Superconducting cosmic strings with exotic spacetime

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The metric for a superconducting cosmic string with constant momentum has been calculated analytically and numerically using the full Einstein field equations. We show that the string metric has mixed components, which for heavy strings ($\eta_\phi \sim 10^{18}$ GeV) describe a spacetime with exotic properties. In the string spacetime particles are deflected (in the z-direction) as they approach the string, effectively isolating the defect from the outside Universe. The inclusion of gravity into models of charged strings and vortons is expected to have significant cosmological implications. [S0556-2821(98)01216-8]

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I. INTRODUCTION

Many grand unified theories (GUTs) predict that the early Universe underwent successive phase transitions. It is conjectured that these phase transitions were accompanied by the formation of topological defects, such as cosmic strings [1] which are described by the Abelian Higgs model [2]. Cosmic strings are of interest due to their unusual properties and cosmological consequences.

Cosmic strings possessing an internal structure may under certain conditions exhibit superconductivity [3]. The origin of superconductivity can be due to bosonic, fermonic [3] or non-Abelian gauge field currents [4]. Superconducting cosmic strings are of interest since the supercurrents are expected to significantly alter string spacetime and string dynamics. Supercurrents in the string can also have a nonzero momentum in the direction of the current flow (i.e., along the string), and this will modify the properties of cosmic strings and string loops. Davis and Shellard [5] demonstrated that charged superconducting loops (i.e., vortons) have sufficient angular momentum to stabilize centrifugally. Vortons have been studied by many authors (see, e.g., [5–10]), and it is thought that these loops have adverse cosmological consequences, particularly for primordial nucleosynthesis [10].

Since superconducting cosmic string loops have significant ramifications for cosmology, the properties of superconducting strings (with zero momentum) have been investigated analytically [11] and numerically [12]. The spacetime of charged superconducting cosmic strings has also been examined in the weak field approximation [13]. However, previous models neglect momentum when constructing the energy-momentum tensor; consequently the spacetime properties of a cosmic string with nonzero momentum have not been examined. In this paper we show that the inclusion of momentum leads to exotic spacetime properties, which may have significant cosmological consequences. We present analytical and numerical solutions to the full Einstein field equations for cosmic strings with constant momentum. The presence of nonzero momentum results in the spacetime metric acquiring mixed metric components, which give rise to oscillatory behavior for g_{tt}, g_{zz} and g_{tz}. The oscillation of the metric components also forms multiple ''horizons'' in r at various distances from the cosmic string. Particles are found to be deflected at these horizons and hence the cosmic string is effectively isolated from the outside Universe.

The organization of this paper is as follows. In Sec. II we examine a weak field approximation to a cosmic string and show that the inclusion of momentum leads directly to mixed metric components. In Sec. III we derive the field equations which govern the spacetime of a superconducting cosmic string with a bosonic supercurrent. The qualitative results are applicable to any string object which has momentum (e.g., fermonic superconductivity). In Sec. IV we present analytical and numerical solutions to the Einstein field equations which elucidate the oscillatory behavior of the metric. The main results and implications of the model are summarized in Sec. V.

II. WEAK FIELD SOLUTION

Insight into how momentum affects the spacetime properties of a cosmic string can be gained by examining a weak field solution. To model a cosmic string with momentum, we modify the energy-momentum tensor used by Vilenkin [14] and Hiscock [15], by adding components, $T_{tz}=T_{zt}=-M\delta(x)\delta(y)$, which describe the momentum per unit length in the string along the z-direction. Adopting Cartesian coordinates (t,x,y,z) with metric signature diag$(+,-,-,-)$, the energy-momentum tensor becomes (in natural units $\hbar=c=1$, with $G=6.72\times10^{-39}$ GeV^{-2}),

$$T_{\mu\nu}=\delta(x)\,\delta(y)\begin{bmatrix} E & 0 & 0 & -M \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -M & 0 & 0 & -S \end{bmatrix}, \tag{1}$$

where E, M and S are the energy, momentum and tension per unit length, respectively. To solve for the metric we employ the weak field approximation

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}, \tag{2}$$

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where $|h_{\mu\nu}|\ll 1$, and utilize the Fock–de Donder gauge, $\partial_\nu h^\nu_\mu - \frac{1}{2}\partial_\mu h^\nu_\nu = 0$. The resulting weak field equations are

$$\frac{1}{2}\Box h_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right), \tag{3}$$

where $T=\eta^{\mu\nu}T_{\mu\nu}$.

Substituting for the energy-momentum tensor (1), the weak field equations reduce to a set of differential equations of the form

$$\nabla^2 h_{tt} = \nabla^2 h_{zz} = 8\pi(E-S)\delta(x)\delta(y) \tag{4a}$$

$$\nabla^2 h_{xx} = \nabla^2 h_{yy} = 8\pi(E+S)\delta(x)\delta(y) \tag{4b}$$

$$\nabla^2 h_{tz} = -16\pi M\delta(x)\delta(y), \tag{4c}$$

where $h_{tz}=h_{zt}$, since $T_{tz}=T_{zt}$. The solution to the weak field equations can be written as

$$h_{tt} = h_{zz} = 4G(E-S)\ln(r/\delta) \tag{5a}$$

$$h_{xx} = h_{yy} = 4G(E+S)\ln(r/\delta) \tag{5b}$$

$$h_{tz} = h_{zt} = -8GM\ln(r/\delta), \tag{5c}$$

where $r^2=x^2+y^2$ and δ denotes the width of the cosmic string. Following [14] we exploit rescaled, cylindrical coordinates defined by

