=(1+Z)^{1/2}\ \mathbf{B},$$
$$q_{\rm phy}=(1+Z)^{-1/2}\ q, \tag{64}$$

we find that such a scaling leaves $q_{\rm phy}\mathbf{E}_{\rm phy}=q\mathbf{E}$ invariant. Thus it is possible to redefine (renormalize) the variables in the theory, thereby taking care of the divergences.

B. Results with the modified weightage factor

With the modified weightage factor, we find that the quantity $\mathcal{L}^{\rm P}_{\rm corr}$ for the constant electromagnetic background is described by the following integral:

$$\mathcal{L}^{\rm P}_{\rm corr}=-\left(\frac{1}{16\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\,e^{-im^2s}\,e^{iL_P^2/s}\left(\frac{qas}{\sinh(qas)}\right)\times\left(\frac{qbs}{\sin(qbs)}\right). \tag{65}$$

The real part of $\mathcal{L}^{\rm P}_{\rm corr}$ is then given by

$${\rm Re}\ \mathcal{L}^{\rm P}_{\rm corr}=-\left(\frac{1}{16\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\cos\,[m^2s-(L_P^2/s)]\times\left(\frac{qas}{\sinh(qas)}\right)\left(\frac{qbs}{\sin(qbs)}\right). \tag{66}$$

Regularizing this quantity by subtracting the flat space contribution (viz. the quantity obtained by setting $a=b=0$ in the above expression), we get

$${\rm Re}\ \bar{\mathcal{L}}^{\rm P}_{\rm corr}=-\left(\frac{1}{16\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\cos[m^2s-(L_P^2/s)]\times\left\{\left(\frac{qas}{\sinh(qas)}\right)\left(\frac{qbs}{\sin(qbs)}\right)-1\right\}. \tag{67}$$

We can now express the effective Lagrangian exactly as we had done earlier. We can write

$$\mathcal{L}^{\rm P}_{\rm eff}=(\mathcal{L}_{\rm em}+{\rm Re}\ \bar{\mathcal{L}}^{\rm P}_{\rm corr})=(\mathcal{L}_{\rm em}+\mathcal{L}^{\rm P}_{\rm div})+({\rm Re}\ \bar{\mathcal{L}}^{\rm P}_{\rm corr}-\mathcal{L}^{\rm P}_{\rm div}), \tag{68}$$

where we $\mathcal{L}^{\rm P}_{\rm div}$ can now be defined as follows:

$$\mathcal{L}^{\rm P}_{\rm div}=-\left(\frac{1}{16\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\cos[m^2s-(L_P^2/s)]\times\left\{-\frac{1}{6}q^2s^2(a^2-b^2)\right\}$$
$$=\frac{Z^{\rm P}}{2}(a^2-b^2)=\frac{Z^{\rm P}}{2}(\mathbf{E}^2-\mathbf{B}^2)=Z^{\rm P}\ \mathcal{L}_{\rm em}. \tag{69}$$

$Z^{\rm P}$ is now a finite quantity described by the integral

$$Z^{\rm P}=\left(\frac{q^2}{48\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\cos(m^2s-L_P^2/s)=\frac{q^2}{6\pi}K_0(2mL_P), \tag{70}$$

where $K_0(2mL_P)$ is the modified Bessel function of order zero. Therefore, we can write

$$\mathcal{L}^{\rm P}_{\rm eff}=(\mathcal{L}_{\rm em}+\mathcal{L}^{\rm P}_{\rm div})+({\rm Re}\ \bar{\mathcal{L}}^{\rm P}_{\rm corr}-\mathcal{L}^{\rm P}_{\rm div})$$
$$=(1+Z^{\rm P})\ \mathcal{L}_{\rm em}+\mathcal{L}^{\rm P}_{\rm finite}, \tag{71}$$

where $\mathcal{L}^{\rm P}_{\rm finite}=({\rm Re}\ \bar{\mathcal{L}}^{\rm P}_{\rm corr}-\mathcal{L}_{\rm div})$ is given by

$$\mathcal{L}^{\rm P}_{\rm finite}=-\left(\frac{1}{16\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\cos(m^2s-L_P^2/s)\times\left\{\left(\frac{qas}{\sinh(qas)}\right)\left(\frac{qbs}{\sin(qbs)}\right)-1+\frac{1}{6}q^2s^2(a^2-b^2)\right\}. \tag{72}$$

We can now redefine the field strengths and the charge just as we had done earlier with $Z^{\rm P}$ instead of Z. $Z^{\rm P}$ is a finite quantity for a non-zero L_P, but diverges logarithmically when L_P is set to zero. Even in the limit $L_P\to 0$, the quantity $\mathcal{L}^{\rm P}_{\rm finite}$ stays finite and the divergence appears only in the expression for $Z^{\rm P}$.

