Evolution of the scale factor with a variable cosmological term

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The evolution of the scale factor $a(t)$ in Friedmann models (those with zero pressure and a constant cosmological term Λ) is well understood, and elegantly summarized in the review of Felten and Isaacman [Rev. Mod. Phys. **58**, 689 (1986)]. Developments in particle physics and inflationary theory, however, increasingly indicate that Λ ought to be treated as a dynamical quantity. We reexamine the evolution of the scale factor with a variable Λ term, and also generalize the treatment to include nonzero pressure. New solutions are obtained and evaluated using a variety of observational criteria. Existing arguments for the inevitability of a big bang (i.e., an initial state with $a=0$) are substantially weakened, and can be evaded in some cases with Λ_0 (the present value of Λ) well below current experimental limits. [S0556-2821(98)02416-3]

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I. INTRODUCTION

The behavior of the cosmological scale factor $a(t)$ in solutions of Einstein's field equations with the Robertson-Walker line element has been the subject of numerous studies. Textbook presentations tend to focus on models in which pressure p is zero and there is no cosmological term ($\Lambda=0$). Some treatments include the *Friedmann models*, in which $p=0$ but $\Lambda\neq0$ [1–5]. Much less attention has been directed at the more general *Lemaître models*, in which pressure p is given in terms of density ρ by an equation of state $p=p(\rho)$, and $\Lambda\neq0$ [6–9].

The possibility of a nonzero Λ term, in particular, has resurfaced lately in connection with the age problem [10]. If Λ is large enough, in fact, the age of the Universe (defined in the standard model [11] as the time elapsed since $a=0$) can in principle become *infinite*, as $a(t)$ never drops below a nonzero minimum value a_* in the past direction. The existence of such "big bangless" oscillating models has been recognized for over sixty years [12]. They have however been dismissed as unphysical on the grounds that the required values of Λ are incompatible with observation [13–16].

We will return to the observational constraints later on, merely noting here that the above arguments, along with nearly all existing astrophysical work on the cosmological term, operate on the assumption that $\Lambda=$ const. Quantum field theorists and others, by contrast, have been treating the cosmological term as a *dynamical* quantity for thirty years [17–22]. Anything which contributes to the energy density ρ_v of the vacuum behaves like a cosmological term via $\Lambda_v = 8\pi G\rho_v$. Many potential sources of fluctuating vacuum energy have now been identified, including scalar fields [23,24], tensor fields [25,26], nonlocal effects [27], wormholes [28], inflationary mechanisms [29] and even cosmological perturbations [30]. Each of these has been claimed to give rise to a negative energy density which grows with time, tending to cancel out any pre-existing positive cosmological term and drive the net value of Λ toward zero. Processes of this kind are among the most promising ways [26] to resolve the longstanding cosmological "constant" problem (see [31] for review).

The purpose of the present paper is to re-examine the evolution of the scale factor when $\Lambda\neq$ const. This has not yet been done in a systematic way. We also expand on most earlier treatments by considering a fairly general equation of state for ordinary matter rather than restricting ourselves to the pressure-free Friedmann models. These two generalizations lead to qualitatively as well as quantitatively new behavior for $a(t)$, and hence for related phenomena such as the age of the Universe, the appearance of gravitational lenses, and the redshifts of distant astronomical objects. They can also allow one to circumvent the abovementioned arguments against oscillating models.

The remainder of the paper is organized as follows: the required assumptions, definitions and dynamical equations are assembled in Sec. II, and applied in Secs. III–VI to models in which Λ varies as a function of time t, the scale factor a, the Hubble parameter H and the deceleration parameter q respectively. In each section we obtain analytic or numerical solutions for the scale factor, paying particular attention to the question of the initial singularity, and discuss observational constraints where appropriate. Conclusions are summarized in Sec. VII.

II. VARIABLE-Λ COSMOLOGY

A. The cosmological "constant"

To begin, we recall why the cosmological term has often been treated as a constant of nature. The Einstein field equations read

$$G_{\mu\nu}+\Lambda g_{\mu\nu}=8\pi GT_{\mu\nu}, \qquad (2.1)$$

where $G_{\mu\nu}\equiv R_{\mu\nu}-Rg_{\mu\nu}/2$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor of matter (we assume units such that $c=1$). Taking the covariant divergence of Eq. (2.1), recalling that the vanishing covariant divergence of the

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Einstein tensor is guaranteed by the Bianchi identities, and assuming that the energy-momentum tensor satisfies the conservation law $\nabla^\nu T_{\mu\nu}=0$, it follows that the covariant divergence of $\Lambda g_{\mu\nu}$ must vanish also, and hence that $\Lambda=$const. This argument, which situates Λ firmly on the left-hand side of the field equations, constitutes a ''geometrical interpretation'' of the cosmological term.

More recently, it has become increasingly common (see, e.g., [32]) to move the cosmological term to the right-hand side of Eq. (2.1):

$$G_{\mu\nu}=8\pi G\tilde{T}_{\mu\nu}, \quad \tilde{T}_{\mu\nu}\equiv T_{\mu\nu}-\frac{\Lambda}{8\pi G}g_{\mu\nu}; \quad (2.2)$$

that is, to interpret Λ as part of the matter content of the universe, rather than a purely geometrical entity. Once this is done, there are no *a priori* reasons why Λ should not vary—as long as the *effective* energy-momentum tensor $\tilde{T}_{\mu\nu}$ satisfies energy conservation [33]:

$$\nabla^\nu\tilde{T}_{\mu\nu}=0. \quad (2.3)$$

B. Dynamical equations

We will make the usual assumptions: a homogeneous and isotropic Universe (i.e., Robertson-Walker line element) and perfect-fluid-like ordinary matter [35] with pressure p and energy density ρ. The definition (2.2) of $\tilde{T}_{\mu\nu}$ then implies that the *effective* energy-momentum tensor also has the perfect fluid form, with effective pressure $\tilde{p}\equiv p-\Lambda/8\pi G$ and energy density $\tilde{\rho}\equiv\rho+\Lambda/8\pi G$ [11]. The field equations (2.2) and law of energy-momentum conservation (2.3) then read

$$\dot{a}^2=\frac{8\pi G}{3}\rho a^2+\frac{\Lambda}{3}a^2-k \quad (2.4)$$

$$\frac{d}{da}\left[\left(\rho+\frac{\Lambda}{8\pi G}\right)a^3\right]=-3\left(p-\frac{\Lambda}{8\pi G}\right)a^2. \quad (2.5)$$

For the equation of state we take

$$p=(\gamma-1)\rho, \quad (2.6)$$

with $\gamma=$const. Previous analytic and numerical studies of the evolution of the scale factor have tended to focus on the dust-like case $\gamma=1$ [2–5], and occasionally also the radiation-like case $\gamma=4/3$ [6–9,36]. We will not restrict ourselves to these values, as many other possibilities have been considered in the literature [37].

Substituting Eq. (2.6) into Eq. (2.5), we find

$$\frac{d}{da}(\rho a^{3\gamma})=-\frac{a^{3\gamma}}{8\pi G}\frac{d\Lambda}{da}. \quad (2.7)$$

When $\Lambda=$const, Eq. (2.7) reverts to the well-known result that density ρ scales as a^{-3} in a pressure-free universe ($\gamma=1$) and a^{-4} in a radiation-dominated universe ($\gamma=4/3$).

Differentiating Eq. (2.4) with respect to time and inserting Eq. (2.7), we obtain

$$\ddot{a}=\frac{8\pi G}{3}\left(1-\frac{3\gamma}{2}\right)\rho a+\frac{\Lambda}{3}a. \quad (2.8)$$

Equation (2.8) shows that a positive density ρ acts to decelerate the expansion, as expected — but only when $\gamma>2/3$. If, on the other hand, the cosmological fluid is such that $\gamma<2/3$, then its density can actually accelerate the expansion, $\ddot{a}>0$ (this is the phenomenon commonly known as *inflation*). Equation (2.8) also confirms that a positive cosmological term contributes positively to acceleration, ''propping up'' the scale factor against the deceleration caused by the matter term ρ. A negative Λ term, on the other hand, acts in the opposite direction and brings about recollapse more quickly.

Combining Eqs. (2.4) and (2.8), we can eliminate ρ to obtain a differential equation for the scale factor in terms of the cosmological term alone:

$$\frac{\ddot{a}}{a}=\left(1-\frac{3\gamma}{2}\right)\left(\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)+\frac{\gamma}{2}\Lambda. \quad (2.9)$$

This differential equation governs the behavior of the scale factor in the presence of a cosmological term Λ, whether or not the latter is constant.

C. Phenomenological Λ-decay laws

The above mentioned sources of negative vacuum energy [23–30] do not, in general, lend themselves to simple expressions for Λ in terms of t,a,H or q. There are some exceptions, including scalar field-based [19,20,34,68] and other theories [42,53,72–74] in which analytic decay laws are derived from modified versions of the Einstein action. In most such papers, however, no exact solution for Λ is obtained; the intent is merely to demonstrate that decay (and preferably near-cancellation) of the effective cosmological term is possible in principle.

In a complementary approach, a number of authors have constructed models of a more phenomenological character, in which specific decay laws are postulated for Λ within general relativity. These theories are incomplete to the extent that they do not include explicit physical mechanisms to govern the exchange of energy between the shrinking cosmological term and other forms of matter [77]. In some models this issue is not addressed at all; in others the energy is assumed to be channelled into production of baryonic matter and/or radiation. The former case can be constrained by observations of the diffuse gamma-ray background, since (assuming the decay process does not violate baryon number) one might expect equal amounts of matter and antimatter to be formed [47,59]. The latter can be constrained by nucleosynthesis arguments [47,49], cosmic microwave background (CMB) anisotropies [47,54,56], the absolute CMB intensity [50,59], and thermodynamical considerations [52,66,76].

We take the point of view here that simple Λ-decay scenarios are worth examining, irrespective of whether they

TABLE I. Examples of phenomenological Λ-decay laws.

Decay law[a]	Reference
$\Lambda \propto t^{-2}$	[19,20,34,38–41]
$\Lambda \propto T^4$	[20]
$\Lambda \propto T^\beta$	[21]
$\Lambda \propto e^{-\beta a}$	[42]
$d\Lambda/dt \propto \Lambda^\beta$	[43]
$\Lambda \propto a^{-2}$	[41,44–46]
$\Lambda \propto a^{-4(1+\epsilon)}$	[47–50]
$\Lambda \propto a^{-m}$	[51–56]
$d\Lambda/dt \propto a H^n \Lambda$	[57]
$d\Lambda/dt \propto H^3$	[57]
$\Lambda \propto C + \beta a^{-m}$	[58,59]
$\Lambda \propto t^{l-2} + \beta t^{2(l-1)}$	[60]
$\Lambda \propto \beta a^{-2} + H^2$	[61,62]
$\Lambda \propto t^{-2} + \beta t^{-2/l}$	[63]
$\Lambda \propto C + e^{-\beta t}$	[63,64]
$\Lambda \propto C + \beta a^{-2} + H^2$	[65]
$\Lambda \propto \beta a^{-m} + H^2$	[66]
$\Lambda \propto H^2$	[67–69]
$\Lambda \propto (1 + \beta H)(H^2 + k/a^2)$	[70]
$\Lambda \propto t^{-1}(\beta + t)^{-1}$	[71]
$d\Lambda/dt \propto \beta \Lambda - \Lambda^2$	[72]
$\Lambda \propto a^{-3}$	[73]
$\Lambda \propto a^{-2} + \beta a^{-4}$	[74]
$\Lambda \propto H^2 + \beta a H(dH/da)$	[75]

[a]T,a,t,H are the temperature, scale factor, time and Hubble parameter respectively; while β, ϵ, l, m and C are constants.

come from extended theories of gravity or phenomenological considerations, for at least four reasons: (1) they have been shown to address a number of pressing problems in cosmology [38–77]; (2) many are independently motivated, e.g., by dimensional arguments, or as limiting cases of more complicated theories; (3) most are simple enough that meaningful conclusions can be drawn about their viability; and (4) successful implementation would point toward the eventual Lagrangian formulation of a more complete theory. For convenience, we have collected together the most common decay laws from the literature and listed them in Table I (by chronological order of appearance).

In the remainder of the paper, we focus on power-law functions of one parameter. Our discussion is however more general than most of those noted in Table I, because we do not fix values of the exponents *a priori*. In particular, we consider decay laws of the following four kinds:

$$\Lambda = \mathcal{A} t^{-l} \tag{2.10}$$

$$\Lambda = \mathcal{B} a^{-m} \tag{2.11}$$

$$\Lambda = \mathcal{C} H^n \tag{2.12}$$

$$\Lambda = \mathcal{D} q^r, \tag{2.13}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, l, m, n$ and r serve as adjustable constants. These four variable-Λ scenarios and their cosmological consequences are explored in Sec. III, Sec. IV, Sec. V and Sec. VI respectively.

D. Definitions

We conclude Sec. II by introducing the terms and definitions which will be needed to connect our solutions to observation. Chief among these are the energy densities of ordinary matter and the cosmological term, expressed in units of the critical density:

$$\Omega \equiv \frac{\rho}{\rho_{crit}}, \quad \lambda \equiv \frac{\Lambda}{3H_0^2}, \quad \rho_{crit} \equiv \frac{3H_0^2}{8\pi G}. \tag{2.14}$$

We will be particularly interested in the values of these parameters at the present time (subscript ''0''):

$$\Omega_0 = \frac{8\pi G \rho_0}{3H_0^2}, \quad \lambda_0 = \frac{\Lambda_0}{3H_0^2}. \tag{2.15}$$

These will constitute our primary free parameters throughout the following sections.

The usefulness of the quantity λ_0 (sometimes denoted Ω_Λ in the literature) is highlighted by using the definitions (2.14) to rewrite the Lemaître Eq. (2.4) in the form $H^2 = H_0^2(\Omega + \lambda) - k/a^2$. At the present time this implies

$$k = a_0^2 H_0^2 (\Omega_0 + \lambda_0 - 1). \tag{2.16}$$

From Eq. (2.16) it is clear that

$$\Omega_0 + \lambda_0 > 1 \Rightarrow k > 0,$$

$$\Omega_0 + \lambda_0 = 1 \Rightarrow k = 0,$$

$$\Omega_0 + \lambda_0 < 1 \Rightarrow k < 0. \tag{2.17}$$

Most cosmologists implicitly choose units for a_0 such that the value of k is normalized to either 0 or ± 1. We will follow Felten and Isaacman [5] in refraining from this, because it is more convenient for our purposes to choose units such that $a_0 \equiv 1$.

In place of Ω and λ, some authors prefer to use the quantities σ and q, defined by

$$\sigma \equiv \frac{4\pi G \rho}{3H^2}, \quad q \equiv -\frac{\ddot{a}a}{\dot{a}^2}. \tag{2.18}$$

The present values of these parameters are fixed with the help of Eqs. (2.8) and (2.14):

$$\sigma_0 = \Omega_0/2, \quad q_0 = (3\gamma/2 - 1)\Omega_0 - \lambda_0. \tag{2.19}$$

Negative values of q_0 can be obtained for large (positive) λ_0, or small γ. Equation (2.19) yields the standard expression [5] for q_0 when $\gamma = 1$.

