Sources for the Majumdar-Papapetrou space-times

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Einstein's field equations are solved exactly for static charged dust distributions. These solutions generalize the Majumdar-Papapetrou metrics. Maxwell's equations lead to the equality of charge and mass densities of the dust distrubition. Einstein's equations reduce to a nonlinear version of Poisson's equation. [S0556-2821(98)07814-X]

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INTRODUCTION

Majumdar-Papapetrou (MP) metrics [1,2] describe the exterior gravitational field of static objects (discrete point sources, line charges, point dipoles, charged dusts, etc.). The line element of the corresponding metric is given by

$$ds^2=-f dt^2+f^{-1}\delta_{ij}dx^i dx^j, \tag{1}$$

where f is a function of x^i and the electromagnetic potential four vector is given by $A_u=(A_0,0,0,0)$. Letting $f=1/\lambda^2$ and $A_0=k/\lambda$ with $k^2=1$ one can then reduce the electrovacuum field equations to the Laplace equation $\nabla^2\lambda=0$. The solution of the Laplace equation corresponding to discrete point sources is given by

$$\lambda=1+\sum_{i=1}^{N}\frac{m_i}{r_i}, \tag{2}$$

$$r_i=[(x-x_i)^2+(y-y_i)^2+(z-z_i)^2]^{1/2}. \tag{3}$$

In this solution the charge and mass of each point source are equal, $e_i=m_i$. These metrics describe multiple extreme black-hole solutions of Einstein's theory of gravity in a conformostatic space-time [3,4]. They are also exact solutions of the low energy limit of string theory [5] which respect supersymmetry [6,7]. There are also other possible solutions of the Laplace equation but the space-time geometries corresponding to these solutions have naked singularities [3]. To this end it is important to find interior solutions to remove such singularities.

An extension of the MP metrics has been studied a long time ago by Das [8]. He considered the charged dust solutions of the static Einstein field equations. His interest was to find the conditions for an extreme case. He showed (in his notation) that the g_{44} component of the metric and A_4 component of the vector potential are related, $g_{44}=a+bA_4+4\pi A_4^2$, in order for the charge ρ_e and mass ρ densities to be equal, $\rho_e=\pm\rho$. He did not investigate the rest of the field equations.

CHARGED DUST CLOUDS

In this work we consider the same problem as Das, a generalization of the MP metrics to the case of a continuous charged dust distribution (a charged perfect fluid with zero pressure). The energy momentum tensor is the sum of the energy momentum tensors of the Maxwell field and a dust distribution. We investigate the complete Einstein field equations for charged dust distributions in a conformostatic space-time. We first show that Einstein field equations reduce to a nonlinear potential equation. We then show that the charge and mass densities are directly related, $\rho_e=\pm\rho$. This relation is not an assumption but arises as a result of the field equations.

Let M be a four dimensional spacetime with the line element (1). Here Latin letters represent the space indices and δ_{ij} is the three dimensional Kronecker delta. In this work we shall use the same convention as in [9]. The only difference is that we use Greek letters for four dimensional indices. Here M is static. It is useful to write the metric tensor in a more elegant form. It is given by

$$g_{\mu\nu}=f^{-1}\eta_{1\mu\nu}-u_\mu u_\nu, \tag{4}$$

where $\eta_{1\mu\nu}=\mathrm{diag}(0,1,1,1)$ and $u_\mu=\sqrt{f}\,\delta_\mu^0$. The inverse metric is given by

$$g^{\mu\nu}=f\eta_2^{\mu\nu}-u^\mu u^\nu, \tag{5}$$

where $\eta_{2\mu\nu}=\mathrm{diag}\,(0,1,1,1)$ and $u^\mu=g^{\mu\nu}u_\nu=-(1/\sqrt{f})\,\delta_0^\mu$. Here u^μ is a timelike four vector, $u^\mu u_\mu=-1$.