In the limit $L_P\to 0$, we can make a rough estimate of the value of $\mathcal{L}^{\rm P}_{\rm finite}$ as follows. The quantity exp $(-m^2s-L_P^2/s)$ is a sharply peaked function about the value $s=(L_P/m)\ll 1$. Therefore, we may, without appreciable error, expand the term in the curly brackets in the above expression for $\mathcal{L}^{\rm P}_{\rm finite}$ in a Taylor series about the point $s=0$, retaining only the first non-zero term. Keeping the limits of integration from 0 to ∞ (and performing a Euclidean rotation in order to evaluate the integral), we obtain

$$\mathcal{L}_{\rm finite}^{\rm P} \approx \left(\frac{1}{16\pi^2}\right)\left(\frac{2L_P^2 q^4}{360 m^2}\right)[7(a^2-b^2)^2+4a^2b^2] K_2(2mL_P), \tag{73}$$

where K_2 is the modified Bessel function of order 2. Expanding K_2 in a series and retaining the terms of least order in L_P, we obtain

$$\mathcal{L}_{\rm finite}^{\rm P} \approx \left(\frac{1}{16\pi^2}\right)\left(\frac{q^4}{360 m^4}\right)[7(a^2-b^2)^2+4a^2b^2](1-L_P^2 m^2). \tag{74}$$

V. CORRECTIONS TO THE THERMAL EFFECTS IN THE RINDLER FRAME

In flat spacetime, the Minkowski vacuum state is invariant only under the Poincaré group, which is basically a set of linear coordinate transformations. Under a non-linear coordinate transformation the particle concept, in general, proves to be coordinate dependent. For example, the quantizations in the Minkowski and the Rindler coordinates are inequivalent [13]. In fact the expectation value of the Rindler number operator in the Minkowski vacuum state proves to be a thermal spectrum. This result is normally obtained in the literature by quantizing the field in the two coordinate systems and then evaluating the expectation value of the Rindler number operator in the Minkowski vacuum state. If we are to evaluate the quantum gravitational corrections to the Rindler thermal spectrum in such a fashion, then we need to know how the metric fluctuations modify the normal modes of the quantum field. But as we have mentioned earlier, we only know how quantum gravitational corrections can be introduced in the effective Lagrangian. Therefore, in this section, we shall first evaluate the effective Lagrangian in the Rindler coordinates and then go on to evaluate the corrections to this effective Lagrangian. The derivation of the Hawking radiation in a black hole spacetime runs along similar lines as the derivation of the Rindler thermal spectrum. Hence the results we present in this section have some relevance to the effects of metric fluctuations on Hawking radiation.

The system we shall consider in this section is a massless scalar field (in 4 dimensions) described by the action

$$\mathcal{S}[\Phi] = \int d^4x \sqrt{-g}\ \mathcal{L}(\Phi) = \int d^4x \sqrt{-g}\left\{\frac{1}{2}\, g_{\mu\nu}\ \partial^\mu\ \Phi \partial^\nu \Phi\right\}. \tag{75}$$

Varying this action, we obtain the equation of motion for Φ to be

$$\hat{H}\Phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g} g^{\mu\nu}\partial_\nu)\Phi = 0. \tag{76}$$

A. Conventional result

1. Effective Lagrangian

The transformations that relate the Minkowski coordinates in flat spacetime to those of an observer who is accelerating uniformly along the x-direction are given by the following relations [14]:

$$t = g^{-1}(1+g\xi)\ \sinh(g\tau),$$
$$x = g^{-1}(1+g\xi)\ \cosh(g\tau), \quad y=y, \quad z=z, \tag{77}$$

where g is a constant. The new coordinates (τ,ξ,y,z) are called the Rindler coordinates. In terms of the Rindler coordinates the flat spacetime line element is then given by

$$ds^2 = (1+g\xi)^2\ d\tau^2 - d\xi^2 - dy^2 - dz^2. \tag{78}$$

Therefore, in the Rindler coordinates, the operator $\hat{H}$ as defined in Eq. (76) is given by

$$\hat{H} \equiv \left\{\frac{1}{(1+g\xi)^2}\partial_\tau^2 - \frac{1}{(1+g\xi)}\partial_\xi[(1+g\xi)\partial_\xi] - \partial_y^2 - \partial_z^2\right\}, \tag{79}$$

where $\partial_x \equiv (\partial/\partial x)$. This operator is invariant under translations along the y- and z-directions. Or, in other words, the kernel corresponds to that of a free particle along these two directions. Exploiting this feature, we can write the quantum mechanical kernel as

$$K(x,x,;s|g_{\mu\nu}) = \left(\frac{1}{4\pi i s}\right)\langle \tau,\xi| e^{-i\hat{H}' s}|\tau,\xi\rangle, \tag{80}$$

where

$$\hat{H}' \equiv \left(\frac{1}{(1+g\xi)^2}\partial_\tau^2 - \frac{1}{(1+g\xi)}\partial_\xi[(1+g\xi)\partial_\xi]\right). \tag{81}$$

On rotating the time coordinate τ to the negative imaginary axis (i.e. on setting $\tau = -i\tau_E$) and changing variables to $u = [g^{-1}(1+g\xi)]$, we find that

$$\hat{H}' \equiv \left(-\frac{1}{g^2u^2}\partial_{\tau_E}^2 - \frac{1}{u}\partial_u[u\partial_u]\right). \tag{82}$$

If we identify u as a radial variable and $g\tau_E$ as an angular variable, then $\hat{H}'$ is similar in form to the Hamiltonian operator of a free particle in polar coordinates (in 2 dimensions). Then, for a constant ξ, the kernel corresponding to the operator $\hat{H}'$ can be written as [15,16]

$$\langle \tau',\xi| e^{-i\hat{H}' s}|\tau',\xi\rangle = \left(\frac{1}{4\pi s}\right)\sum_{n=-\infty}^{\infty}\ \exp\left\{-\frac{i}{4s}(1+g\xi)^2 \times(\tau-\tau'+2\pi i n g^{-1})^2\right\}. \tag{83}$$