III. Λ AS A FUNCTION OF TIME

A. Interpretation of the time co-ordinate

We begin with the oldest, and probably the most straightforward implementation of the variable-Λ idea; namely, one in which the cosmological term is a simple power-law function of time, as set out in Eq. (2.10):

$$\Lambda = \mathcal{A}\tau^{-l}, \qquad (3.1)$$

where, for later convenience, we measure time in units of Hubble times ($\tau \equiv H_0 t$). The case $l=2$ has previously been considered by several authors. Each, however, has imposed supplementary conditions. Endō, Fukui and others [19,34] operate in the context of a modified Brans-Dicke theory. Canuto *et al.* [20] assume invariance under changes of scale, while Lau [38] adopts the Dirac large-number hypothesis (with a time-varying gravitational ''constant'' G) from the outset. Berman [39] requires that the density ρ of ordinary matter *also* scale as τ^{-2}, and that the deceleration parameter q be a constant. Beesham [40] restricts his treatment to Bianchi type I models with variable G. Lopez and Nanopoulos [41] take Λ to have the same dependence on the *scale factor* ($\Lambda \propto a^{-2}$) as on time, for late times at least. (These latter authors also make the important claim that a Λ-decay ansatz of this kind could follow from certain versions of string theory.) The question of the initial singularity is not addressed in any of these papers. In this paper, we examine the properties of models with the form (3.1) in a more comprehensive way.

Several conceptual issues should be dealt with before we proceed. First, since we are interested in oscillating as well as traditional ''big bang'' models, we are obliged to broaden the conventional definition of cosmic time [78], in which it is set to zero at the moment when $a=0$. In those cases where the scale factor never vanishes, we choose here to measure τ instead from the most recent moment when $da/d\tau=0$ (the ''big bounce''). In either case we refer to this as the ''initial moment'' and denote it by $\tau = \tau_* \equiv 0$. (Here and elsewhere, the subscript ''*'' will indicate quantities taken at this time.)

Secondly, Eq. (3.1) implies that $\Lambda \to \infty$ in the limit $\tau \to 0$, which may not be realistic. Λ may go to infinity at some other time (τ_∞, say) than the initial moment. Or, more likely, its divergent behavior may be cut off at some critical temperature (at time τ_c, say) by a phase transition or similar mechanism, above which Λ is effectively constant [21]. So a more plausible formulation of Eq. (3.1) might take the form

$$\Lambda = \begin{cases} \Lambda_c & \text{when } \tau < \tau_c, \\ \mathcal{A}(\tau - \tau_\infty)^{-l} & \text{when } \tau \geq \tau_c, \end{cases} \qquad (3.2)$$

where continuity across the cutoff time τ_c implies that $\Lambda_c \equiv \mathcal{A}(\tau_c - \tau_\infty)^{-l}$. However, the fact that the dynamical equations of Sec. II contain no explicit time dependence means that we can shift our time co-ordinate via $(\tau - \tau_\infty) \to \tau$ with impunity. The decay law (3.2) then reverts to the form (3.1), for all $\tau \geq \tau_c$ at least. In practice we will assume that Eq. (3.1) holds for all times of interest. For earlier times the

standard (Lemaître) solution applies, with $\Lambda = \Lambda_c = \text{const}$. If $\tau_c \approx \tau_\infty$, then Eq. (3.1) holds all the way to $\tau \approx 0$.

B. Riccati's equation

We are now in a position to study the evolution of the scale factor. Switching dependent variables from a to H:

$$H \equiv \frac{\dot{a}}{a} = H_0\left(\frac{da}{a\,d\tau}\right), \qquad (3.3)$$

we obtain for the differential equation (2.9):

$$\frac{dH}{d\tau} = \left(\frac{-3\gamma}{2H_0}\right)H^2 + \left(\frac{\gamma}{2H_0}\right)\Lambda + \left(1 - \frac{3\gamma}{2}\right)\frac{k}{H_0 a^2}. \quad (3.4)$$

If we restrict ourselves to spatially flat universes ($k=0$), then the last term drops off, leaving a special case of Riccati's equation:

$$\frac{dH}{d\tau} = \mathcal{P}(\tau)H^2 + \mathcal{Q}(\tau)H + \mathcal{R}(\tau), \qquad (3.5)$$

where $\mathcal{P} \equiv -3\gamma/2H_0$, $\mathcal{Q} \equiv 0$ and $\mathcal{R}(\tau) \equiv (\gamma/2H_0)\Lambda(\tau)$. We will adopt this restriction for the remainder of Sec. III. This has also been imposed in most of the special cases studied so far [19,38,39,41].

Solving Riccati's equation by standard methods [79], we change dependent variables from H to x via:

$$H \equiv -\frac{1}{\mathcal{P}x}\frac{dx}{d\tau} = \left(\frac{2H_0}{3\gamma}\right)\frac{dx}{x\,d\tau}, \qquad (3.6)$$

whereupon Eq. (3.4) takes the form [putting $k=0$ and inserting Eq. (3.1) for $\Lambda(\tau)$]:

$$\tau^l \frac{d^2x}{d\tau^2} - \alpha x = 0, \qquad (3.7)$$

with

$$\alpha \equiv \frac{3\gamma^2 \mathcal{A}}{4H_0^2}. \qquad (3.8)$$

This is now linear, as desired. We will solve for $x(\tau)$ in the cases $l=1,2,3,4$.

Once $x(\tau)$ is found, the Hubble parameter $H(\tau)$ follows immediately from the definition (3.6). Moreover, the scale factor $a(\tau)$ is also known, as may be verified by comparing Eqs. (3.3) and (3.6) to yield

$$a(\tau) = [x(\tau)]^{2/3\gamma}. \qquad (3.9)$$

The constant α given by Eq. (3.8) can be fixed in terms of observational quantities as follows. Applying the decay law (3.1) to the present epoch $\tau = \tau_0$, and using the definition (2.15) of λ_0, we find that $\mathcal{A} = 3H_0^2\lambda_0\tau_0^l$, which can be substituted into Eq. (3.8) to yield

$$\alpha = (3\gamma/2)^2 \lambda_0 \tau_0^l. \tag{3.10}$$

With the restriction $k=0$, Eq. (2.17) implies that Ω_0 is given by $(1-\lambda_0)$ throughout Sec. III.

C. The case $l=1$

We now proceed to the first case. The differential equation (3.7) reads

$$\tau \frac{d^2x}{d\tau^2} - \alpha x = 0, \tag{3.11}$$

where α is given by Eq. (3.10):

$$\alpha = (3\gamma/2)^2 \lambda_0 \tau_0. \tag{3.12}$$

Following standard techniques [79], we switch independent variables from τ to $z \equiv 2\sqrt{-\alpha\tau}$, whereupon

$$z^2 \frac{d^2x}{dz^2} - z \frac{dx}{dz} + z^2 x = 0. \tag{3.13}$$

This is transformable to Bessel's equation, with general solution $x(z) = c_1 z J_1(z) + c_2 z Y_1(z)$, where $J_1(z)$ and $Y_1(z)$ are Bessel and Neumann functions of order one. Equation (3.9) then gives for the scale factor:

$$a(\tau) = \tau^{1/3\gamma}[c_1 J_1(z) + c_2 Y_1(z)]^{2/3\gamma}, \tag{3.14}$$

where we have absorbed a factor of $2\sqrt{-\alpha}$ into c_1, c_2. The Hubble parameter is found by putting $x(z)$ into Eq. (3.6):

$$H(\tau) = H_0 \sqrt{-\lambda_0 \left(\frac{\tau_0}{\tau} \right)} \left[\frac{c_1 J_0(z) + c_2 Y_0(z)}{c_1 J_1(z) + c_2 Y_1(z)} \right], \tag{3.15}$$

where $J_0(z)$ and $Y_0(z)$ are Bessel and Neumann functions of order zero. We note from the definition (3.12) that $z(\tau)$, and hence $a(\tau)$ and $H(\tau)$, can only be real (for positive times) if $\lambda_0 \le 0$, which would imply a negative cosmological constant. While this possibility has been considered in some contexts [73,80], we will see shortly that it leads in the present theory to unrealistically short ages for the Universe. Therefore the case $l=1$ is probably not realized in nature.

We can impose the following boundary conditions at the present epoch:

$$a(\tau_0) = a_0 \equiv 1, \quad H(\tau_0) = H_0. \tag{3.16}$$

Substituting these expressions into Eqs. (3.14) and (3.15), it is straightforward to solve for c_1 and c_2:

$$c_1 = \frac{\sqrt{-\lambda_0}Y_0(z_0) - Y_1(z_0)}{\sqrt{-\lambda_0}\tau_0[J_1(z_0)Y_0(z_0) - J_0(z_0)Y_1(z_0)]} \tag{3.17}$$

$$c_2 = \frac{-\sqrt{-\lambda_0}J_0(z_0) + J_1(z_0)}{\sqrt{-\lambda_0}\tau_0[J_1(z_0)Y_0(z_0) - J_0(z_0)Y_1(z_0)]}, \tag{3.18}$$

where:

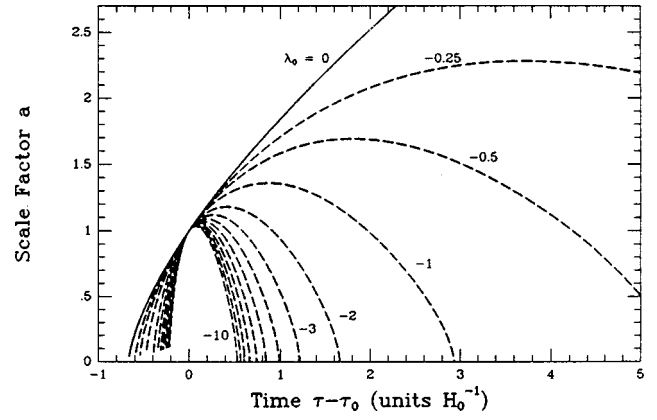

FIG. 1. Evolution of the scale factor for flat models with $\Lambda \propto \tau^{-1}$ and $\gamma=1$. Values of λ_0 are labelled beside each curve, and $\Omega_0 = 1-\lambda_0$ in each case.

$$z_0 \equiv 3\gamma\tau_0\sqrt{-\lambda_0}. \tag{3.19}$$

To keep $a(\tau)$ finite at $\tau=0$, we require $c_2 = 0$, since $Y_1(z)$ diverges logarithmically at $z=0$. This constitutes our third boundary condition. In conjunction with Eq. (3.18) it implies that $J_1(z_0) - \sqrt{-\lambda_0}J_0(z_0) = 0$. This equation may be solved numerically for τ_0 as a function of λ_0, with the help of the definition (3.19). The results can then be substituted back into Eq. (3.17) to fix the value of c_1. With c_1 and c_2 both known, $a(\tau)$ and $H(\tau)$ are given by Eqs. (3.14) and (3.15) respectively.

The evolution of the scale factor for this case is illustrated in Fig. 1 for various values of λ_0, assuming $\gamma=1$ (after Felten and Isaacman [5]). In particular, we have followed these authors in plotting the scale factor a as a function of $(\tau-\tau_0)$, rather than τ for each curve. This has the effect of shifting all the curves so that they intersect at $(0,1)$, which marks the present epoch. [Recall that we have chosen units such that $a_0 = 1$, Eq. (2.16).]

We have plotted for four Hubble times into the future, and one Hubble time into the past. It may be seen that the $\lambda_0 = 0$ curve (solid line) intersects the time axis at $(\tau-\tau_0) = -2/3$, which confirms the well-known rule that the age of a flat universe with no cosmological constant is $\tau_0 = 2/3$. The models with $\lambda_0 < 0$ (dashed lines) are younger than this, which considerably diminishes their attractiveness. The $\lambda_0 = -1$ model, for instance, has $\tau_0 = 0.48$, while the $\lambda_0 = -3$ model has $\tau_0 = 0.35$. If we use a current widely-accepted value of $H_0 = 73 \pm 10$ km s^{-1} Mpc^{-1} [81] for the Hubble parameter, then (recalling that $t = \tau/H_0$) we see that the age of the Universe in these models can be no more than 7.4 and 5.4 billion years old respectively. This conflicts badly with estimated globular cluster ages, which are thought to be at least 9.6 billion years old in some cases [82]. The situation improves slightly if one switches to the lower values for H_0 which are reported by some authors [83]. If $H_0 = 55 \pm 10$ km s^{-1} Mpc^{-1}, then the maximum possible age for these two models increases to 10.4 and 7.6 billion years respectively. One can safely rule out models with $\lambda_0 < -1$ on this basis, while models in the range $-1 < \lambda_0 \le 0$ remain marginally viable at best.

Figure 1 tells us that, while flat models with $\lambda_0 = 0$ will continue to expand indefinitely as usual, those with negative values of λ_0 will experience eventual recollapse. This can be understood by looking at Eq. (2.8), which shows that there are two contributions to the deceleration: one (which goes as $-\rho a$) due to ordinary matter and the other (which goes as $+\Lambda a$) due to the negative cosmological constant. Because the density ρ of matter rapidly thins out with expansion, the first contribution alone is not enough to close the Universe when $k=0$. The second contribution, however, is diluted much more slowly (Λ drops off as only τ^{-1} in this case), and is therefore sufficient to turn the expansion around eventually, no matter how small its value at the present time. Thus, models with $\lambda_0 = -1$ and $\lambda_0 = -3$ encounter the "big crunch" after only 2.94 and 1.21 Hubble times respectively.

D. The case $l = 2$

For this case, Eq. (3.7) takes the form:

$$\tau^2 \frac{d^2x}{d\tau^2} - \alpha x = 0, \tag{3.20}$$

with α given by Eq. (3.10) as follows:

$$\alpha = (3\gamma/2)^2 \lambda_0 \tau_0^2. \tag{3.21}$$

This is a special case of Euler's differential equation. Applying standard methods [79], we switch independent variables via $y \equiv \ln\tau$ to recast Eq. (3.20) in the form:

$$\frac{d^2x}{dy^2} - \frac{dx}{dy} - \alpha x = 0. \tag{3.22}$$

This now has constant coefficients, as desired. There are three subcases, according as λ_0 is greater than, equal to, or less than $-1/(3\gamma\tau_0)^2$.