The Ricci scalar and Einstein tensor components corresponding to the conformostatic metric (1) are given by

$$R=\frac{1}{2f}(2f\nabla^2 f-3(\nabla f)^2), \tag{6}$$

$$G_{00}=f\nabla^2 f-\frac{5}{4}(\nabla f)^2,\quad G_{0i}=0, \tag{7}$$

$$G_{ij}=-\frac{1}{2f^2}\partial_i f\partial_j f+\frac{(\nabla f)^2}{4f^2}\delta_{ij}, \tag{8}$$

where ∇^2 denotes the three dimensional Laplace operator in Cartesian flat coordinates. The Maxwell antisymmetric tensor and the corresponding energy momentum tensor are respectively given by

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$$F_{\mu\nu}=\nabla_{\nu}A_{\mu}-\nabla_{\mu}A_{\nu}, \tag{9}$$

$$M_{\mu\nu}=\frac{1}{4\pi}\left(F_{\mu\alpha}F^{\alpha}_{\nu}-\frac{1}{4}F^2 g_{\mu\nu}\right), \tag{10}$$

where $F^2=F^{\mu\nu}F_{\mu\nu}$. The current vector j^{μ} is defined as

$$\nabla_{\nu}F^{\mu\nu}=4\pi j^{\mu}. \tag{11}$$

Here we should assume $A_i=0$. Then

$$M_{00}=\frac{f}{8\pi}(\nabla A_0)^2,$$

$$M_{0i}=0, \quad M_{ij}=-\frac{1}{4\pi f}\partial_i A_0 \partial_j A_0+\frac{1}{8\pi f}(\nabla A_0)^2\delta_{ij}, \tag{12}$$

and

$$4\pi j^0=f\nabla(f^{-1}\nabla A_0), \quad j^i=0. \tag{13}$$

The Einstein field equations for a charged dust distribution are given by

$$G_{\mu\nu}=8\pi T_{\mu\nu}=8\pi M_{\mu\nu}+(8\pi\rho)u_{\mu}u_{\nu}, \tag{14}$$

where ρ is the energy density of the charged dust distribution and the four velocity of the dust is the same vector u^{μ} appearing in the metric tensor. We find that $j^{\mu}=-\rho_e u^{\mu}$, where

$$\rho_e=\frac{1}{4\pi}f^{3/2}\nabla(f^{-1}\nabla A_0) \tag{15}$$

is the charge density of the dust distribution. Let λ be a real function depending on the space coordinates x^i. Einstein's equation $G_{ij}=8\pi T_{ij}$ forces us to choose $A_0=k\sqrt{f}$ or

$$f=\frac{1}{\lambda^2}, \quad A_0=\frac{k}{\lambda}, \tag{16}$$

where $k=\pm 1$. Then the remaining field equation $G_{00}=8\pi T_{00}$ reduces simply to the following equations:

$$\nabla^2\lambda+4\pi\rho\lambda^3=0, \tag{17}$$

$$\rho_e=k\rho. \tag{18}$$

These equations represent the Einstein and Maxwell equations, respectively. In particular the first equation (17) is a generalization of the Poisson's potential equation in Newtonian gravity. In the Newtonian approximation $\lambda=1+V$, Eq. (17) reduces to the Poisson equation, $\nabla^2 V+4\pi\rho=0$. Hence for any physical mass density ρ of the dust distribution we solve the equation (17) to find the function λ. This determines the space-time metric completely. As an example for a constant mass density $\rho=\rho_0>0$ we find that

$$\lambda=\frac{a}{2\sqrt{\pi\rho_0}}cn(l_i x^i). \tag{19}$$

Here l_i is a constant three vector, $a^2=l_i l^i$ and cn is one of the Jacobi elliptic functions with modulus square equals $\frac{1}{2}$. This is a model universe which is filled by an (extreme) charged dust with a constant mass density.

INTERIOR SOLUTIONS [10]

In an asymptotically flat space-time, the function λ asymptotically obeys the boundary condition $\lambda\to 1$. In this case we can establish the equality of mass and charge $e=\pm m_0$, where $m_0=\int\rho\sqrt{-g}d^3x$. For physical considerations our extended MP space-times may be divided into inner and outer regions. The inner and outer regions are defined as the regions where $\rho_i>0$ and $\rho=0$, respectively. Here $i=1,2,\ldots,N$, where N represents the number of regions. The gravitational fields of the outer regions are described by any solution of the Laplace equation $\nabla^2\lambda=0$, for instance by the MP metrics. As an example the extreme Reissner-Nordström (RN) metric (for $r>R_0$), $\lambda=1+m_0/r$, may be matched to a metric with