Therefore, the complete quantum mechanical kernel corresponding to the operator $\hat{H}$ (in the coincidence limit) is given by

$$K(x,x;s|g_{\mu\nu})=\left(\frac{1}{16\pi^2 is^2}\right)\sum_{n=-\infty}^{\infty}\exp(i\beta^2n^2/4s)$$
$$=\left(\frac{1}{16\pi^2 is^2}\right)\left\{1+2\sum_{n=1}^{\infty}\exp(i\beta^2n^2/4s)\right\},\tag{84}$$

where $\beta=[2\pi g^{-1}(1+g\xi)]$. Substituting this kernel in expression (4) and setting $m=0$, we find that

$$\mathcal{L}_{\rm corr}=-\left(\frac{1}{32\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\left\{1+2\sum_{n=1}^{\infty}\exp(i\beta^2n^2/4s)\right\}.\tag{85}$$

On regularization, i.e. on subtracting the quantity $\mathcal{L}^0_{\rm corr}$ [given by Eq. (14) with $m=0$] from the above expression, we obtain that

$$\overline{\mathcal{L}}_{\rm corr}=(\mathcal{L}_{\rm corr}-\mathcal{L}^0_{\rm corr})=-\left(\frac{1}{16\pi^2}\right)\sum_{n=1}^{\infty}\int_0^\infty\frac{ds}{s^3}\exp(i\beta^2n^2/4s).\tag{86}$$

The integral over s can be expressed in terms of Gamma functions (see, e.g., Ref. [11], p. 934), so that

$$\overline{\mathcal{L}}_{\rm corr}=\left(\frac{\Gamma(2)}{\pi^2\ \beta^4}\right)\sum_{n=1}^{\infty}n^{-4}=\left(\frac{\Gamma(2)}{\pi^2\ \beta^4}\right)\zeta(4)=\left(\frac{\pi^2}{90\ \beta^4}\right),\tag{87}$$

where we have made use of the fact that $\zeta(4)=(\pi^4/90)$ (cf. Ref. [7], p. 334).

Two points need to be noted regarding the above result. First, $\overline{\mathcal{L}}_{\rm corr}$ corresponds to the total energy radiated by a blackbody at a temperature β^{-1}. Second, there arises no imaginary part to the effective Lagrangian which clearly implies that the thermal effects in the Rindler frame arise due to vacuum polarization and not due to particle production.

2. Power spectrum from the propagator

The Feynman propagator (in Lorentzian space) corresponding to an operator $\hat{H}$ is given by [cf. Eq. (1)]

$$G_{\rm F}(x,x')=-i\int_0^\infty ds\ K(x,x';s|g_{\mu\nu}),\tag{88}$$

where $K(x,x';s|g_{\mu\nu})$ is given by Eq. (5). (Note that since we are considering a massless scalar field, we have set $m=0$.) For the Rindler coordinates we are considering here, the propagator is obtained by substituting the kernel (84) in the expression for the propagator as given by Eq. (88). If we set $\xi=\xi'=0$, $y=y'$ and $z=z'$, we find that the propagator is given by

$$G_{\rm F}(x,x')\equiv G_{\rm F}(\Delta\tau)$$
$$=-\left(\frac{1}{16\pi^2}\right)\int_0^\infty\frac{ds}{s^2}\sum_{n=-\infty}^{n=\infty}\exp-[i(\Delta\tau+i\beta n)^2/4s]$$
$$=\left(\frac{i}{4\pi^2}\right)\sum_{n=-\infty}^{n=\infty}(\Delta\tau+2\pi ing^{-1})^{-2},\tag{89}$$

where $\Delta\tau=(\tau-\tau')$. Also, since we have set $\xi=0$, $\beta=(2\pi/g)$. [Compare this result with Eq. (3.66) in Ref. [17].] Fourier transforming this propagator with respect to $\Delta\tau$, we find that

$$\mathcal{P}(\Omega)\equiv\left|\int_{-\infty}^{\infty}d\Delta\tau\ e^{-i\Omega\Delta\tau}\ G_{\rm F}(\Delta\tau)\right|=\left(\frac{1}{2\ \pi}\right)\left(\frac{\Omega}{e^{\beta\Omega}-1}\right);\tag{90}$$

i.e., the resulting power spectrum is a thermal spectrum with a temperature β^{-1}.

B. Results with the modified weightage factor

1. The modified effective Lagrangian

Let us now evaluate the effective Lagrangian in the Rindler frame with the modified weightage factor. Substituting the kernel (84) in the expression (6) for $\mathcal{L}^{\rm P}_{\rm corr}$ and setting $m=0$, we obtain that

$$\mathcal{L}^{\rm P}_{\rm corr}=-\left(\frac{1}{32\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\ e^{iL_P^2/s}\left\{1+2\sum_{n=1}^{\infty}\exp\ (i\beta^2n^2/4s)\right\}.\tag{91}$$

On regularization, i.e. on subtracting the quantity $\mathcal{L}^{\rm P0}_{\rm corr}$ (with $m=0$) from $\mathcal{L}^{\rm P}_{\rm corr}$, we find that

$$\overline{\mathcal{L}}^{\rm P}_{\rm corr}=(\mathcal{L}^{\rm P}_{\rm corr}-\mathcal{L}^{\rm P0}_{\rm corr})$$
$$=-\left(\frac{1}{16\pi^2}\right)\int_0^\infty\frac{ds}{s^3}\ e^{iL_P^2/s}\sum_{n=1}^{\infty}\exp(i\beta^2n^2/4s)$$
$$=\left(\frac{1}{\pi^2}\right)\sum_{n=1}^{\infty}(\beta^2n^2+4L_P^2)^{-2}$$
$$=\left(\frac{1}{\pi^2\beta^4}\right)\sum_{n=1}^{\infty}[n^2+(4L_P^2/\beta^2)]^{-2}.\tag{92}$$