1. The subcase $\lambda_0 > -1/(3\gamma\tau_0)^2$

Since we expect on observational grounds that λ_0 is probably positive, this is the most physical of the subcases. Solution of Eq. (3.22) for $x(y)$ and hence $x(\tau)$ is straightforward. The scale factor and Hubble parameter are found from Eqs. (3.9) and (3.6) to be:

$$a(\tau) = \tau^{1/3\gamma}(c_1\tau^{m_0} + c_2\tau^{-m_0})^{2/3\gamma} \tag{3.23}$$

$$H(\tau) = \frac{2H_0}{3\gamma}\left[\frac{m_1 c_1 \tau^{m_0} + m_2 c_2 \tau^{m_0}}{\tau(c_1\tau^{m_0} + c_2\tau^{-m_0})}\right], \tag{3.24}$$

where $m_0 \equiv \frac{1}{2}\sqrt{1 + (3\gamma\tau_0)^2\lambda_0}$. It is clear that $a(\tau)$ diverges at $\tau=0$ for $\lambda_0 > 0$ unless $c_2 = 0$, which we consequently adopt as a boundary condition (as in the $l=1$ case). Equations (3.23) and (3.24) simplify to

$$a(\tau) = (c_1\tau^{m_1})^{2/3\gamma} \tag{3.25}$$

$$H(\tau) = \frac{2H_0}{3\gamma}\left[\frac{m_1}{\tau}\right], \tag{3.26}$$

where $m_1 \equiv 1/2 + m_0$. We then apply the boundary conditions (3.16) at $\tau = \tau_0$. Equation (3.26) with $H(\tau_0) = H_0$ implies that:

$$\tau_0 = 2m_1/3\gamma. \tag{3.27}$$

Substituting this result into Eq. (3.25), with $a(\tau_0) = 1$, we find that $c_1 = (3\gamma/2m_1)^{m_1}$, which can be put back into Eq. (3.25) to yield the following expression for $a(\tau)$:

$$a(\tau) = \left(\frac{\tau}{\tau_0}\right)^{2m_1/3\gamma}. \tag{3.28}$$

The scale factor expands as a simple power-law function of time. This is consistent with previous special cases obtained for $l = 2$: Endō and Fukui's $a(\tau) \propto \tau^{2n/3(n-1)}$ [19], Berman and Som's $a(\tau) \propto \tau^{1/m}$ [34], Lau and Beesham's $a(\tau) \propto \tau^{1/3}$ [38,40], Berman's $a(\tau) \propto \tau^{2/3}$ [39], and Lopez and Nanopoulos' $a(\tau) \propto \tau$ [41].

In conjunction with the definitions of m_0 and m_1, Eq. (3.27) fixes the age of the Universe at:

$$\tau_0 = \frac{2}{3\gamma(1-\lambda_0)}, \tag{3.29}$$

from which we draw two conclusions: first that all models satisfying the boundary conditions obey $\lambda_0 < 1$; and secondly, that the age of the Universe $\tau_0 \to \infty$ as $\lambda_0 \to 1$. The initial singularity can thus be pushed back arbitrarily far into the past. We also find a lower limit on the age of the Universe in these models by noting that $m_1 = (m_0 + 1/2) > 1/2$. Inserting this into Eq. (3.27) produces the result:

$$\tau_0 > 1/3\gamma. \tag{3.30}$$

In other words, assuming zero pressure ($\gamma = 1$), all models in this case have survived for at least one-third of a Hubble time. Adopting a recent observational upper limit of 83 km s^{-1} Mpc^{-1} [81], this implies a minimum age of at least 3.9 billion years. Finally, putting Eq. (3.30) back into the expression (3.29) for τ_0, we find $\lambda_0 > -1$, which fixes the critical value of λ_0 separating this subcase from the other two.

2. The subcase $\lambda_0 = -1/(3\gamma\tau_0)^2$

Solution of Eq. (3.22) is also straightforward, and one finds from Eqs. (3.9) and (3.6) the following general expressions for $a(\tau)$ and $H(\tau)$:

$$a(\tau) = \tau^{1/3\gamma}(c_3 + c_4\ln\tau)^{2/3\gamma} \tag{3.31}$$

$$H(\tau) = \frac{2H_0}{3\gamma}\left[\frac{(c_3 + 2c_4) + c_4\ln\tau}{2\tau(c_3 + c_4\ln\tau)}\right], \tag{3.32}$$

where c_3, c_4 are arbitrary constants. To keep $a(\tau)$ finite at $\tau = 0$ we require $c_4 = 0$, so that:

$$a(\tau)=(c_3\sqrt{\tau})^{2/3\gamma} \tag{3.33}$$

$$H(\tau)=\frac{H_0}{3\gamma\tau}. \tag{3.34}$$

Inserting $H(\tau_0)=H_0$ into Eq. (3.34), we find for the age of the Universe in this model:

$$\tau_0=1/3\gamma, \tag{3.35}$$

which is exactly the limit $\lambda_0\to-1$ in Eq. (3.29). Equation (3.35) also corresponds to the lower limit allowed by Eq. (3.30), as one might expect.

Substituting the age (3.35) into Eq. (3.33), meanwhile, and imposing $a(\tau_0)=1$ as usual, we find that $c_3=\sqrt{3\gamma}$. Inserted back into Eq. (3.33), this yields:

$$a(\tau)=\left(\frac{\tau}{\tau_0}\right)^{1/3\gamma}, \tag{3.36}$$

which joins smoothly onto the result (3.28) from the previous subcase.

3. The subcase $\lambda_0<-1/(3\gamma\tau_0)^2$

Solution of Eq. (3.22) is also straightforward and leads to the following general expressions for $a(\tau)$ and $H(\tau)$:

$$a(\tau)=\tau^{1/3\gamma}[c_5\sin(m_3\ln\tau)+c_6\cos(m_3\ln\tau)]^{2/3\gamma} \tag{3.37}$$

$$H(\tau)=\frac{2H_0}{3\gamma}\left\{\frac{(c_5-2m_3c_6)\sin(m_3\ln\tau)+(c_6+2m_3c_5)\cos(m_3\ln\tau)}{2\tau[c_5\sin(m_3\ln\tau)+c_6\cos(m_3\ln\tau)]}\right\}, \tag{3.38}$$

where $m_3\equiv\frac{1}{2}\sqrt{-(3\gamma\tau_0)^2\lambda_0-1}$ and c_5,c_6 are arbitrary constants. Application of the boundary conditions (3.16) gives c_5 and c_6 in terms of τ_0:

$$c_5=\frac{1}{\sqrt{\tau_0}}\sin(m_3\ln\tau_0)+\frac{1}{m_3}\left[\left(\frac{3\gamma}{2}\right)\sqrt{\tau_0}-\frac{1}{2\sqrt{\tau_0}}\right]\cos(m_3\ln\tau_0) \tag{3.39}$$

$$c_6=\frac{1}{\sqrt{\tau_0}}\cos(m_3\ln\tau_0)-\frac{1}{m_3}\left[\left(\frac{3\gamma}{2}\right)\sqrt{\tau_0}-\frac{1}{2\sqrt{\tau_0}}\right]\sin(m_3\ln\tau_0). \tag{3.40}$$

As usual, we need a third boundary condition to fix the value of τ_0. Unlike the previous two subcases, we cannot keep $a(\tau)$ finite at $\tau=0$ by setting one of c_5,c_6 to zero. Instead we have adopted a numerical approach, searching iteratively for the value of τ_0 consistent with the boundary conditions [i.e., with the requirement that either $a(\tau)$ or $H(\tau)$ go smoothly to zero as $\tau\to0$]. The values of c_5 and c_6 then follow from Eqs. (3.40) and (3.40) respectively.

The evolution of the scale factor for this case is illustrated in Fig. 2, which has exactly the same format as Fig. 1, except that we have plotted for three Hubble times into the past instead of one. Figure 2 exhibits a richer variety of solutions than Fig. 1. The most noticeable difference is the existence of solutions for positive λ_0 (short-dashed lines). Of particular interest is the limiting case $\lambda_0=1$, which only approaches $a=0$ asymptotically as $\tau\to-\infty$. This case is not very realistic, however, as it has zero density (since $\Omega_0=1-\lambda_0$). It is in fact the empty de Sitter model. The same solution is found for these values of λ_0 and Ω_0 in conventional Lemaître cosmology with $\Lambda=$const [5].

Figure 2 therefore shows that we cannot avoid the big bang in a theory with $\Lambda\propto\tau^{-2}$ and $k=0$. We can, however, significantly extend the age of the Universe. Suppose we

choose $\lambda_0=0.5$, for example; a value compatible with the tightest observational bounds thus far [84]. Figure 2 shows that this model would have come into being 1.33 Hubble times ago [see also Eq. (3.29) above]. Even if we adopt the upper limit of 83 km s^{-1} Mpc^{-1} for H_0 [81], this translates into an age of 15.7 billion years — more than enough time for the oldest globular clusters to form [82]. By way of comparison, a *constant*-Λ model with $\lambda_0=\Omega_0=0.5$ has an age of only 0.83 Hubble times.

Flat models with no cosmological constant are represented in Fig. 2 by the curve labelled $\lambda_0=0$ (solid line). As in the $l=1$ case, these have an age of 2/3 Hubble times. Models with a negative λ_0 (long-dashed lines) all have shorter ages, as in the $l=1$ case. The difference is that they are now *even shorter*, because the negative cosmological term is driven to high negative values more quickly in the past direction when $l=2$. Thus, the age of the $\lambda_0=-1$ model has dropped from 0.48 to 0.33 Hubble times, while
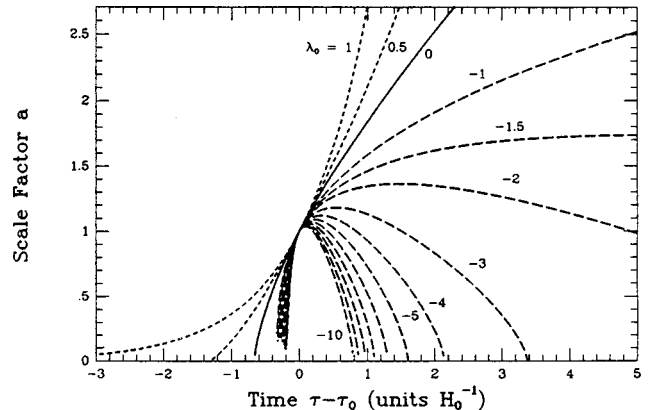

FIG. 2. Evolution of the scale factor for flat models with $\Lambda\propto\tau^{-2}$ and $\gamma=1$. Values of λ_0 are labelled beside each curve, and $\Omega_0=1-\lambda_0$ in each case.

that of the $\lambda_0 = -3$ model is now only $\tau_0 = 0.30$, down from 0.35 in Fig. 1.

Figure 2 indicates that the $\lambda_0 < 0$ models tend toward eventual recollapse, as they did in the $l=1$ case. However, this process now takes much longer. In other words, while the $l=2$ models are younger, their life expectancies are considerably greater. This can be understood by means of the same argument as before (Sec. III C). The larger value of l means that the contribution of the cosmological term to the deceleration drops off more quickly in the future direction, thereby postponing recollapse for a longer period of time. Thus, models with $\lambda_0 = -1$ and $\lambda_0 = -3$ now survive for $t \geqslant 5$ and $t = 3.37$ Hubble times respectively.

E. The case $l=3$

For this case, Eq. (3.7) reads

$$\tau^3 \frac{d^2x}{d\tau^2} - \alpha x = 0, \tag{3.41}$$

where α is given by Eq. (3.10):

$$\alpha = (3\gamma/2)^2 \lambda_0 \tau_0^3. \tag{3.42}$$

Following standard techniques [79], we switch independent variables from τ to $z \equiv 1/\tau$, whereupon:

$$z \frac{d^2x}{dz^2} + 2 \frac{dx}{dz} - \alpha x = 0. \tag{3.43}$$

This is now in a similar form to Eq. (3.11) in the $l=1$ case, and it can be solved in the same manner [79]. Changing independent variables again, from z to $y \equiv 2\sqrt{-\alpha z}$, Eq. (3.43) takes the form:

$$y^2 \frac{d^2x}{dy^2} + 3y \frac{dx}{dy} + y^2 x = 0. \tag{3.44}$$

This is again transformable to Bessel's equation, but with a different general solution $x(y) = y^{-1}[c_1 J_1(y) + c_2 Y_1(y)]$, where c_1, c_2 are arbitrary constants. Equation (3.9) then gives $a(\tau)$, as usual:

$$a(\tau) = \tau^{1/3\gamma}[c_1 J_1(y) + c_2 Y_1(y)]^{2/3\gamma}, \tag{3.45}$$

where $y(\tau) = 2\sqrt{-\alpha/\tau}$ and we have absorbed a factor of $2\sqrt{-\alpha}$ into c_1, c_2. The Hubble parameter is found as usual by putting $x(y)$ into Eq. (3.6):

$$H(\tau) = H_0 \sqrt{-\lambda_0 \left(\frac{\tau_0}{\tau}\right)^3} \left[\frac{c_1 J_2(y) + c_2 Y_2(y)}{c_1 J_1(y) + c_2 Y_1(y)}\right], \tag{3.46}$$

where $J_2(y)$ and $Y_2(y)$ are Bessel and Neumann functions of order two. As in the $l=1$ case, these solutions are real-valued (for $\tau > 0$) only if the cosmological term is negative.

In conjunction with the boundary conditions (3.16), Eqs. (3.45) and (3.46) give for c_1 and c_2:

$$c_1 = \frac{\sqrt{-\lambda_0} Y_2(y_0) - Y_1(y_0)}{\sqrt{-\lambda_0} \tau_0 [J_1(y_0) Y_2(y_0) - J_2(y_0) Y_1(y_0)]} \tag{3.47}$$

$$c_2 = \frac{-\sqrt{-\lambda_0} J_2(y_0) + J_1(y_0)}{\sqrt{-\lambda_0} \tau_0 [J_1(y_0) Y_2(y_0) - J_2(y_0) Y_1(y_0)]}, \tag{3.48}$$

where $y_0 = z_0$ is given by Eq. (3.19). We require one additional boundary condition to fix τ_0. Unfortunately, as in the previous subcase, the procedure is complicated by the fact that both terms in Eq. (3.45) diverge at $\tau = 0$, whereas we expect that the scale factor as a whole should behave smoothly there.

We can make this more precise by employing the asymptotic expressions for $J_1(y)$ and $Y_1(y)$ at large y (i.e., small τ). We find (for $\tau \ll 1$)

$$a(\tau) \approx \tau^{1/2\gamma} \left[C_1 \sin\left(\frac{\omega_0}{\sqrt{\tau}}\right) + C_2 \cos\left(\frac{\omega_0}{\sqrt{\tau}}\right) \right]^{2/3\gamma}, \tag{3.49}$$

where $C_1 \equiv C_0^{2/3\gamma}(c_1 - c_2)$, $C_2 \equiv -C_0^{2/3\gamma}(c_1 + c_2)$, $C_0 \equiv 1/\sqrt{\pi\omega_0}$ and $\omega_0 \equiv 3\gamma\tau_0\sqrt{-\lambda_0\tau_0}$. This goes smoothly to zero for $\tau \to 0$. In order to find the correct (i.e., self-consistent) value of τ_0, we use the same numerical approach as in the $l=2$ case. Once τ_0 is obtained, the values of c_1 and c_2 follow from Eqs. (3.47) and (3.48) respectively.

The evolution of the scale factor for this case is illustrated in Fig. 3, which has exactly the same format as Fig. 1. The $\lambda_0 = 0$ model (solid line) has an age of 2/3 Hubble times, while those with negative values of λ_0 (long-dashed lines) all have shorter ages, as usual. The age of the $\lambda_0 = -3$ model, for example, has dropped from 0.30 to just 0.27 Hubble times. It is unlikely that any of these models could describe the real universe, given the observational constraints on H_0 and τ_0 (Sec. III C).