$$\lambda=a\frac{\sin(br)}{r}, \quad r<R_0 \tag{20}$$

describing the gravitational field of an inner region filled by a spherically symmetric charged dust distribution with a mass density

$$\rho=\rho(0)\left[\frac{br}{\sin(br)}\right]^2. \tag{21}$$

Here $\rho(0)=1/4\pi a^2$, $r^2=x_i x^i$, and a and b are constants to be determined in terms of the radius R_0 of the boundary and total mass m_0 [or in terms of $\rho(0)$]. They are given by

$$bR_0=\sqrt{\frac{3m_0}{R_0}}, \quad a\sin bR_0=R_0+m_0. \tag{22}$$

In this way one may eliminate the singularities of the outer solutions by matching them to an inner solution with a physical mass density.

We can extend the above solution to N charged dust spheres by letting the mass density $\rho=b^2/4\pi\lambda^2$ where we have the complete solution of Eq. (17),

$$\lambda=\sum_{l,m}a_{l,m}j_l(br)Y_{l,m}(\theta,\phi), \tag{23}$$

where $j_l(br)$ are the spherical Bessel functions which are given by

$$j_l(x)=(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{\sin x}{x}\right) \tag{24}$$

and $Y_{l,m}$ are the spherical harmonics. The constants $a_{l,m}$ are determined when this solution is matched to an outer solution with $\nabla^2\lambda=0$. The interior solution given above for the extreme RN metric $(N=1)$ with density (21) correspond to $l=0$.

CONCLUDING REMARKS

In this work we have solved the Einstein field equations in a conformostatic space-time for a charged dust distribution. We reduced the problem to a nonlinear Poisson type of potential equation (17). Any physically reasonable solution of this equation gives an interior solution to an exterior MP metric with naked singularities. In this way we remove the naked singularities of the MP metrics by matching them to the metrics of extremely charged dust distributions. We have given some explicit exact solutions corresponding to some specific mass densities. In particular we have given an interior solution of the extreme Reissner-Nordström (RN) metric.

In reality such objects may or may not exist. If they exist, such charged dust clouds cannot be detected directly, because they may transmit light rays. On the other hand, they may be observed through the bending of light rays passing very close to them. The measure of deflection angles will be of the same order of magnitude of the deflection angle of the null geodesics in the Schwarzschild geometry with the same total mass m_0. The reason for this becomes obvious when we write the RN metric in the Schwarzschild coordinates

$$ds^2 = -\left(1-\frac{m_0}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1-\frac{m_0}{r}\right)^2} + r^2 d\Omega^2. \qquad (25)$$

For a charged dust cloud with $m_0/R_0<1$ and $r>R_0$ we have $(1-m_0/r)^2 \to (1-2m_0/r)$. Thus in the neighborhood of the charged dust cloud, the exterior metric becomes closer to the Schwarzschild metric. We then expect the same effects as in the Schwarzschild geometry on the test particles in the neighborhood of extremely charged dust clouds. The measure of the deflection angle may be much larger if these light rays are also transmitted through the dust clouds.

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[1] S. D. Majumdar, Phys. Rev. **72**, 390 (1947).
[2] A. Papapetrou, Proc. R. Soc. London **A51**, 191 (1947).
[3] J. B. Hartle and S. W. Hawking, Commun. Math. Phys. **26**, 87 (1972).
[4] D. Kramer, H. Stephani, M. McCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).
[5] E. Bergshoeff, R. Kallosh, and T. Ortin, Nucl. Phys. **B478**, 156 (1996).
[6] G.W. Gibbons and C.M. Hull, Phys. Lett. **109B**, 190 (1982).
[7] K. P. Tod, Phys. Lett. **121B**, 241 (1983).
[8] A. Das, Proc. R. Soc. London **A267**, 1 (1962).
[9] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge, England, 1973).
[10] For more new solutions, see, M. Gürses, *Extremely Charged Static Dust Distributions in General Relativity*, in the Proceedings of the International Seminar on Mathematical Cosmology, Potsdam, Germany, 1998, edited by H.-J. Schmidt and M. Rainer (World Scientific, Singapore, to be published).