Using the relation (cf. Ref. [8], Vol. I, p. 687)

$$\sum_{n=0}^{\infty}(n^2+\alpha^2)^{-2}=\left(\frac{1}{\alpha^4}\right)+\sum_{n=1}^{\infty}(n^2+\alpha^2)^{-2}$$
$$=\left(\frac{1}{2\alpha^4}\right)+\left(\frac{\pi}{4\alpha^3}\right)\coth(\pi\alpha)$$
$$+\left(\frac{\pi^2}{4\alpha^2}\right)\ {\rm csch}^2(\pi\alpha),\tag{93}$$

we can express $\bar{\mathcal{L}}_{\rm corr}^{\rm P}$ in a closed form as follows:

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P}=\left\{-\left(\frac{1}{32\pi^2 L_P^4}\right)+\left(\frac{1}{32\pi\beta L_P^3}\right)\coth(2\pi L_P/\beta)+\left(\frac{1}{16\beta^2 L_P^2}\right)\operatorname{csch}^2(2\pi L_P/\beta)\right\}. \quad (94)$$

Making use of the series expansions (cf. Ref. [11], p. 36)

$$\coth(\pi x)=\left(\frac{1}{\pi x}\right)+\left(\frac{2x}{\pi}\right)\sum_{n=1}^{\infty}(x^2+n^2)^{-1} \quad (95)$$

and

$$\operatorname{csch}^2(\pi x)=\left(\frac{1}{\pi^2 x^2}\right)+\left(\frac{2}{\pi^2}\right)\sum_{n-1}^{\infty}\left\{\frac{x^2-n^2}{(x^2+n^2)^2}\right\}, \quad (96)$$

we find that, as $L_P\to 0$

$$\bar{\mathcal{L}}_{\rm corr}^{\rm P}\to\left\{\left(\frac{\pi^2}{90\beta^4}\right)-L_P^2\left(\frac{8\pi^4}{945\beta^6}\right)\right\}. \quad (97)$$

2. Power spectrum from the modified propagator

The propagator with the modified weightage factor is given by

$$G_{\rm F}^{\rm P}(x,x')=-i\int_0^\infty ds\; e^{iL_P^2/s}\;K(x,x';s|g_{\mu\nu}). \quad (98)$$

In the Rindler coordinates, we find that, if we set $\xi=\xi'=0$, $y=y'$ and $z=z'$, the modified propagator reduces to

$$\begin{aligned}G_{\rm F}^{\rm P}(\Delta\tau)&=-\left(\frac{1}{16\pi^2}\right)\int_0^\infty\frac{ds}{s^2}\\&\quad\times\sum_{n=-infty}^{n=\infty}\exp\left\{-i\left[\frac{1}{4s}(\Delta\tau+i\beta n)^2-\frac{L_P^2}{s}\right]\right\}\\&=\left(\frac{i}{4\pi^2}\right)\sum_{n=-\infty}^{n=\infty}[(\Delta\tau+2\pi ing^{-1})^{-2}-4L_P^2]^{-1},\end{aligned} \quad (99)$$

where, as before, $\Delta\tau=(\tau-\tau')$. Fourier transforming this modified propagator with respect to $\Delta\tau$, we obtain that

$$\begin{aligned}\mathcal{P}^{\rm P}(\Omega)&\equiv\left|\int_{-\infty}^{\infty}d\Delta\tau\; e^{-i\Omega\Delta\tau}\;G_{\rm F}^{\rm P}(\Delta\tau)\right|\\&=\left(\frac{1}{2\pi}\right)\left(\frac{|\sin(2\Omega L_P)|}{2\Omega L_P}\right)\left(\frac{\Omega}{e^{\beta\Omega}-1}\right).\end{aligned} \quad (100)$$

This modified spectrum shows an appreciable deviation from the Planckian form only when $\beta\simeq L_P$. But when $\beta\simeq L_P$, the semiclassical approximation we are working with will anyway cease to be valid.

VI. CORRECTIONS TO THE GRAVITATIONAL LAGRANGIAN

The system we shall consider in this section consists of a scalar field Φ interacting with a classical gravitational field described by the metric tensor $g_{\mu\nu}$. It is described by the action

$$\begin{aligned}\mathcal{S}[g_{\mu\nu},\Phi]&=\int d^Dx\sqrt{-g}\;\mathcal{L}(g_{\mu,\nu},\Phi)\\&=\int d^Dx\sqrt{-g}\left\{\frac{1}{16\pi G}(R-2\Lambda)+\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi-\frac{1}{2}m^2\Phi^2-\frac{1}{2}\xi R\Phi^2\right\},\end{aligned} \quad (101)$$

where R is the scalar curvature of the spacetime, Λ is the cosmological constant and G is the gravitational constant. Setting the parameter $\xi=0$ or $\xi=(1/6)$ corresponds to a minimal or conformal coupling of the scalar field to the gravitational background respectively. We are interested in finding quantum corrections to the purely gravitational part of the total Lagrangian. This will be done as usual in the framework of the semiclassical theory by considering the one-loop effective action formalism. In the conventional derivation, divergences arise in the expression for $\mathcal{L}_{\rm corr}$. There are three divergent terms, two of which are absorbed into the cosmological constant Λ and the gravitational constant G, and thus Einstein's theory is reinterpreted suitably. The third divergent term cannot be so absorbed. Extra terms will have to be introduced into the gravitational Lagrangian in order to absorb this divergence [17]. When the duality principle is used, however, no divergences occur. The cosmological constant and the the gravitational constant are modified by the addition of finite terms which are seen to diverge in the limit $L_P\to 0$, thus recovering the standard result.