The main difference between Fig. 3 and its predecessors occurs at large times, where we observe that the curves all straighten out, and show no sign of leading to a recollapse of the scale factor. The explanation for this is that the (negative)
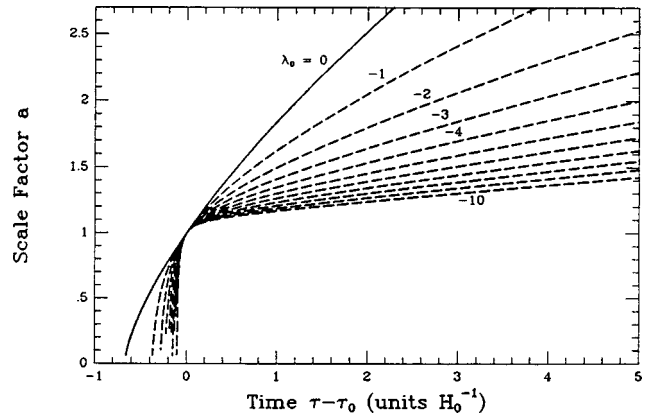

FIG. 3. Evolution of the scale factor for flat models with $\Lambda \propto \tau^{-3}$ and $\gamma = 1$. Values of λ_0 are labelled beside each curve, and $\Omega_0 = 1 - \lambda_0$ in each case.

cosmological term is now decaying so quickly with time that, like ordinary matter, it is no longer sufficient to turn the expansion around.

F. The case $l=4$

For this case, Eq. (3.7) reads

$$\tau^4 \frac{d^2x}{d\tau^2} - \alpha x = 0, \tag{3.50}$$

with α given by Eq. (3.10) as follows:

$$\alpha = (3\gamma/2)^2 \lambda_0 \tau_0^4. \tag{3.51}$$

Switching independent variables via $z \equiv 1/\tau$ as in the previous case, we find that Eq. (3.50) takes the form:

$$z \frac{d^2x}{dz^2} + 2 \frac{dx}{dz} - \alpha z x = 0. \tag{3.52}$$

Employing standard methods [79], we switch dependent variables from z to y via $dy/dz \equiv zx(z)$. It may then be verified that $y(z)$ satisfies the familiar equation:

$$\frac{d^2y}{dz^2} = \alpha y. \tag{3.53}$$

The solutions of Eq. (3.53) are well known; there are three subcases to consider, according as α [and hence λ_0, Eq. (3.51)] is positive, zero, or negative.

1. The subcase $\lambda_0 > 0$

Since we expect on observational grounds that λ_0 is probably positive, this is the most physical of the three. Solution of Eq. (3.53) for $y(z)$ and hence $x(z)$ is straightforward. The scale factor and Hubble parameter are given by Eqs. (3.9) and (3.6):

$$a(\tau) = \tau^{2/3\gamma} \left[c_1 \exp\left(\frac{\sqrt{\alpha}}{\tau}\right) + c_2 \exp\left(\frac{-\sqrt{\alpha}}{\tau}\right) \right]^{2/3\gamma} \tag{3.54}$$

$$H(\tau) = \frac{2H_0}{3\gamma} \left[\frac{c_1(1 - \sqrt{\alpha}/\tau)\exp(\sqrt{\alpha}/\tau) + c_2(1 + \sqrt{\alpha}/\tau)\exp(-\sqrt{\alpha}/\tau)}{c_1 \tau \exp(\sqrt{\alpha}/\tau) + c_2 \tau \exp(-\sqrt{\alpha}/\tau)} \right], \tag{3.55}$$

where c_1, c_2 are arbitrary constants. It is clear that $a(\tau)$ diverges at $\tau = 0$ unless $c_1 = 0$, which we consequently assume. Equations (3.54) and (3.55) simplify:

$$a(\tau) = \left[c_2 \tau \exp\left(\frac{-\sqrt{\alpha}}{\tau}\right) \right]^{2/3\gamma} \tag{3.56}$$

$$H(\tau) = \frac{2H_0}{3\gamma} \left(1 + \frac{\sqrt{\alpha}}{\tau} \right) \frac{1}{\tau}. \tag{3.57}$$

We then apply the boundary conditions (3.16) at $\tau = \tau_0$, as usual. Equation (3.57) with $H(\tau_0) = H_0$ fixes the age at:

$$\tau_0 = \frac{2}{3\gamma(1 - \sqrt{\lambda_0})}. \tag{3.58}$$

Substituting this result into Eq. (3.56) with $a(\tau_0) = 1$, we find that $c_2 = (1/\tau_0) \exp(\sqrt{\alpha}/\tau_0)$, which can be put back into Eq. (3.56) to yield this expression for $a(\tau)$:

$$a(\tau) = \left[\left(\frac{\tau}{\tau_0} \right) \exp\left(\frac{\sqrt{\alpha}}{\tau_0} - \frac{\sqrt{\alpha}}{\tau} \right) \right]^{2/3\gamma}. \tag{3.59}$$

We can draw a number of conclusions from Eq. (3.58): first, that $\lambda_0 < 1$; and secondly, that the age of the Universe $\tau_0 \to \infty$ as $\lambda_0 \to 1$. This is reminiscent of the $l = 2$ case, and in

fact Eq. (3.58) is almost identical to Eq. (3.27), the only difference being that λ_0 in the denominator has been replaced by $\sqrt{\lambda_0}$. Therefore, for the same value of λ_0, the $l = 4$ models are longer-lived by a factor of $(1 - \lambda_0)/(1 - \sqrt{\lambda_0})$. This is due to the fact that, for positive Λ, the higher value of l means that the cosmological term exerts a greater repulsive force in the past direction, and is consequently able to push the big bang back more effectively. As in the $l = 2$ case, we also find a lower limit on the age of these models. This is, from Eq. (3.58):

$$\tau_0 > 2/3\gamma. \tag{3.60}$$

This is twice as long as in the $l = 2$ case, but here the reason is simply that this lower limit corresponds to the case $\lambda_0 = 0$ (not $\lambda_0 = -1$ as before). With dust-like conditions ($\gamma = 1$) and the upper bound $H_0 \leqslant 83$ km s^{-1} Mpc^{-1} [81], we now find a minimum age of at least 7.9 billion years.

2. The subcase $\lambda_0 = 0$

For this subcase Eq. (3.53) is trivial. Using Eqs. (3.9) and (3.6) we find immediately that:

$$a(\tau) = (c_3 + c_4\tau)^{2/3\gamma} \tag{3.61}$$

$$H(\tau) = \frac{2H_0}{3\gamma} \left(\frac{c_4}{c_3 + c_4\tau} \right), \tag{3.62}$$

where c_3, c_4 are arbitrary constants. For the first time we have a scale factor with the potential to go smoothly to some finite value *other than* zero at $\tau = 0$. Let us pursue this possibility and see if a nonsingular solution is possible. Instead of the boundary condition $a(\tau_0) = 1$, we impose $a(0) = a_*$, where a_* is the minimum value of the scale factor. In Eq. (3.61) this implies that $c_3 = a_*^{3\gamma/2}$. Inserting this result back into Eq. (3.61) and applying the usual condition $a(\tau_0) = 1$, we obtain $c_4 = (1 - c_3)/\tau_0$. Substituting this into Eq. (3.62) and applying the third boundary condition $H(\tau_0) = H_0$, we obtain for the age of the Universe:

$$\tau_0 = 2(1 - c_3)/3\gamma. \tag{3.63}$$

This result matches onto that of the previous subcase, Eq. (3.58), only if $c_3 = 0$, which also implies that $a_* = 0$. The present case is therefore singular at $\tau = 0$, like all the others studied in this section. The age of the Universe is given by Eq. (3.63) as $\tau_0 = 2/3\gamma$. Putting these results back into Eq. (3.61), we find:

$$a(\tau) = \left(\frac{\tau}{\tau_0} \right)^{2/3\gamma}. \tag{3.64}$$

This is just the standard $k = 0$ solution with no cosmological term, as might have been expected.

3. The subcase $\lambda_0 < 0$

Solution of Eq. (3.53) is again straightforward and leads via Eqs. (3.9) and (3.6) to:

$$a(\tau) = \tau^{2/3\gamma} \left[c_5 \sin\left(\frac{\sqrt{-\alpha}}{\tau} \right) + c_6 \cos\left(\frac{\sqrt{-\alpha}}{\tau} \right) \right]^{2/3\gamma} \tag{3.65}$$

$$H(\tau) = \frac{2H_0}{3\gamma} \left\{ \frac{[c_5 + (\sqrt{-\alpha}/\tau)c_6]\sin(\sqrt{-\alpha}/\tau) + [c_6 - (\sqrt{-\alpha}/\tau)c_5]\cos(\sqrt{-\alpha}/\tau)}{c_5\tau\sin(\sqrt{-\alpha}/\tau) + c_6\tau\cos(\sqrt{-\alpha}/\tau)} \right\},$$

where c_5, c_6 are arbitrary constants. Application of the boundary conditions (3.16) fixes these constants in terms of τ_0:

$$c_5 = \frac{1}{\sqrt{-\alpha}} \left[\beta_0 \sin\beta_0 + \cos\beta_0 \left(1 - \frac{3\gamma}{2}\tau_0 \right) \right] \tag{3.66}$$

$$c_6 = \frac{1}{\sqrt{-\alpha}} \left[\beta_0 \cos\beta_0 - \sin\beta_0 \left(1 - \frac{3\gamma}{2}\tau_0 \right) \right], \tag{3.67}$$

where $\beta_0 \equiv (3\gamma/2)\tau_0\sqrt{-\lambda_0}$. As usual, we require one additional boundary condition to fix the value of τ_0. The situation is again complicated by the fact that both terms in Eq. (3.65) diverge at $\tau = 0$, whereas $a(\tau)$ itself goes smoothly to zero there. [In fact, Eq. (3.65) has exactly the same form as the asymptotic expression (3.49) in the $l = 3$ case.] We therefore have recourse once again to the numerical method described in Sec. III D 3. Once τ_0 is obtained in this way, the values of c_5 and c_6 are fixed by Eqs. (3.66) and (3.67).

The evolution of the scale factor for this case is illustrated in Fig. 4, which has exactly the same format as Fig. 2. Several features may be noted. To begin with, we see that models with $\Lambda \propto \tau^{-4}$ are qualitatively the same as those with $\Lambda \propto \tau^{-2}$ for positive λ_0, and qualitatively similar to those with $\Lambda \propto \tau^{-3}$ for negative λ_0.

There are important quantitative differences, however. Models with positive λ_0 are significantly older. With $\lambda_0 = 0.5$, for example, τ_0 is now 2.28 Hubble times — older by a factor of 1.71 times than the equivalent $l = 2$ model, ex-

actly as predicted in the discussion following Eq. (3.59). This is once again due to the fact that, with $l = 4$, a positive Λ-term increases in size very rapidly in the past direction. Negative-λ_0 models, on the other hand, are once again younger. The age of the $\lambda_0 = -3$ model, for instance, has dropped from 0.27 to just 0.25 Hubble times. And finally, in the future direction, we see that there is no longer very much distinction between the $\lambda_0 < 0$ and $\lambda_0 = 0$ models, compared to Fig. 3. The contribution of the cosmological term to the deceleration of the scale factor now drops off so quickly that it rapidly becomes irrelevant.

IV. Λ AS A FUNCTION OF THE SCALE FACTOR

A. Previous work

We now move on to consider decay laws of the form set out in Eq. (2.11):

$$\Lambda = \mathcal{B}a^{-m}. \tag{4.1}$$

The scale factor may be more natural than time as an independent variable, to the extent that many physical quantities (such as temperature) depend more simply on a than t. Nearly half of the decay laws listed in Table I contain terms of the form (4.1). The case $m = 2$ has been singled out for the most attention [41,44–46], and is motivated by some dimensional arguments [41,45]. A second group of authors has concentrated on values of $m \approx 4$ [47–50], for which the Λ term behaves like ordinary radiation. It has been shown that, for certain kinds of Λ decay, the lower of these two values of m is thermodynamically more stable [52]. A third idea, that
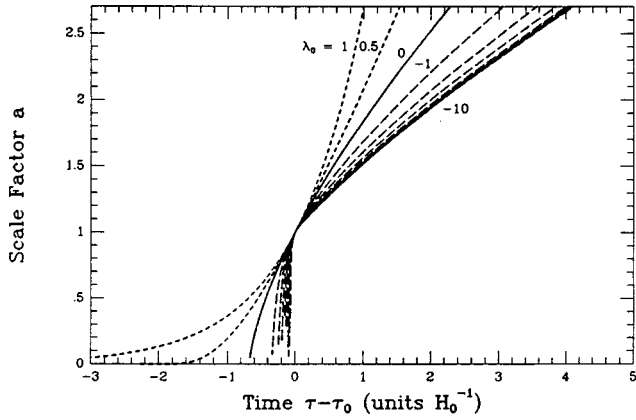
FIG. 4. Evolution of the scale factor for flat models with $\Lambda \propto \tau^{-4}$ and $\gamma = 1$. Values of λ_0 are labelled beside each curve, and $\Omega_0 = 1 - \lambda_0$ in each case.

the cosmological term scales with a like ordinary matter ($m = 3$), follows from one interpretation of an intriguing new scale-invariant extension of general relativity [73].

There are also some studies in which the value of m is not fixed *a priori*. Ages of general-m models have been calculated and agree with observation if $m < 3$ [51]. The power spectrum of matter density perturbations does not appear to be greatly modified by a decaying Λ term, at least for $0 \leqslant m \leqslant 2$ [54]. Lensing statistics favor models with $m \geqslant 1$ [55], or $m \geqslant 1.6$ when combined with other tests involving CMB anisotropies and the classical magnitude-redshift relation for high-redshift supernovae [56]. Other aspects of models in which Λ decays as a^{-m} have been discussed by several authors [53,58,59], although no specific numerical bounds are set on m.

The question of the initial singularity has so far received little attention in theories of this kind [85]. Some explicitly nonsingular solutions have been constructed, all with $m = 2$ [44]. In one other case it is noted in passing that the existence of an initial singularity would require $0 < m < 4$ under some circumstances [59]. The remaining authors either do not mention the issue, or (as in one case [51]) rule out *a priori* the possibility of nonsingular solutions. In this paper, we take a broader view and examine all possible solutions for the scale factor, including those in which it takes a non-zero minimum value. Moreover we will extend the discussion, not only to general m, but to general γ as well [where γ characterizes the equation of state of ordinary matter, Eq. (2.6)]. If m and γ are thought of as defining a parameter space, then we determine, first, the extent to which the space is singularity-free; and secondly, the extent to which it is observationally viable.

B. Evolution of the scale factor

We begin with the dynamical equations (2.4)–(2.9). In particular, we consider the energy conservation law (2.7), which, with the decay law (4.1), becomes:

$$\frac{d}{da}(\rho a^{3\gamma}) = \left(\frac{m\mathcal{B}}{8\pi G}\right) a^{3\gamma - (m+1)}. \tag{4.2}$$

Integrating, we find for the matter energy density:

$$\rho(a) = \rho_0 a^{-3\gamma} f(a), \tag{4.3}$$

where we have set $a(t_0) = a_0 = 1, \rho(a_0) = \rho_0$, and defined:

$$f(a) \equiv 1 + \kappa_0 \times \begin{cases} \dfrac{m(a^{3\gamma - m} - 1)}{3\gamma - m} & \text{if } m \neq 3\gamma \\ 3\gamma \ln(a) & \text{if } m = 3\gamma \end{cases} \tag{4.4}$$

$$\kappa_0 \equiv \mathcal{B}/8\pi G\rho_0. \tag{4.5}$$

When $m = 0$, then $f(a) = 1$ and Eq. (4.3) yields the usual result that ρ scales as a^{-3} in a pressure-free universe ($\gamma = 1$) and a^{-4} in a radiation-dominated universe ($\gamma = 4/3$). The new parameter κ_0 can be fixed in terms of observable quantities by means of the decay law (4.1), which gives $\mathcal{B} = \Lambda_0 = 3H_0^2\lambda_0$ [with $a_0 = 1$ and λ_0 defined as usual by Eq. (2.15)]. Substituting this result into Eq. (4.5), we find:

$$\kappa_0 = \lambda_0/\Omega_0, \tag{4.6}$$

where Ω_0 is defined by Eq. (2.15). The parameter κ_0 is simply the ratio of energy density in the cosmological term to that in ordinary matter at the present epoch.