A. Conventional result

In what follows, we shall first briefly outline the conventional approach for calculating $\mathcal{L}_{\rm corr}$ and then use the modified weightage factor to compute the corrections to the gravitational Lagrangian.

Varying the above action with respect to Φ, we obtain that

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)+m^2+\xi R\right)\Phi=0. \quad (102)$$

Comparing this equation of motion with Eq. (3), it is easy to identify that

$$\hat{H}\equiv\left(\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)+m^2+\xi R\right). \quad (103)$$

In D dimensions, the quantum mechanical kernel $K(x,x';s|g_{\mu\nu})$ [cf. Eq. (5)], corresponding to the operator $\hat{H}$ above, can be written as [17,18]

$$K(x,x';s|g_{\mu\nu}) = \left(\frac{i}{(4\pi i s)^{D/2}}\right) e^{i\sigma(x,x')/2s}\Delta^{1/2}(x,x')\ F(x,x';s), \quad (104)$$

where

$$\sigma(x,x') = \frac{1}{2}\int_0^s ds' \left\{ g_{\mu\nu}\frac{dx^\mu}{ds'}\frac{dx^\nu}{ds'}\right\}^{1/2} \quad (105)$$

is the proper arc length along the geodesic from x' to x and $\Delta^{1/2}(x,x')$ is the Van Vleck determinant given by

$$\Delta^{1/2}(x,x') = (-[-g(x)]^{-1/2} \det[\partial_\mu\partial_\nu\sigma(x,x')] \times[-g(x')]^{-1/2}). \quad (106)$$

The function $F(x,x';s)$ can be written down in an asymptotic expansion

$$F(x,x';s) = \sum_{n=0}^{\infty} a_n\ (is)^n = a_0 + a_1(x,x')\ (is) + a_2(x,x')\ (is)^2 + \dots, \quad (107)$$

where the leading term a_0 is unity since F must reduce to unity in flat spacetime.

Substituting the quantum mechanical kernel above in the expression for $\mathcal{L}_{\rm corr}$ given by Eq. (4), we obtain that

$$\mathcal{L}_{\rm corr} = -\lim_{x\to x'}\left(\frac{\Delta^{1/2}(x,x')}{32\ \pi^2}\right)\int_0^\infty \frac{ds}{s^3}\ e^{-im^2 s} e^{i\sigma(x,x')/2s} \times\{1 + a_1(x,x')\ (is) + a_2(x,x')\ (is)^2 + \dots\}. \quad (108)$$

In the coincidence limit, $\sigma(x,x')$ vanishes and one can easily see that the integrals over the first three terms in the square brackets diverge. The integrals over the remaining terms involving a_3, a_4 and so on are finite in this limit. Therefore, the divergent part of $\mathcal{L}_{\rm corr}$ is given by

$$\mathcal{L}^{\rm div}_{\rm corr} = -\left(\frac{1}{32\ \pi^2}\right)\int_0^\infty \frac{ds}{s^3}\ e^{-im^2 s}\ \{1 + a_1(x,x)\ (is) + a_2(x,x)\ (is)^2\}, \quad (109)$$

where the coefficients a_1 and a_2 are given by the relations [17]

$$a_1(x,x) = \left(\frac{1}{6} - \xi\right)R \quad (110)$$

and

$$a_2(x,x) = \frac{1}{6}\left(\frac{1}{5} - \xi\right)g^{\mu\nu}R_{;\mu;\nu} + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2 + \frac{1}{180}R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - \frac{1}{180}R_{\mu\nu}R^{\mu\nu}. \quad (111)$$

Since a_1 and a_2 depend only on $R_{\mu\nu\lambda\rho}$ and its contractions, they are purely geometrical in nature. The divergences arise because of the ultraviolet behavior of the field modes. These short wavelengths probe only the local geometry in the neighborhood of x and are not sensitive to the global features of the spacetime and are independent of the quantum state of the field Φ. Since the divergent part of the effective Lagrangian is purely geometrical, it can be regarded as the correction to the gravitational part of the Lagrangian. The divergence corresponding to the first term in the square brackets can be added to the cosmological constant, thus regularizing it, while the divergence due to the second term can be absorbed into the gravitational constant, giving rise to the renormalized gravitational constant which is finite. The third term which involves a_2 contains derivatives of the metric tensor of order 4 and this term represents a correction to Einstein's theory which contains derivatives of order 2 only. Therefore one needs to introduce extra terms into the gravitational Lagrangian so that these divergences can be absorbed into suitable constants [17].