Substitution of Eqs. (4.1) and (4.3) into the Lemaître equation (2.4) yields:

$$\frac{da}{d\tau} = a[\Omega_0 a^{-3\gamma} f(a) - (\Omega_0 + \lambda_0 - 1)a^{-2} + \lambda_0 a^{-m}]^{1/2}, \tag{4.7}$$

where we have made use of the definitions (2.15), recalled that $\dot{a}/a = (H_0/a)da/d\tau$, and selected the positive root since redshifts rather than blueshifts are observed.

At this point we could choose integer values of m and attempt to solve analytically for $a(\tau)$, as in Sec. III. Detailed analyses have been carried out along these lines for the case $m = 0$; i.e., for a constant cosmological term [86]. It is doubtful that they can be usefully extended to the general situation in which $m \neq 0$. We opt instead to solve the problem numerically, following the lead of Felten and Isaacman [5]. The time derivative of Eq. (4.7) is:

$$\frac{d^2a}{d\tau^2} = \left(1 - \frac{3\gamma}{2}\right)\Omega_0 a^{1-3\gamma} f(a) + \lambda_0 a^{1-m}. \tag{4.8}$$

[This could equally well have been obtained from Eq. (2.8).] We substitute Eqs. (4.7) and (4.8) into a Taylor expansion for the scale factor:

$$a_k \approx a_{k-1} + \left(\frac{da}{d\tau}\right)_{k-1} \Delta\tau + \frac{1}{2}\left(\frac{d^2a}{d\tau^2}\right)_{k-1} (\Delta\tau)^2. \tag{4.9}$$

This can be integrated numerically backwards in time to determine whether or not a model with given values of $\{m, \gamma, \Omega_0, \lambda_0\}$ eventually reaches $a = 0$. We have tested the procedure for the case of a constant cosmological term ($m = 0$) and dust-like equation of state ($\gamma = 1$), and our results
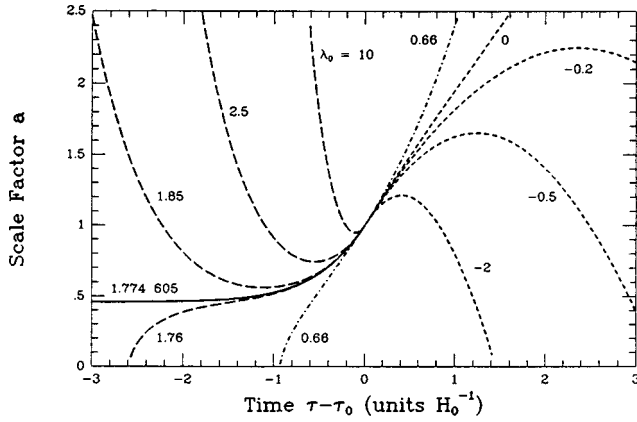
FIG. 5. Evolution of the scale factor for models with $m=0, \gamma =1$, $\Omega_0=0.34$, and values of λ_0 labelled beside each curve (after Felten and Isaacman [5]).

in this case confirm those of Felten and Isaacman [5]. Figure 5 depicts a group of examples with $\Omega_0=0.34$ (typical of large-scale observations [87]) and various values of λ_0 (labelled beside each curve). Note that the difference between this figure and the ones in the previous section is that we now include models of all *three* kinds: closed (long-dashed lines), flat (dash-dotted lines) and open (short dashes). (To keep the diagram from being too crowded, we show only models with the same value Ω_0 of the matter density.) Figure 5 indicates that negative values of λ_0 can lead to recollapse in open, as well as flat universes (cf. Sec. III). Of greater interest, however, is the fact that models with λ_0 above a *critical value* λ_* ($=1.774\,605$ in this case) avoid the big bang, undergoing a finite ''big bounce'' instead. Models with slightly less than this critical value (e.g., $\lambda_0=1.76$ in this case) are of the ''coasting Lemaître'' kind: they begin in a singular state but go through an extended phase in which the scale factor is nearly constant. Models with *exactly* $\lambda_0 =\lambda_*$ (shown in Fig. 5 with a solid line) are perhaps the most interesting of all. As time $\tau \to -\infty$, they neither plunge to zero size nor bounce back up to infinite size, but level off indefinitely at a constant value $a=a_*$ ($=0.46$ in this case). These are nonsingular Eddington-Lemaître models, asymptotic to the static Einstein universe in the infinite past.

All these features of the $m=0, \gamma=1$ models have been discussed at greater length elsewhere [5]. Our purpose here is to generalize the discussion to arbitrary values of m and γ.

C. Critical values of λ_0

In particular, we wish to obtain general expressions for the critical value λ_* of the lambda parameter λ_0 and minimum size a_* of the scale factor, given any class of models $\{m, \gamma, \Omega_0\}$. As discussed in Sec. III A, we are interested in models for which $da/d\tau \to 0$ at some time in the past. This occurs, for example, at the moment of the ''bounce'' in all the oscillating models shown in Fig. 5. The *critical* case is distinguished by that fact that not only $da/d\tau$, but also $d^2a/d\tau^2$ vanishes at this point [5,14]. We therefore set $da/d\tau = d^2a/d\tau^2 = 0$ in Eqs. (4.7) and (4.8). This yields:
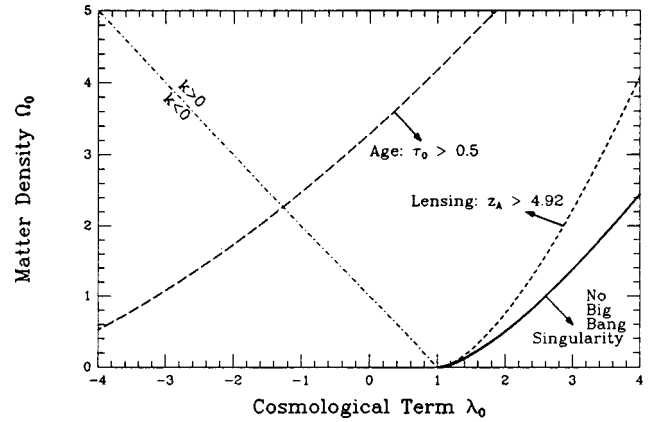

FIG. 6. Phase space diagram showing constraints on models with $m=0$ and $\gamma=1$ (after Lahav *et al.* [15]).

$$\lambda_* = \frac{(3\gamma-2)(3\gamma-m)\Omega_0}{3\gamma(2-m)a_*^{3\gamma-m}+(3\gamma-2)m}, \quad (4.10)$$

where a_*, the minimum value of the scale factor, is found by solving:

$$3\gamma(3\gamma-m)\Omega_0 a_*^2 + 3\gamma(2-m)(1-\Omega_0)a_*^{3\gamma} - (3\gamma-2)$$

$$\times (3\gamma\Omega_0-m)a_*^m = 0. \quad (4.11)$$

In general Eq. (4.11) has to be solved numerically, but in the case $m=0, \gamma=1$ it reduces to a cubic equation [5]. Figure 6 is a phase space portrait of this case, with each point on the diagram corresponding to a choice of Ω_0 and λ_0 (after Lahav *et al.* [15]). The critical values λ_* are represented in this figure by a heavy solid line. The region to the right of this curve corresponds to universes with $\lambda_0 > \lambda_*$; that is, with no big bang. Also shown in Fig. 6 is a straight dash-dotted line representing the boundary between open and closed universes; models on this line have $k=0$ ($\Omega_0=1-\lambda_0$) while those on the left and right have $k<0$ and $k>0$ respectively (Sec. II D).

We now have the tools we need to investigate models with arbitrary values of m and γ. The idea will be to use phase space diagrams like Fig. 6 to determine how much of the parameter space (1) corresponds to models with a non-zero minimum scale factor; and (2) agrees with observational constraints. We can then confirm whether a given model with $\{m, \gamma, \Omega_0\}$ does in fact avoid the big bang by carrying out the numerical integration described in Sec. IV B, and plotting the results on evolution diagrams like Fig. 5.

D. Observational constraints

1. Upper bounds on λ_0

We pause first to take stock of some of the experimental constraints that have been placed on models with nonzero cosmological terms. Most immediate are direct upper bounds on λ_0 from a variety of methods, most of them assuming that $\Omega_0+\lambda_0=1$. Until recently, these have typically been of order ~ 1 [15]. Additional methods, however, have become
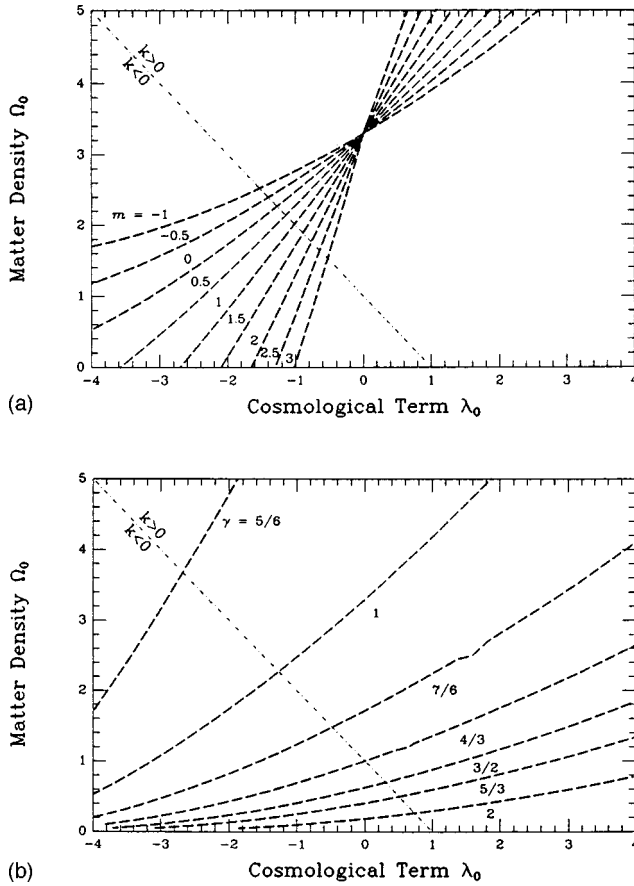
FIG. 7. The age constraint $\tau_0 > 0.5$: (a) as a function of m, assuming $\gamma = 1$; and (b) as a function of γ, assuming $m = 0$.

available in the past few years. CMB fluctuations, for instance, have produced an upper limit of $\lambda_0 \lesssim 0.86$ [88]. Gravitational lens statistics give a tighter bound of $\lambda_0 < 0.66$ [89], and observations of type Ia supernovae appear to reduce this still further, to $\lambda_0 < 0.51$ [84]. All of the above are described as 95% confidence level measurements. On the other hand, a *lower* limit of $\lambda_0 > 0.53$ has been obtained from the galactic luminosity density — *also* at 95% confidence [90].

Complicating the picture somewhat is the fact that several other observational data are well explained by substantial values of λ_0, including the lack of observed small-scale dark matter, the expectation that inflation should lead to near-flatness, and especially the high *age* of the Universe inferred from models of stellar evolution. This "cosmic concordance" [89] at one time led to calls for $\lambda_0 \sim 0.8$ [91], although this has since dropped to 0.5–0.7 [92].

Thus, the observational situation is not yet settled. It is clear, however, that the nonsingular models in Fig. 5, which require $\lambda_0 > 1.77$, are almost certainly unphysical. It remains to be seen if the same conclusion applies when $m \neq 0$ and/or $\gamma \neq 1$.

2. Age of the Universe

A lower limit on λ_0 derives from the age of the Universe, $t_0 = \int_0^{a_0} da/\dot{a}$. In our units of Hubble times:

$$\tau_0 \equiv H_0 t_0 = \int_0^1 \frac{da}{da/d\tau}. \qquad (4.12)$$

If we use a recent value of 73 ± 10 km s^{-1} Mpc^{-1} for H_0 [81] in conjunction with the oldest globular cluster age of $t_0 \gtrsim 9.6$ Gyr [82], then $\tau_0 \gtrsim 0.62$; that is, the Universe is at least 0.62 Hubble times old. Of course, as noted in Sec. III C, the true status of these two parameters is still a subject of some controversy. A lower value of $H_0 = 55 \pm 10$ km s^{-1} Mpc^{-1} [83] leads to $\tau_0 \gtrsim 0.44$. We choose an intermediate value of 0.5. In conjunction with our expression (4.7) for $da/d\tau$, this implies:

$$\tau_0 = \int_1^\infty \frac{dv}{[\Omega_0 v^{2+3\gamma} g(v) - (\Omega_0 + \lambda_0 - 1)v^4 + \lambda_0 a^{2+m}]^{1/2}}$$
$$> 0.5, \qquad (4.13)$$

where we have changed variables of integration from a to $v \equiv a^{-1} = 1 + z$ for computational reasons, and:

$$g(v) \equiv 1 + \kappa_0 \times \begin{cases} \dfrac{m(v^{m-3\gamma} - 1)}{3\gamma - m} & \text{if } m \neq 3\gamma, \\ -3\gamma \ln(v) & \text{if } m = 3\gamma. \end{cases} \qquad (4.14)$$

Equation (4.13) reduces to the standard results [2,15] in the case $m = 0, \gamma = 1$.

Numerical solution of Eq. (4.13) produces a lower limit on λ_0 as a function of $\{m, \gamma, \Omega_0\}$. This age constraint is shown in Fig. 6 (for the case $m = 0, \gamma = 1$) as a long-dashed line. Its position matches that in a similar plot by Lahav *et al.* [15]. The region to the left of this curve corresponds to universes younger than half a Hubble time. Big bangless models, of course, are not constrained by this; they are infinitely old (by definition), the heavy solid line being precisely the boundary where $\tau_0 \to \infty$. The main impact of the age constraint is to rule out models with a *negative* cosmological term.

In Fig. 7, we show the effects of varying the parameters m and γ respectively on this age constraint. It may be seen that altering the value of m changes the slope of the curve, but does not otherwise greatly affect the age, even over the range $-1 \leq m \leq 3$. Altering the value of γ, on the other hand, has a larger effect. In particular, the "harder" the equation of state (i.e., the larger the value of γ), the further this constraint encroaches on the available parameter space. This is in accord with the well-known fact that a radiation-dominated universe ($\gamma = 4/3$), for example, is a short-lived one.