B. Results with the modified weightage factor

Let us now evaluate the corrections to the gravitational Lagrangian with the modified weightage factor. Substituting the kernel (104) into Eq. (6), we obtain

$$\mathcal{L}^{\rm P}_{\rm corr} = -\left(\frac{1}{32\ \pi^2}\right)\int_0^\infty \frac{ds}{s^3}\ e^{-im^2 s}\ e^{iL_P^2/s}\{1 + a_1(x,x)\ (is) + a_2(x,x)\ (is)^2 + \dots\}$$
$$= \left(\frac{m^4}{32\ \pi}\right)^2\left\{\left(\frac{2}{L_P^2\ m^2}\right) K_2(2L_P m) + \left(\frac{2}{L_P\ m^3}\right) K_1(2L_P m)a_1(x,x) + \left(\frac{2}{m^4}\right)K_0(2L_P m)a_2(x,x) + \dots\right\}, \quad (112)$$

where K_0, K_1 and K_2 are the modified Bessel functions of orders 0, 1 and 2, respectively. The effective Lagrangian for the classical gravitational background is therefore given by

$$\mathcal{L}_{\rm eff}^{\rm P}=(\mathcal{L}_{\rm grav}+\mathcal{L}_{\rm corr}^{\rm P})=\frac{1}{16\pi G_{\rm corr}}R-\frac{1}{8\pi G}\Lambda_{\rm corr}+\frac{1}{16\pi^2}K_0(2L_Pm)a_2(x,x)+\ \dots, \tag{113}$$

where

$$\Lambda_{\rm corr}=\Lambda-\left(\frac{m^2G}{2\pi L_P^2}\right)K_2(2L_Pm)\quad \text{and}\quad \frac{1}{G_{\rm corr}}=\frac{1}{G}+\left(\frac{m}{\pi L_P}\right)\left(\frac{1}{6}-\xi\right)K_1(2L_Pm). \tag{114}$$

Invoking the principle of path duality corresponds to multiplying the kernel $K(x,x;s|g_{\mu\nu})$ by the factor $\exp(iL_P^2/s)$, where L_P^2 is the square of the Planck length. But as mentioned in the Introduction, it can so happen that the fundamental length is (ηL_P) where η is a numerical factor of order unity. Therefore, if we replace L_P in the above equations by (ηL_P) and since $L_P^2\equiv G$ by definition, the formula for $G_{\rm corr}$ can equivalently be written as

$$\frac{1}{G_{\rm corr}}=\frac{1}{G}\left[1+\frac{m\sqrt{G}}{\eta\pi}\left(\frac{1}{6}-\xi\right)K_1(2\eta\sqrt{G}m)\right]. \tag{115}$$

In the limit $(\eta m)\to 0$, using the power series expansion for the functions $K_2(2\eta\sqrt{G}m)$ and $K_1(2\eta\sqrt{G}m)$, we can write the corrections to G and Λ as follows:

$$\Lambda_{\rm corr}=\Lambda-\frac{m^2}{2\pi\eta^2}\left(\frac{1}{2\eta^2Gm^2}-\frac{1}{2}\right)=\Lambda+\frac{m^2}{4\pi\eta^2}-\frac{1}{4\pi\eta^4G} \tag{116}$$

and

$$\frac{1}{G_{\rm corr}}=\frac{1}{G}\left[1+\frac{1}{2\pi\eta^2}\left(\frac{1}{6}-\xi\right)\right]. \tag{117}$$

For the case when the scalar field is assumed to be coupled minimally to the gravitational background (i.e. when $\xi=0$) the correction to G reduces to

$$\frac{1}{G_{\rm corr}}=\frac{1}{G}\left[1+\frac{1}{12\pi\eta^2}\right], \tag{118}$$

whereas in the conformally coupled case (i.e. when $\xi=1/6$) the correction to G vanishes identically.

VII. CORRECTIONS TO THE TRACE ANOMALY

The problem concerning the renormalization of the expectation value of the energy-momentum tensor in curved spacetime is considerably more involved than the corresponding problem in Minkowski spacetime. This concerns the role of the energy-momentum tensor $T_{\mu\nu}$ in gravity. In flat spacetime only energy differences are meaningful and therefore infinite constants like the energy of the vacuum can be subtracted out without any problem. In curved spacetime, however, energy is a source of gravity. Therefore, one is not free to rescale the zero point of the energy scale in an arbitrary manner. In the semi-classical theory of gravity, one can carry out the renormalization of $\langle T_{\mu\nu}\rangle$ in a unique way using different methods like the ζ-function renormalization technique, dimensional regularization and other methods. Since $\langle T_{\mu\nu}\rangle$ can be obtained from the effective action by functionally differentiating with respect to the metric tensor, the renormalization procedure is therefore connected with the renormalization of the effective action which was described in the previous section. Upon specializing to theories where the classical action is invariant under conformal transformations, it can be shown that the trace of the *classical* energy-momentum tensor is zero. But when the renormalized expectation value of the trace is calculated, however, it is found to be non-zero. This is the conformal or trace anomaly. It essentially arises because of the divergent terms present in the effective action. When the principle of path integral duality is applied, no divergences appear and hence one would expect the trace anomaly to vanish. But because a fixed length scale appears in the problem, the trace anomaly is still nonzero.

A. Conventional result

In this section we derive the formula relating the trace of the energy momentum tensor and the effective action. We then apply the duality principle and derive an explicit formula for the trace anomaly. In the limit of $L_P\to 0$, it is shown that the usual divergences appear which when renormalized using dimensional regularization and zeta function regularization techniques yield the usual formula for the trace anomaly.

Consider a scalar field that is coupled to a classical gravitational background as described by the action (101). The energy-momentum of such a scalar field can be obtained by varying the action with respect to the metric tensor $g_{\mu\nu}$ as follows [19]:

$$\begin{aligned}T^{\mu\nu}&\equiv\left(\frac{2}{\sqrt{-g}}\right)\left(\delta\frac{\mathcal{S}}{\delta}g_{\mu\nu}\right)\\ &=\partial^\mu\Phi\,\partial^\nu\Phi-\frac{1}{2}g^{\mu\nu}\partial^\alpha\partial_\alpha\Phi-\frac{1}{2}g^{\mu\nu}m^2\Phi^2\\ &\quad+\xi\left\{g^{\mu\nu}\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta)\Phi^2\right.\\ &\quad\left.-(\Phi^2)^{;\mu;\nu}+G^{\mu\nu}\Phi^2\right\},\end{aligned} \tag{119}$$

where $G^{\mu\nu}=(R^{\mu\nu}-(1/2)g^{\mu\nu}R)$. Using the field equations (102), the trace of $T^{\mu\nu}$ is

$$T^{\mu}_{\mu}=(6\xi-1)\partial^{\mu}\Phi\partial_{\mu}\Phi+\xi(6\xi-1)R\Phi^2+(6\xi-2)m^2\Phi^2. \tag{120}$$

For the conformally invariant case, i.e. when $\xi=1/6$ and $m=0$, the trace vanishes. Thus, if the action is invariant under conformal transformations of the metric, the classical energy-momentum tensor is traceless. Since conformal transformations are essentially a rescaling of lengths at each spacetime point x, the presence of a mass and therefore the existence of a fixed length scale in the theory will break the conformal invariance. On the other hand, the trace of the renormalized expectation value of the energy momentum tensor does not vanish in the conformal limit. In the semiclassical domain, the expectation value of $T^{\mu\nu}$ is given by

$$\langle T^{\mu\nu}\rangle=\frac{2}{\sqrt{-g(x)}}\frac{\delta\mathcal{S}_{\rm corr}}{\delta g_{\mu\nu}}, \tag{121}$$

where

$$\mathcal{S}_{\rm corr}=\int d^4x\sqrt{-g}\mathcal{L}_{\rm corr}.$$

Consider the change in $\mathcal{S}_{\rm corr}$ under an infinitesimal conformal transformation

$$g_{\mu\nu}\to\tilde{g}_{\mu\nu}=\Omega^2(x)g_{\mu\nu}, \tag{122}$$

with

$$\Omega^2(x)=1+\epsilon(x) \quad \text{and} \quad \delta g_{\mu\nu}=\epsilon(x)g_{\mu\nu}.$$

Regarding $\mathcal{S}_{\rm corr}$ as a functional of $\Omega^2(x)$, with $g_{\mu\nu}$ being a given function, we obtain

$$\mathcal{S}_{\rm corr}[(1+\epsilon)g_{\mu\nu}] =\mathcal{S}_{\rm corr}[g_{\mu\nu}]+\int d^4x\frac{\delta\mathcal{S}_{\rm corr}[\Omega^2(x)g_{\mu\nu}]}{\delta\Omega^2(x)}\bigg|_{\Omega^2(x)=1}\epsilon(x). \tag{123}$$

Thus

$$\frac{\delta\mathcal{S}_{\rm corr}[g_{\mu\nu}]}{\delta g_{\mu\nu}}g_{\mu\nu}=\frac{\delta\mathcal{S}_{\rm corr}[\Omega^2(x)g_{\mu\nu}]}{\delta\Omega^2(x)}\bigg|_{\Omega^2(x)=1}. \tag{124}$$

Therefore

$$\langle T^{\mu}{}_{\mu}\rangle=\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}_{\rm corr}[g_{\mu\nu}]}{\delta g_{\mu\nu}}g_{\mu\nu} =\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}_{\rm corr}[\Omega^2(x)g_{\mu\nu}]}{\delta\Omega^2(x)}\bigg|_{\Omega^2(x)=1}. \tag{125}$$

Now, $\mathcal{S}_{\rm corr}$ is given by the formula

$$\mathcal{S}_{\rm corr}[g_{\mu\nu}]=-\frac{i}{2}\int d^4x\sqrt{-g}\int_0^{\infty}\frac{ds}{s}\langle x|e^{-is\hat{H}}|x\rangle. \tag{126}$$

It can be shown that under a conformal transformation [17],

$$H(x)\to\tilde{H}(x)=\Omega^{-3}(x)H(x)\Omega(x), \tag{127}$$

where $H(x)$ is given by Eq. (103) in the conformal limit with $\xi=1/6$ and $m=0$. The corresponding relation satisfied by the operator $\hat{H}$ under a conformal transformation is

$$\hat{\tilde{H}}=\Omega^{-1}\hat{H}\Omega^{-1} \tag{128}$$

while the trace operator ''Tr'' defined by the relation

$$\mathrm{Tr}\,(\hat{H})=\int d^4x\sqrt{-g}\langle x|\hat{H}|x\rangle \tag{129}$$

remains invariant [18].

Using the above results it is easy to show that

$$\mathrm{Tr}\,(e^{-is\hat{\tilde{H}}})=\mathrm{Tr}\,(e^{-is\Omega^{-1}\hat{H}\Omega^{-1}})=\mathrm{Tr}\,(e^{-is\Omega^{-2}\hat{H}}). \tag{130}$$

Using the above formula for $\mathcal{S}_{\rm corr}$, and the above results, we find that, under the infinitesimal transformation (122),

$$\mathcal{S}_{\rm corr}[\Omega^2g_{\mu\nu}]=-\frac{i}{2}\int d^4x\sqrt{-g}\int_0^{\infty}\frac{ds}{s}\langle x|e^{-is\Omega^{-2}\hat{H}}|x\rangle. \tag{131}$$

The above expression for $\mathcal{S}_{\rm corr}$ is clearly divergent near $s=0$ as shown in the previous section. Making a change of variable $s\to s'=s\Omega^{-2}$, it appears that

$$\mathcal{S}_{\rm corr}[\Omega^2g_{\mu\nu}]=-\frac{i}{2}\int d^4x\sqrt{-g}\int_0^{\infty}\frac{ds'}{s'}\langle x|e^{-is'\hat{H}}|x\rangle \equiv\mathcal{S}_{\rm corr}[g_{\mu\nu}]. \tag{132}$$

But such a change of variable is not valid since the integral is divergent. To make sense of such an integral, we can resort to various techniques to determine the trace anomaly. Following Refs. [17,18], using the ζ function approach, it can be shown that the trace of the energy-momentum tensor is equal to $[a_2(x,x)/(4\pi)^{-2}]$.