3. Gravitational lensing and the antipode

For closed models, the most stringent constraint on λ_0 comes from gravitational lensing, which requires that the "antipode" be further away than the most distant normally lensed object [93]. The antipode is the point where the radial coordinate $\omega \equiv \int_t^{t_0} dt/a(t) = \int_a^{a_0} da/\dot{a}a$ takes the value π [6]. Using our expression (4.7) for $da/d\tau$, together with Eq. (2.16), we obtain (taking $a_0 = 1$ as usual):

$$\omega = \sqrt{\Omega_0 + \lambda_0 - 1}$$

$$\times \int_1^{1+z} \frac{dv}{[\Omega_0 v^{3\gamma} g(v) - (\Omega_0 + \lambda_0 - 1)v^2 + \lambda_0 a^m]^{1/2}}, \tag{4.15}$$

where we have again changed integration variables from a to v. The standard formula [2] is recovered when $m = 0$ and $\gamma = 1$.

At present, the furthest known normally lensed object is a pair of lensed galaxies at $z_l = 4.92$ [94]. We therefore require:

$$z_A(\Omega_0, \lambda_0, m, \gamma) > 4.92, \tag{4.16}$$

where $z_A(\Omega_0, \lambda_0, m, \gamma)$ is defined by Eq. (4.15) with $\omega = \pi$. Numerical solution of this equation yields an upper limit on λ_0 as a function of $\{m, \gamma, \Omega_0\}$. This lensing constraint is shown in Fig. 6 (for the case $m = 0, \gamma = 1$) as a short-dashed line. Its position is close to that in previous plots [15,93] employing a smaller value $z_l = 3.27$ (our constraints are slightly stronger). The region to the right of this curve corresponds to universes incompatible with the lensing observations. Since this includes the entirety of nonsingular parameter space, we can see that oscillating models with constant Λ and zero pressure are ruled out, as noted previously by these authors.

In Fig. 8, we show the effects of varying the parameters m and γ respectively on this lensing constraint. As before, it is seen that harder values of γ lead to tighter constraints on the available parameter space. However, the situation with regard to m is altered quite dramatically. In particular, the higher the value of m, the *weaker* the lensing constraint becomes. As we will see, this significantly improves the prospects for viable big bangless models.

4. The maximum redshift constraint

A fourth observational constraint, which must be satisfied *only* by nonsingular models, concerns the maximum observable redshift $z_* = a_*^{-1} - 1$ in a universe with a minimum scale factor a_*. This must obviously be larger than the greatest redshift z_{obs} actually observed. Thus the nonsingular
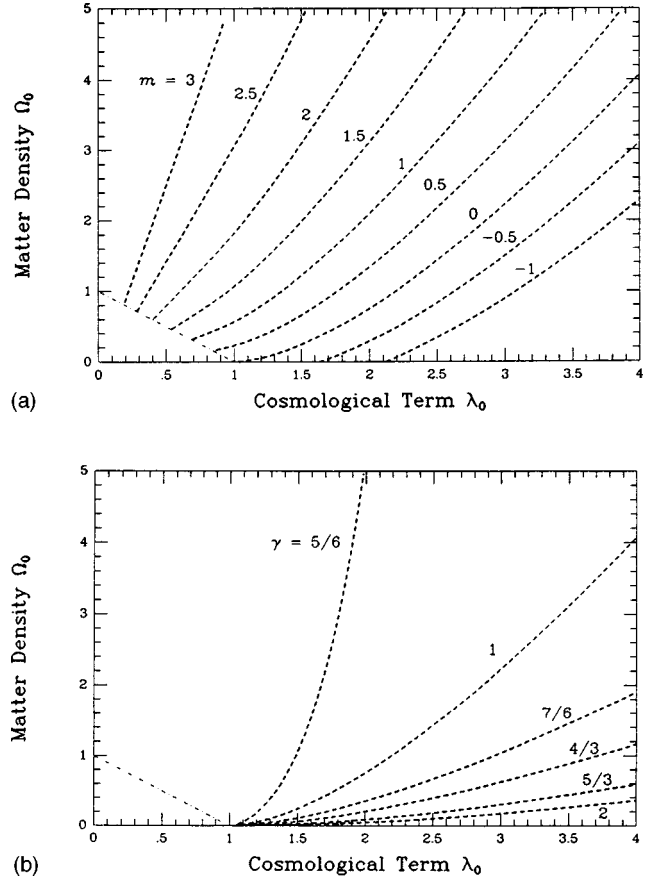

(a)

(b)

FIG. 8. The lensing constraint $z_A > 4.92$: (a) as a function of m, assuming $\gamma = 1$; and (b) as a function of γ, assuming $m = 0$.

models in Fig. 5, which never get smaller than $a_* = 0.46$, cannot accommodate redshifts greater than $z_* = 1.2$. This disagrees with observations such as those of the lensed galaxies mentioned above.

The condition $z_* > z_{obs}$ can be reformulated as an upper limit on the *matter density* of the Universe. Assuming that $da/d\tau = 0$ and $d^2a/d\tau^2 \geq 0$ at $z = z_*$ for nonsingular models, Eqs. (4.7) and (4.8) can be combined to read (for $m \neq 3\gamma$):

$$\Omega_0 \leq \frac{2 - m + (m/3\gamma)(3\gamma - 2)(1 + z_*)^{3\gamma - m}}{2 - m - (3\gamma - m)(1 + z_*)^{3\gamma - 2} + (3\gamma - 2)(1 + z_*)^{3\gamma - m}}. \tag{4.17}$$

Equation (4.17) reduces to earlier expressions of Börner and Ehlers [14] when $m = 0$ and $\gamma = 1$ or 4/3. These authors have then argued as follows: given that quasar redshifts have been observed out to $z_{obs} > 4$, we know that the Universe has $z_* > 4$. This constraint with Eq. (4.17) implies (assuming $m = 0$ and $\gamma = 1$ or 4/3) that $\Omega_0 \lesssim 0.018$, which is contrary to observation. This indicates that our Universe could not have been nonsingular.

Let us see how the above conclusion changes when we generalize the situation to values of $m \neq 0$. Moreover, we will *strengthen* the argument by noting that some distant galaxies have now been assigned photometric redshifts as high as $z_{obs} > 6$ [95], implying that $z_* > 6$. The resulting upper limits on density Ω_0 are listed in Table II for various values of m and γ. From this table we see that the new photometric redshifts tighten the Börner-Ehlers constraint

TABLE II. Börner-Ehlers-type upper limits on matter density Ω_0 for various values of m and γ, assuming $z_* > 6$.

γ:	2/3	5/6	1	7/6	4/3	5/3	2
m:							
0	∞	0.033	0.006	0.002	0.000	0.000	0.000
1/4	∞	0.16	0.10	0.079	0.067	0.053	0.044
1/2	∞	0.31	0.20	0.16	0.14	0.11	0.090
3/4	∞	0.49	0.33	0.26	0.22	0.17	0.14
1	∞	0.75	0.48	0.38	0.32	0.25	0.20
5/4	∞	1.1	0.71	0.55	0.46	0.35	0.29
3/2	∞	1.9	1.1	0.87	0.71	0.54	0.44
7/4	∞	4.2	2.4	1.8	1.4	1.0	0.84
2	∞	∞	∞	∞	∞	∞	∞

noticeably: as long as $m=0$, a nonsingular Universe requires $\Omega_0 \lesssim 0.006$ (if $\gamma = 1$) or $\Omega_0 < 0.001$ (if $\gamma = 4/3$). These numbers, of course, are too low to describe the real universe. However, Table II demonstrates that much higher densities are possible in singularity-free models with *variable* Λ. For example, retaining pressure-free conditions ($\gamma = 1$), we see that if $m = 1$, then the matter density in a nonsingular universe must satisfy $\Omega_0 < 0.48$. This value is not unreasonable at all; in fact it is well above most dynamical measurements, which suggest $\Omega_0 \approx 0.3$ [87]. The constraint is similarly loosened if we move toward softer equations of state with $\gamma < 1$, such as those that have been proposed in [37].

E. Viable oscillating models

We now demonstrate that models with $m \neq 0$ are capable of satisfying all the constraints discussed above. Figure 9 is a phase space diagram like Fig. 6, but plotted for a range of nonzero values of m. We have assumed $\gamma = 1$ as before, but there is now a different line of critical values $\lambda_*(m, \gamma, \Omega_0)$ for each value of m. Like Fig. 6, Fig. 9 shows that much of the nonsingular parameter space (below and to the right of the heavy solid lines) is eliminated because it does not overlap with the regions allowed by the lensing constraint (above and to the left of the lighter dashed lines). This is especially
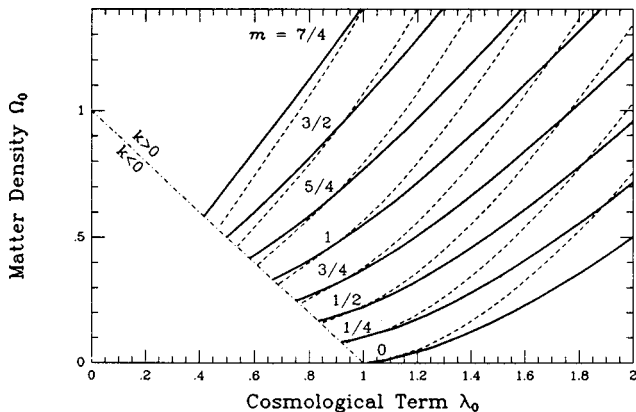

FIG. 9. Enlarged view of the phase space diagram, Fig. 6, now plotted for various values of m between 0 and 2 (labelled beside each pair of curves), assuming $\gamma = 1$.
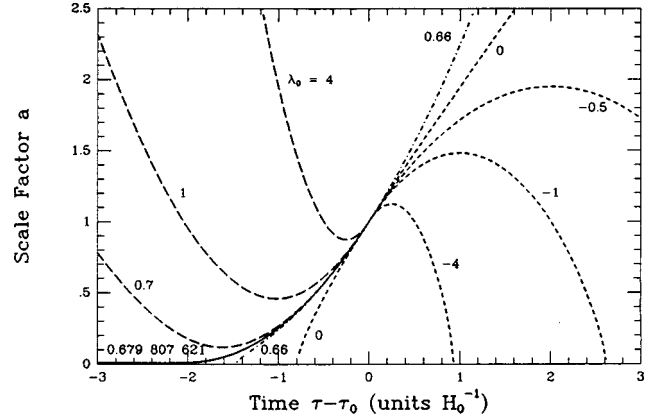

FIG. 10. Evolution of the scale factor for universes with $m = 1, \gamma = 1$ and $\Omega_0 = 0.34$. Compare Fig. 5.

true for small values of m. *With increasing m*, however, significant triangle-shaped regions of parameter space appear near $k = 0$. Thus for $m = 1$, there are allowed models with $\Omega_0 \sim 0.3$ and λ_0 between about 0.7 and 0.9. These are very close to the values favored by observation [87,92], so we focus on this case. The lensing constraint imposes the lower bound $\Omega_0 > 0.31$. If we take $\Omega_0 = 0.34$ as a specific example (as before), then by tracing horizontally across Fig. 9 we find that lensing also places an upper bound $\lambda_0 \lesssim 0.72$ on the cosmological term. From Eq. (4.10) the critical value for this case turns out to be $\lambda_* = 0.68$, which is marginally consistent with the upper limits on λ_0 mentioned earlier. Therefore $m = 1$ models with $\Omega_0 = 0.34$ and $0.68 \lesssim \lambda_0 \lesssim 0.72$ are both realistic and singularity free.

For larger values of m, the range of acceptable λ_0 values is broader, but one is also driven to higher values of Ω_0. With $m = 3/2$, for example, we find that viable nonsingular models occur only for $\Omega_0 > 0.46$, and that at $\Omega_0 = 0.51$ they lie in the range $0.51 \lesssim \lambda_0 \lesssim 0.57$. At $m = 7/4$ and $\Omega_0 = 0.59$ this broadens to $0.42 \lesssim \lambda_0 \lesssim 0.50$ (low enough to satisfy even the supernova constraint [84]). As $m \rightarrow 2$, we find that $a_* \rightarrow 0$, so that nonsingularity is lost. This also happens in the limit $\gamma \rightarrow 2/3$ and, interestingly, as $\lambda_* \rightarrow 1 - \Omega_0$. (It is for this latter reason that all the nonsingular solutions in Fig. 9 have $k > 0$.)

Unusual values of γ, although interesting and possibly relevant at early times, do not alter these results in any fundamental way. In practice we find that the parameter space allowed by observation shrinks slightly for harder values of γ, like 4/3, but grows significantly for softer values such as those in the range $0.4 < \gamma < 1$ [37].

To verify that models with the properties described above can in fact avoid the big bang, Fig. 10 shows the evolution of the scale factor when $m = 1, \gamma = 1$ and $\Omega_0 = 0.34$, obtained by numerical integration using the Taylor expansion (4.9), as before. Various values of λ_0 are labelled beside the appropriate curves. This plot has exactly the same format as Fig. 5, with long dashes corresponding to $k > 0$, short ones to $k < 0$, and the dash-dotted line corresponding to $k = 0$.

Figure 10 confirms that when λ_0 takes on the critical value λ_* ($=0.679\,807\,621$ in this case), the scale factor a evolves back to a constant value a_*, as expected (solid line). More importantly, the small *size* of this minimum value, $a_*=0.0097$, means that we can now accommodate observed redshifts up to $z_*=102$, well beyond the furthest objects yet seen. The model is thus compatible with all observational data. It cannot, however, accommodate larger redshifts like that attributed to the last scattering surface ($z_{lss}\sim1100$), let alone the era of nucleosynthesis, which occurred at temperatures $T_{nuc}\sim10^{10}$ K, or — since $T\propto(1+z)$ — redshifts z_{nuc} $\sim10^{10}$. In fact, with $z_*=102$ the $m=1$ model heats up to no more than about $T_*=103T_0=281$ K, which is just under room temperature. That is to say, if the Universe is described by a nonsingular variable-Λ theory with $m=1$, then the CMB and the abundance of the light elements cannot be accounted for in the conventional way.

Unconventional explanations have been proposed for both these phenomena, but are not widely accepted [96]. A more conservative approach might be to retain the traditional hot, dense early phase in the context of variable-Λ theory by moving to larger values of m. As discussed above, this leads to smaller values of a_* and hence larger values of z_* and T_*. In this picture the high temperature of the early universe is a result, not of an initial big bang singularity, but of a very deep ''big bounce.'' With $m=1.5$ and $\Omega_0=0.51$, for instance, we find that $a_*=1.7\times10^{-4}$, corresponding to a maximum redshift of 5900 and a ''bounce temperature'' of 16,000 K — more than sufficient for recombination. At $m=1.8$, this latter number climbs to 5.7×10^9 K — hot enough for nucleosynthesis. (An analysis like that below Fig. 9 shows that models with $m=1.8$ and $\Omega_0=0.61$, for example, are observationally viable if $0.41\leqslant\lambda_0\leqslant0.49$.) With $m=1.9$, one can entertain bounce temperatures as high as 2.9×10^{13} K (2.5 GeV), approaching the realm of electroweak unification. (Models with $m=1.9$ and $\Omega_0=0.67$ agree with observation if $0.39\leqslant\lambda_0\leqslant0.48$.) As $m\rightarrow2$, in fact, one finds that $a_*\rightarrow0$ and $T_*\rightarrow\infty$. As remarked above, however, these higher values of m come at the modest observational price of higher matter densities. The lensing constraint sets a limit of $\Omega_0>0.54$ for $m=1.8$ models, for example, and $\Omega_0>0.57$ for $m=1.9$ models.

V. Λ AS A FUNCTION OF THE HUBBLE PARAMETER

A. Previous work

We have seen in Sec. IV that variable-Λ theories in which $\Lambda\propto a^{-m}$ appear to offer the possibility of avoiding the big bang singularity without violating any observational constraints. The parameter space occupied by viable models, however, remains small. As discussed in Sec. IV E, one reason for this is that the oscillating solutions are all closed, whereas observational evidence tends to favor an open universe [96]. In this section we shift our attention to decay laws of the form set out in Eq. (2.12):

$$\Lambda=\mathcal{C}H^n. \tag{5.1}$$

Among other things we will find that a cosmological term of this kind allows for a nonzero minimum scale factor in *open* models.

Terms of the form (5.1) are nearly as common as those studied in Sec. IV, appearing in almost a third of the decay laws listed in Table I. The value $n=2$, favored on dimensional [61] and other grounds [67], is adopted almost universally [61,62,65–69]. Other powers of H are considered in three cases, but only in combinations such as $\exp(-\int aH^n dt)$ [57], $a^{-2}H$ [70], and $aH(dH/da)$ [75]. The question of the initial singularity has received little attention in these papers, apart from one explicitly nonsingular [62] and one asymptotically de Sitter–like solution [70].

B. Riccati's equation

Substituting the decay law (5.1) into the differential equation (2.9), we obtain:

$$\frac{dH}{da}=\left(\frac{\gamma\mathcal{C}}{2a}\right)H^{n-1}-\left(\frac{3\gamma}{2a}\right)H-\left(\frac{3\gamma-2}{2a}\right)\left(\frac{k}{a^2}\right)H^{-1}, \tag{5.2}$$

where we have used the fact that $\ddot{a}/a=aH(dH/da)+H^2$. This is nonlinear, but bears some resemblance to the Riccati equation (3.5). We switch dependent variables from H to $x\equiv H^r$, where r is an arbitrary constant whose value will be chosen in a moment. In terms of x, Eq. (5.2) takes the form

$$\frac{dx}{da}=\left(\frac{r\gamma\mathcal{C}}{2a}\right)x^l-\left(\frac{3r\gamma}{2a}\right)x-\left[\frac{r(3\gamma-2)k}{2a^3}\right]x^m, \tag{5.3}$$

where $l\equiv(n+r-2)/r$ and $m\equiv(r-2)/r$.

When $n=4$, we see that the choice $r=2$ puts $l=2$ and $m=0$, whereupon Eq. (5.3) takes the form:

$$\frac{dx}{da}=\mathcal{P}(a)x^2+\mathcal{Q}(a)x+\mathcal{R}(a), \tag{5.4}$$

with

$$\mathcal{P}(a)\equiv\frac{\gamma\mathcal{C}}{a}, \quad \mathcal{Q}(a)\equiv-\frac{3\gamma}{a}, \quad \mathcal{R}(a)\equiv\frac{(2-3\gamma)k}{a^3}. \tag{5.5}$$

This is Riccati's equation (3.5), as desired.

When $n=2$, on the other hand, the choice $r=2$ leads to $l=1$ and $m=0$, whereupon Eq. (5.3) reduces to

$$\frac{dx}{da}+\mathcal{S}(a)x=\mathcal{T}(a), \tag{5.6}$$

with:

$$\mathcal{S}(a)\equiv\frac{\gamma(3-\mathcal{C})}{a}, \quad \mathcal{T}(a)\equiv\frac{(2-3\gamma)k}{a^3}. \tag{5.7}$$

This case is linear, and may be solved easily.

For other values of n, a check reveals that no apparent choice of r will put Eq. (5.3) into a form which can be solved analytically for x (i.e., for H). We therefore concentrate on the cases $n=2$ and $n=4$ for the time being, leaving the others to future numerical analysis. In both cases the Hubble parameter is given by $H=\sqrt{x}$ (since $r=2$).

C. The case $n=2$

Multiplying through by a factor of $\exp[\int S(a)da]$ $=a^{\gamma(3-C)}$ puts Eq. (5.6) into exact form, which may be integrated directly for $x(a)$ and hence $H(a)$:

$$H(a)=\left\{\left[\frac{(2-3\gamma)k}{\gamma(3-C)-2}\right]a^{-2}+C_0a^{\gamma(C-3)}\right\}^{1/2}, \quad (5.8)$$

where C_0 is a constant of integration. Using the boundary condition $H(a_0)=H(1)=H_0$ to eliminate C_0, we obtain:

$$H(a)=H_0[\alpha a^{\gamma(C-3)}+\beta a^{-2}]^{1/2}, \quad (5.9)$$

where

$$\alpha\equiv 1-\beta, \quad \beta\equiv\left[\frac{2-3\gamma}{\gamma(3-C)-2}\right]\frac{k}{H_0^2}. \quad (5.10)$$

The parameters α,β and C can all be fixed in terms of observable quantities at the present time $t=t_0$, as follows. With $n=2$, the decay law (5.1) gives $C=\Lambda_0/H_0^2=3\lambda_0$, where we have used the definition (2.15). Substituting this result into Eqs. (5.10) and (5.9), we find:

$$\alpha=\left[\frac{(3\gamma/2-1)\Omega_0-\lambda_0}{(3\gamma/2)(1-\lambda_0)-1}\right], \quad \beta=1-\alpha, \quad (5.11)$$

$$\frac{da}{d\tau}=[\alpha a^{2-3\gamma(1-\lambda_0)}+\beta]^{1/2}, \quad (5.12)$$

where we have used Eq. (2.16) and set $a_0=1$ as usual.

D. Evolution of the scale factor

For special values of γ and λ_0, it may be possible to solve Eq. (5.12) in terms of elliptic (or simpler) integrals. For general purposes, however, we take the same approach as in Sec. IV and evolve the scale factor numerically in terms of its first and second derivatives. The latter of these is given by:

$$\frac{d^2a}{d\tau^2}=\alpha\left[1-\frac{3\gamma}{2}(1-\lambda_0)\right]a^{1-3\gamma(1-\lambda_0)}. \quad (5.13)$$

Equations (5.12) and (5.13) can be substituted into the Taylor expansion (4.9) and integrated backward numerically to determine the behavior of the scale factor.

We also wish to determine the conditions under which the Universe evolves backward to a nonzero minimum scale factor, $a=a_*$. Setting $da/d\tau=0$ (at $a=a_*$), we find from Eq. (5.12) that:

$$a_*^{3\gamma(1-\lambda_0)-2}=-\alpha/\beta. \quad (5.14)$$

It is convenient to distinguish two cases, according to whether the exponent on a_* is positive or negative. Using Eqs. (5.11), we obtain:

$$a_*=\begin{cases}\left[\dfrac{2\lambda_0-(3\gamma-2)\Omega_0}{(3\gamma-2)(1-\Omega_0-\lambda_0)}\right]^{1/[3\gamma(1-\lambda_0)-2]}\\ \quad\text{if}\quad\lambda_0<\lambda_c\\[2mm] \left[\dfrac{(3\gamma-2)(1-\Omega_0-\lambda_0)}{2\lambda_0-(3\gamma-2)\Omega_0}\right]^{1/[2-3\gamma(1-\lambda_0)]}\\ \quad\text{if}\quad\lambda_0>\lambda_c,\end{cases} \quad (5.15)$$

where we have defined $\lambda_c\equiv 1-2/3\gamma$ (leaving the case $\lambda_0=\lambda_c$ aside for the time being).

From Eq. (5.15) we draw a number of important conclusions: (1) Spatially *flat* solutions ($\Omega_0+\lambda_0=1$) have either $a_*=\infty$ (if $\lambda_0<\lambda_c$) or $a_*=0$ (if $\lambda_0>\lambda_c$). The former case is not interesting. The latter case is de Sitter–like, with the initial singularity pushed back into the infinite past. We have encountered this kind of solution before (Sec. III D 3).

(2) With the modest assumption that $\gamma>2/3$ (i.e., normal, non-inflationary matter), we notice that *closed* solutions (i.e., $\lambda_0>1-\Omega_0$) must satisfy $\lambda_0<(3\gamma/2-1)\Omega_0$, while *open* ones ($\lambda_0<1-\Omega_0$) obey $\lambda_0>(3\gamma/2-1)\Omega_0$. These conclusions follow from requiring that the terms in square brackets be positive; i.e., from the requirement that a_* be a real number. (The exception in which the exponent is an even integer $\wp$ occurs only for special values of the lambda parameter, $\lambda_0=1-2/3\gamma-1/6\wp\gamma$, and will not be considered further here.)

(3) Requiring that $0<a_*<1$, we learn that both the numerators and denominators in Eq. (5.15) must be positive. Comparing their relative absolute magnitudes, we distinguish two possibilities: (a) if $\lambda_0<\lambda_c$, then both the numerator and denominator must be positive, since otherwise $(3\gamma-2)(1-\Omega_0-\lambda_0)<2\lambda_0-(3\gamma-2)\Omega_0$, which reduces to $\lambda_0>\lambda_c$, contrary to the hypothesis. On the other hand, (b) if $\lambda_0>\lambda_c$, then we *also* find that both numerator and denominator must be positive, since otherwise $(3\gamma-2)(1-\Omega_0-\lambda_0)>2\lambda_0-(3\gamma-2)\Omega_0$, which reduces to $\lambda_0<\lambda_c$, again contrary to the hypothesis. Therefore both the numerators and denominators are positive in all cases.

(4) It follows from the conclusion (3), in conjunction with the assumption $\gamma>2/3$, that:

$$\lambda_0<1-\Omega_0. \quad (5.16)$$

In other words, to realistically describe the present Universe, models with a nonzero minimum scale factor must, in the present theory, be *open*. While some nonsingular open solutions have been found in theories employing scalar fields [97] and higher-order curvature terms [98], we are not aware of precedents for this in theories based on the cosmological term.

(5) By combining Eq. (5.16) with the conclusion (2), we infer that:
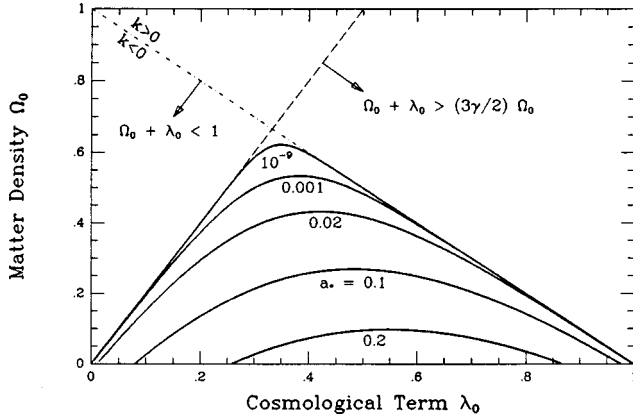
$$\lambda_0>(3\gamma/2-1)\Omega_0, \quad (5.17)$$

FIG. 11. Phase space diagram for the case $n=2$ with $\gamma=1$, showing contours of equal minimum size a_*.

FIG. 12. Evolution of the scale factor for models with $n=2, \gamma=1$, $\Omega_0=0.3$, and values of λ_0 labelled beside each curve.

which sets a lower limit on the size of the cosmological term.

(6) Equations (5.16) and (5.17) together impose an upper limit:

$$\Omega_0 < 2/3\gamma, \qquad (5.18)$$

on the matter density of the Universe.

E. Minimum values of the scale factor

The information contained in Eqs. (5.15)–(5.18) is summarized in Fig. 11, which is a phase space plot like Fig. 9, but enlarged to show only the range of interest, $0 \leqslant \Omega_0 \leqslant 1$ and $0 \leqslant \lambda_0 \leqslant 1$. Models with (λ_0, Ω_0) are represented by points on this diagram, as usual. The critical values of λ_0 in this theory define the upper edges of the triangular region at the base of the diagram; i.e., the region bounded by the curves $\Omega_0 + \lambda_0 < 1$ (dash-dotted line) and $\Omega_0 + \lambda_0 > (3\gamma/2)\Omega_0$ (dashed line). All models between these curves are nonsingular, with real values of a_* in the range $0 < a_* < 1$, as we have stipulated. Figure 11 is plotted for $\gamma = 1$.

Using Eq. (5.15), we have plotted contours of equal minimum scale factor a_* in this region (heavy solid lines). Any point along one of the contours corresponds to an oscillating model with the labelled value of a_*. Following the discussion in Sec. IV E, we would like to find models in which a_* is as small as possible, in order to obtain the largest possible maximum observable redshift $z_{obs} \leqslant a_*^{-1} - 1$. For instance, to be compatible with observations of quasars ($z_{obs} \approx 10$), a_* must be less than about 0.1. If we wish to explain the CMB as relic radiation from the last scattering surface at $z_{lss} \approx 1100$, then we require a smaller minimum scale factor, $a_* < 0.001$. And to meet the demand that the early Universe heat up to nucleosynthesis temperatures $[T_{nuc} \approx T_0(a_0/a_*) \approx 10^{10}$ K$]$, our model must satisfy

$$a_* < 10^{-9} \quad \text{(nucleosynthesis)} . \qquad (5.19)$$

Figure 11 demonstrates that the present theory can readily satisfy this constraint. Any model lying along the curve labelled 10^{-9} will be capable, in principle, of reaching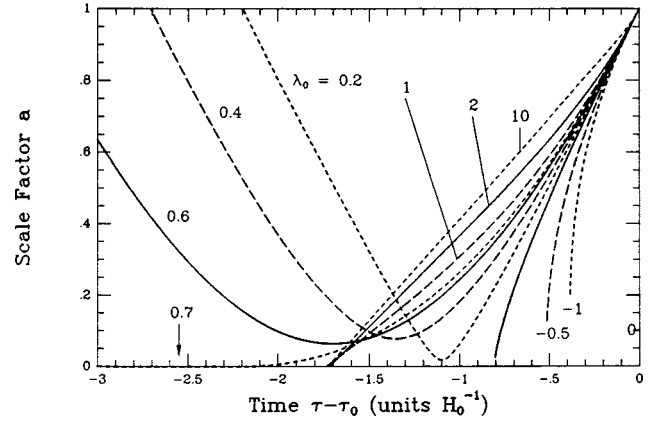 these temperatures near the "big bounce." This confirms comments made by several authors [99] that there is no reason in principle why the oscillations in a nonsingular model cannot be deep enough to account for all the evidence which is usually taken as proof that the Universe began in a singularity. Moreover, the only observational constraint which seriously limited the variable-Λ models in the last section — the lensing constraint — does not apply in this section because there is no antipode in an open universe.

To meet the condition (5.19), models in this theory must lie close to the upper edge of the triangular region in Fig. 11. To an extent this is "fine-tuning." However, it also allows us to make very definite predictions (as in Sec. IV) about the values of λ_0 that would be required in a realistic oscillating model. As an example, let us consider the observationally favored value of $\Omega_0 = 0.3$ [87], and let us assume $\gamma = 1$ as usual. Tracing horizontally across the line defined by $\Omega_0 = 0.3$ in Fig. 11, we can see that the nucleosynthesis condition (5.19) is met by only *two* values of the cosmological term: $\lambda_0 \approx 0.15$ and $\lambda_0 \approx 0.7$. Of these, the larger value is only marginally viable, being very close to the observational upper bounds described in Sec. IV D 1. The smaller value, however, is perfectly acceptable from an observational standpoint.