B. Results with the modified weightage factor

The expression for $\mathcal{S}_{\rm corr}[\Omega^2g_{\mu\nu}]$ with the modified weightage factor is given by

$$\mathcal{S}_{\rm corr}[\Omega^2g_{\mu\nu}]=-\frac{i}{2}\int d^4x\sqrt{-g}\int_0^{\infty}\frac{ds}{s}e^{iL_P^2/s}\langle x|e^{-is\Omega^{-2}\hat{H}}|x\rangle. \tag{133}$$

This expression has no divergences. Changing the variable $s\to s'=s\Omega^{-2}$ (which is valid now) we obtain

$$\mathcal{S}_{\rm corr}[\Omega^2 g_{\mu\nu}] = -\frac{i}{2}\int d^4x\sqrt{-g}\int_0^\infty \frac{ds'}{s'} e^{(iL_P^2/\Omega^2 s')} \times \langle x|e^{-is'\hat{H}}|x\rangle. \tag{134}$$

Using the formula for the trace of the energy momentum tensor (125), we find that

$$\langle T^\mu{}_\mu\rangle = -L_P^2\int_0^\infty \frac{ds}{s^2} e^{iL_P^2/s}\langle x|e^{-is\hat{H}}|x\rangle. \tag{135}$$

Using the formula for the propagator given in Eq. (104) in 4 dimensions and carrying out the integral over s, we get

$$\langle T^\mu{}_\mu\rangle = \frac{2L_P^2}{(4\pi)^2}\sum_{n=0}^\infty a_n\left(\frac{L_P}{m}\right)^{n-3} K_{(n-3)}(2L_P m) \tag{136}$$

where K_n is the usual modified Bessel function of order n. In the limit $L_P\to 0$, the expression above reduces to

$$\lim_{L_P\to 0}\langle T^\mu{}_\mu\rangle = \lim_{L_P\to 0}\frac{2}{(4\pi)^2}\left[\frac{1}{L_P^4}+\frac{a_1-m^2}{2L_P^2}\right] + \frac{1}{2(4\pi)^2}(2a_2-2m^2a_1+m^4). \tag{137}$$

The terms present in the square brackets represent the divergences that are present in the evaluation of the energy-momentum tensor without using the duality principle. These divergences need to be regularized by other methods like the ζ function approach mentioned earlier. The finite part that remains is the last term that, in the conformal limit, reduces to $[a_2(x,x)/(4\pi)^{-2}]$. Thus, we recover the standard result in the limit of $L_P\to 0$.

VIII. DISCUSSION

In this paper, we evaluated the quantum gravitational corrections to some of the standard quantum field theoretic results using the "principle of path integral duality." We find that the main feature of this duality principle is that it is able provide an ultra-violet cutoff at the Planck energy scales, thereby rendering the theory finite. Another key feature of this approach is that the prescription is completely Lorentz invariant. Hence we were able to obtain finite but Lorentz invariant results for otherwise divergent expressions.

The obvious drawbacks of the approach are the following:

(i) The prescription of path integral duality is essentially an *ad hoc* prescription. It is not backed by a theoretical framework which is capable of replacing the conventional quantum field theory at the present juncture. Hence, the prescription only tells us as to how we can modify the kernels and the associated Green's functions. To obtain any result with the prescription of path integral duality we have to first relate the result to the kernel or Green's function, modify the kernel and thereby obtain the final result.

In spite of this constraint we have been able to show in this paper that concrete computations can be done and specific results can be obtained. As regards the *ad hocness* of the prescription, it should be viewed as a first step in the approach to quantum gravity based on a general physical principle. Its relation to zero-point length and the emergence of analogous duality principles in string theories, for example, makes one hopeful that it can be eventually put on a firmer foundation.

(ii) The modified kernel (based on the principle of path integral duality) may not be obtainable from the standard framework of field theory based on unitarity, microscopic causality and locality. (We have no rigorous proof that this is the case; however, it is quite possible since standard field theories based on the above principles are usually divergent.)

It is not clear to us whether such principles will be respected in the fully quantum gravitational regime. It is very likely that the continuum field theory which we are accustomed to will be drastically modified at Planck scales. If that is the case, it is quite conceivable that the quantum gravitational corrections also leave a trace of the breakdown of continuum field theory even when expressed in such a familiar language. As an example, consider an attempt to study and interpret quantum mechanics in terms of classical trajectories. Any formulation will lead to some contradictions like, for example, the breakdown of differentiability for the path. This arises because we are attempting to interpret physical principles using an inadequate formalism.

The next logical step will be to attempt to derive the path integral duality from a deeper physical principle using appropriate mathematical methods. This should throw more light on, for example, the connection between path integral duality and zero-point length which at the moment remains a mystery. We hope to address it in a future publication.

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