In general, the theory predicts that the most likely value of λ_0 is either *just below*:

$$1 - \Omega_0, \qquad (5.20)$$

or else *just above*:

$$(3\gamma/2 - 1)\Omega_0; \qquad (5.21)$$

which is to say, just above $\Omega_0/2$ in a dust-like universe ($\gamma = 1$). The former situation might be preferable to some on theoretical grounds [92], while the latter is in better agreement with the increasingly stringent observational upper limits on λ_0 (Sec. IV D 1).

To confirm that models with these features really do avoid the big bang, the solutions can be evolved backward in time as before, using Eqs. (5.12) and (5.13) with the Taylor series expansion (4.9). The results of this procedure are shown in Fig. 12 for the case $\Omega_0 = 0.3$ and $\gamma = 1$. This diagram is an

enlarged version of the evolution plots in Sec. IV (Figs. 5 and 10), showing only the past three Hubble times. Values of λ_0 are marked beside the appropriate curves. Several features can be noted.

First, the initial singularity is avoided for any value of λ_0 between 0.15 and 0.7, as expected on the basis of the phase space diagram, Fig. 11. In the limiting case where $\lambda_0 = 0.7$ exactly, which is spatially flat, we see that $a_* = 0$ (de Sitter–like behavior), as expected based on the discussion following Eq. (5.15).

Secondly, Fig. 12 confirms that the value of a_* is smallest near the critical values of λ_0: 0.064 for the $\lambda_0 = 0.6$ case (just below 0.7), and 0.018 for the $\lambda_0 = 0.2$ case (just above 0.15). These numbers have been chosen for illustrative purposes; smaller values of a_* (with consequently larger bounce temperatures) are obtained by letting λ_0 approach the critical values more closely.

Thirdly, this evolution plot gives us some information about the ages of the models; that is, the elapsed time since the big bang (or the big bounce, as appropriate). It may be seen that, within the range $0.15 < \lambda_0 < 0.7$, larger values of λ_0 correspond to older universes, as usual: 1.10 Hubble times for $\lambda_0 = 0.2$, and 1.70 Hubble times for $\lambda_0 = 0.6$. Even if H_0 takes on its largest currently acceptable value of 83 km s^{-1} Mpc^{-1} [81], the ages of the two models are 13.0 and 20.1 billion years respectively — well above the globular cluster limit of 9.6 Gyr [82].

F. The case $n = 4$

We proceed to the other case of interest, $n = 4$, which consists of the Riccati equation (5.4) for $x(a)$. This can be solved using standard techniques [79]. Switching dependent variables from x to y via $x = (-1/\mathcal{P}y)\, dy/da$, we obtain:

$$\frac{d^2y}{da^2} + \zeta a^{-1}\frac{dy}{da} + \eta a^{-4}y = 0, \qquad (5.22)$$

where:

$$\zeta \equiv 1 + 3\gamma, \qquad \eta \equiv \gamma\mathcal{C}(2 - 3\gamma)k. \qquad (5.23)$$

The parameter η can be connected to observation as follows. With $n = 4$, the decay law (5.1) gives $\mathcal{C} = \Lambda_0/H_0^4 = 3\lambda_0/H_0^2$. Substituting this result into the second of Eqs. (5.23), we find

$$\eta \equiv 3\gamma(2 - 3\gamma)\lambda_0(\Omega_0 + \lambda_0 - 1), \qquad (5.24)$$

where we have used Eq. (2.16) and set $a_0 = 1$ as usual.

Equation (5.22) is linear as desired, but not straightforward because of its variable coefficients. We can recast it in normal form by changing independent variables from a to $z \equiv \int e^{-\phi(a)}da$, where $\phi(a) \equiv \int (\zeta a^{-1})da = \zeta\ln a$. This procedure leads to the following differential equation for $y(z)$:

$$z^\mu \frac{d^2y}{dz^2} + \nu y = 0, \qquad (5.25)$$

where $z = (1 - \zeta)^{-1}a^{1-\zeta}$ and:

$$\mu \equiv \frac{2(2 - \zeta)}{1 - \zeta}, \qquad \nu \equiv \eta(1 - \zeta)^{-\mu}. \qquad (5.26)$$

Somewhat surprisingly, Eq. (5.25) has the same form as the differential equation (3.7) governing the solutions of the $\Lambda \propto \tau^{-l}$ models in Sec. III. (The two are identical if we put $z \to \tau$, $y \to x$, $\mu \to l$, and $\nu \to -\alpha$.)

So we could in principle bring over all the results of Sec. III, for the cases $\mu = 1, 2, 3, 4$ at least. However, combining the definitions (5.23) and (5.26), we find that $\mu = (2/3\gamma) \times (3\gamma - 1)$; or $\gamma = [3(1 - \mu/2)]^{-1}$. Therefore solutions obtained in this way would correspond to equations of state with γ-values of $2/3, \infty, -2/3$ and $-1/3$ respectively. These do not describe realistic forms of matter, at least not in the present universe [37]. Conversely, values of γ that *are* reasonable (such as $\gamma = 1$ or $4/3$) correspond to non-integral values of μ (such as $4/3$ and $3/2$ respectively). It is doubtful that Eq. (5.25) can be solved analytically in these cases. We therefore leave the possibility that $n = 4$ for future numerical analysis.

VI. Λ AS A FUNCTION OF THE DECELERATION PARAMETER

A. Evolution of the scale factor

We turn finally to the last of our phenomenological decay laws, Eq. (2.13), writing it in the form:

$$\Lambda = \mathcal{D}\left(\frac{\ddot{a}}{a}\right)^r. \qquad (6.1)$$

As far as we are aware, no such dependence has previously been considered for the cosmological term. However, it is a natural extension of the other decay scenarios considered so far. There is no fundamental difference between the first and second derivatives of the scale factor that would preclude the latter from acting as an independent variable if the former is acceptable.

Substituting the decay law (6.1) into Eq. (2.9), we find:

$$\mathcal{D}\left(aH\frac{dH}{da} + H^2\right)^r = \left(3 - \frac{2}{\gamma}\right)\left(H^2 + \frac{k}{a^2}\right) + \frac{2}{\gamma}\left(aH\frac{dH}{da} + H^2\right). \qquad (6.2)$$

We adopt the value $r = 1$ for the remainder of Sec. VI, since we would like to solve for the Hubble parameter in analytic form in order to make use of of the Taylor expansion (4.9). Equation (6.2) then takes the form:

$$\frac{dH}{da} = \frac{\gamma}{a}\left(\frac{3 - \mathcal{D}}{\gamma\mathcal{D} - 2}\right)H + \frac{k}{a^3}\left(\frac{3\gamma - 2}{\gamma\mathcal{D} - 2}\right)H^{-1}. \qquad (6.3)$$

As in Sec. V B, let us make a change of dependent variables from H to $x \equiv H^s$, where s is an arbitrary constant. Equation (6.3) then takes the form:

$$\frac{dx}{da} = \frac{s\gamma}{a}\left(\frac{3 - \mathcal{D}}{\gamma\mathcal{D} - 2}\right)x + \frac{sk}{a^3}\left(\frac{3\gamma - 2}{\gamma\mathcal{D} - 2}\right)x^{(s-2)/s}. \qquad (6.4)$$

If we choose $s = 2$, this is reduced to the linear form (5.6), with:

$$\mathcal{S}(a) \equiv 2\gamma \left(\frac{\mathcal{D} - 3}{\gamma \mathcal{D} - 2} \right) a^{-1},$$

(6.5)

$$\mathcal{T}(a) \equiv 2k \left(\frac{3\gamma - 2}{\gamma \mathcal{D} - 2} \right) a^{-3}.$$

Multiplying through by a factor of $\exp[\int \mathcal{S}(a) da] = a^{2\gamma(\mathcal{D} - 3)/(\gamma \mathcal{D} - 2)}$ and solving exactly as in Sec. V C, we obtain for the Hubble parameter:

$$H(a) = [C_0 a^{-2\gamma(\mathcal{D} - 3)/(\gamma \mathcal{D} - 2)} - k a^{-2}]^{1/2},$$

(6.6)

where C_0 is a constant of integration. Imposing the boundary condition $H(a_0) = H(1) = H_0$, we find with the help of Eq. (2.16) that $C_0 = H_0^2 + k = H_0^2(\Omega_0 + \lambda_0)$. We can also fix $\mathcal{D}$ in terms of observable quantities. With $r = 1$, the decay law (6.1) gives $\mathcal{D} = \Lambda_0(\ddot{a}/a)_{t=t_0} = -\Lambda_0/H_0^2 q_0 = -3\lambda_0/q_0$, where q_0 is the present value of the deceleration parameter, and we have used the definitions (2.15) and (2.18). Substituting this result into Eq. (6.6) along with Eq. (2.16), and recalling that $H = (H_0/a) da/d\tau$, we find:

$$\frac{da}{d\tau} = [\alpha a^\xi + \beta]^{1/2},$$

(6.7)

where:

$$\alpha = (\Omega_0 + \lambda_0), \quad \beta = 1 - \alpha,$$

(6.8)

$$\xi \equiv \frac{(2 - 3\gamma) q_0}{q_0 + (3\gamma/2)\lambda_0}.$$

This expression, together with its time derivative:

$$\frac{d^2 a}{d\tau^2} = \frac{\xi}{2} (\Omega_0 + \lambda_0) a^{\xi - 1},$$

(6.9)

can be substituted into the Taylor expansion (4.9).

B. Minimum values of the scale factor

As in Sec. V D, we require that oscillating models satisfy $da/d\tau = 0$ at $a = a_*$. In conjunction with Eq. (6.7), this implies:

$$a_*^\xi = \left(\frac{\Omega_0 + \lambda_0 - 1}{\Omega_0 + \lambda_0} \right).$$

(6.10)

Let us write this out explicitly using the last of Eqs. (6.8). As with Eq. (5.15), we will find it convenient to distinguish two possible cases:

$$a_* = \begin{cases} \left(\dfrac{\Omega_0 + \lambda_0}{\Omega_0 + \lambda_0 - 1} \right)^{[(1 + (3\gamma\lambda_0/2q_0))/(3\gamma - 2)]} & \text{if } q_0 > q_c \\[2em] \left(\dfrac{\Omega_0 + \lambda_0 - 1}{\Omega_0 + \lambda_0} \right)^{[(-(3\gamma\lambda_0/2q_0) - 1)/(3\gamma - 2)]} & \text{if } q_0 < q_c, \end{cases}$$

(6.11)

where we have defined $q_c \equiv -(3\gamma/2)\lambda_0$ (leaving the case $q_0 = q_c$ aside for the time being).

We can draw a number of useful conclusions from the form of Eq. (6.11). First, (1) that spatially *flat* solutions ($\Omega_0 + \lambda_0 = 1$) again have either $a_* = \infty$ (if $q_0 > q_c$) or $a_* = 0$ (if $q_0 < q_c$). This is just as in the previous section (Sec. V D).

(2) Secondly, requiring real values for a_* (subject to the same proviso about even-numbered integer exponents as in Sec. V D), we can conclude that *closed* solutions (i.e., $\lambda_0 > 1 - \Omega_0$) must satisfy $\lambda_0 > -\Omega_0$, while *open* ones ($\lambda_0 < 1 - \Omega_0$) obey $\lambda_0 < -\Omega_0$. It follows that, if λ_0 is a positive quantity, as observations almost certainly indicate (Sec. IV D 2), then models with a nonzero minimum scale factor must, in the present theory, be *closed*. We will assume that both these conditions hold in the remainder of Sec. VI.

(3) Thirdly, requiring that $0 < a_* < 1$ as before, we learn that the deceleration parameter q_0 satisfies

$$q_0 < q_c.$$

(6.12)

This follows from the fact that $|\Omega_0 + \lambda_0 - 1|$ cannot be greater than $|\Omega_0 + \lambda_0|$ (assuming that Ω_0 and λ_0 are both positive).

(4) Finally, combining the conclusion (3) with the definition of q_c, we infer that

$$q_0 < -(3\gamma/2)\lambda_0.$$

(6.13)

Assuming as we are that the cosmological term is positive, this implies that the deceleration parameter must be *negative* for a universe filled with normal matter ($\gamma > 2/3$).

Unfortunately, the deceleration parameter q_0 remains among the most poorly-constrained quantities in observational cosmology. Nevertheless, it is fair to say that the majority opinion among cosmologists holds that q_0 is probably positive [100]. The most recent experimental determination, obtained from type Ia supernovae, leads to a value of $q_0 = 0.385 \pm 0.36$ [101]. Since oscillating models in the present theory not only have $q_0 < 0$, but $k > 0$ as well, they are somewhat disfavored in comparison to those of Sec. IV and Sec. V; and we judge that this is a reasonable place to halt our investigation for the time being.

VII. CONCLUSIONS

We have examined the evolution of the scale factor $a(t)$ in the presence of a variable cosmological term Λ, and also extended existing treatments by adopting a fairly general equation of state for ordinary matter.

A number of new exact solutions for $a(t)$ have been obtained in cases where $\Lambda \propto t^{-l}$. These models are singular, but can be significantly older than those in which $\Lambda = $ const. For

odd values of l, the cosmological term must be negative (or zero) if the scale factor is to be real-valued. Our conclusions may not extend to cases in which $k \neq 0$; this will require more detailed numerical analysis.

For a cosmological term that scales as $\Lambda \propto a^{-m}$, we have solved numerically for the scale factor as a function of time, and found that there are closed models which are compatible with observation and contain no big bang. This is in sharp contrast to the situation where $\Lambda = \text{const}$, for which experimental evidence firmly establishes the existence of an initial singularity. (The variation effectively allows one to obtain a large Λ term where it is most important — near the ''big bounce'' — without the price of a large cosmological constant at present times.) This appears not to have been widely appreciated, probably because variable cosmological terms have so far been studied almost exclusively in the context of the cosmological ''constant'' problem. We have obtained constraints from experimental upper limits on Λ_0, as well as requirements of sufficient age, normal gravitational lensing at high redshifts, and others. As specific numerical examples, oscillating models with zero pressure, $\Omega_0 = \{0.34, 0.51, 0.61, 0.67\}$ and λ_0 in the ranges $\{0.68–0.72, 0.51–0.57, 0.41–0.49, 0.39–0.48\}$ are observationally viable if $m = 1, 1.5, 1.8$ or 1.9 respectively. If the bounce is to be deep enough to generate the temperatures required by conventional nucleosynthesis, then $m \gtrsim 1.8$.

We have also solved numerically for the scale factor when $\Lambda \propto H^n$. In this case we have found *open* models which can account for the observational data despite their lack of an initial singularity. In particular, oscillating models with zero pressure and values of λ_0 either just above $\Omega_0/2$ or just below $1 - \Omega_0$ are viable if $\Omega_0 < 2/3$ and $n = 2$. (We have investigated only the cases $n = 2$ and 4 in detail.) If $\Omega_0 \approx 0.3$ and $\lambda_0 \approx 0.15$ (or 0.7), for example, then the most recent ''big bounce'' could have been deep enough to account for phenomena such as the cosmic microwave background radiation and light element synthesis in a model with $n = 2$.

For a cosmological term that depends on the *deceleration parameter* via $\Lambda \propto q^r$, we have solved only the case $r = 1$. Closed oscillating models are possible, but require that q be negative at the present time, if the cosmological term is positive.

ACKNOWLEDGMENTS

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