$$[1+4G(E-S)\ln(r/\delta)]dt^2 = d\nu^2 \tag{6a}$$

$$[1-4G(E-S)\ln(r/\delta)]dz^2 = d\zeta^2 \tag{6b}$$

$$[1-4G(E+S)\ln(r/\delta)]dr^2 = d\rho^2 \tag{6c}$$

$$[1-4G(E+S)\ln(r/\delta)]r^2 = [1-4G(E+S)]\rho^2. \tag{6d}$$

In terms of the rescaled coordinates, the weak field approximation to the metric for a straight cosmic string with nonzero constant momentum becomes

$$ds^2 = d\nu^2 - d\zeta^2 - d\rho^2 - [1-4G(E+S)]\rho^2 d\theta^2 - \frac{16GM\ln(r/\delta)}{\sqrt{1-[4G(E-S)\ln(r/\delta)]^2}}d\nu d\zeta. \tag{7}$$

From Eq. (7), it is evident that the existence of nonzero momentum leads to a conical metric with a mixed g_{tz} metric component. Mixed metric components involving timelike and spacelike coordinates will affect particle geodesics, in which case we would expect particles to be dragged along the z-direction. To describe the motion of a photon we calculate the null geodesic for the ζ coordinate by varying the metric with respect to an affine parameter τ, i.e.,

$$\dot{\zeta} = -\frac{A\Phi+B}{1+\Phi^2}, \tag{8}$$

where an overdot denotes differentiation with respect to the affine parameter, A and B are constants and

$$\Phi = \frac{16GM\ln(r/\delta)}{\sqrt{1-[4G(E-S)\ln(r/\delta)]^2}}. \tag{9}$$

In general, $\dot{\zeta}\neq 0$ and hence photons in the vicinity of a string which has nonzero momentum will be dragged along the z-direction.

For GUT scale cosmic strings $G(E+S)\sim 10^{-6}$, and hence the weak field approximation should be an adequate description of the string metric. However, it is expected that the metric properties will alter significantly in the full Einstein field equations. In Sec. III we explore this conjecture by analyzing the field equations of general relativity for a cylindrically symmetric system with nonzero momentum.

III. FIELD EQUATIONS

To describe a superconducting cosmic string (with nonzero momentum), we employ a model of bosonic superconducting cosmic strings due to Witten [3]. The numerical evaluation of the metric is made tractable by considering a cosmic string with a neutral current, which does not couple to a gauge field. It is important to emphasize that the metric properties discussed in this paper are a consequence of the nonzero momentum of the string and the concomitant mixed metric components. The model of a superconducting string with a neutral bosonic supercurrent is chosen purely for convenience, and the results are applicable to fermonic superconducting cosmic strings.

The Lagrangian for a superconducting cosmic string with a neutral bosonic current is given by

$$\mathcal{L} = |D_\mu\phi|^2 + |\nabla_\mu\sigma|^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\phi,\sigma), \tag{10}$$

where ∇_μ is the conventional covariant derivative, ϕ is the complex Higgs field, σ describes the bosonic current field, $D_\mu=\nabla_\mu - ieA_\mu$ is the gauge covariant derivative and the electromagnetic field tensor is given by $F_{\mu\nu}=\nabla_\mu A_\nu - \nabla_\nu A_\mu$. The potential is

$$V(\phi,\sigma) = \frac{\lambda_\phi}{4}(|\phi|^2-\eta_\phi^2)^2 + \frac{\lambda_\sigma}{4}(|\sigma|^2-\eta_\sigma^2)^2 + \beta|\phi|^2|\sigma|^2 - \frac{\lambda_\sigma}{4}\eta_\sigma^4, \tag{11}$$

where λ_ϕ, η_ϕ, λ_σ and η_σ are constants which define the form of the potential for the string and current fields. The two fields are coupled together through a parameter β.

To ensure that the Higgs field undergoes symmetry breaking first, we introduce the constraint $\lambda_\phi\eta_\phi^4 > \lambda_\sigma\eta_\sigma^4$. Requiring the current field to vanish outside the string introduces a second constraint, $2\beta/\lambda_\sigma \geq \eta_\sigma^2/\eta_\phi^2$. A final constraint, $\beta/\lambda_\sigma < \lambda_\sigma\eta_\sigma^4/4\lambda_\phi\eta_\phi^4$, ensures that $|\sigma|=0$ is unstable in the presence of a string [16].

We generalize Eq. (7) to describe the metric for a stationary cylindrically symmetric string with constant momentum, i.e.,

$$ds^2 = \Lambda(r)dt^2 - 2\Phi(r)dtdz - dr^2 - \Psi(r)^2 d\theta^2 - \Omega(r)dz^2, \tag{12}$$

where the string is orientated along the z-direction and $g_{tz} = g_{zt}$. Invoking an argument due to Garfinkle [17] allows the metric component g_{rr} to be set to unity without loss of generality.

To calculate the vortex solution for a superconducting cosmic string with constant momentum, we exploit the cylindrical symmetry to write

$$\phi = \eta_\phi f(r)\exp(in\theta) \tag{13a}$$

$$A_\theta = \frac{1-\sqrt{\lambda_\phi}\eta_\phi a_\theta(r)}{e\Psi(r)} \tag{13b}$$

$$\sigma = \eta_\phi s(r)\exp[i(kz-\omega t)], \tag{13c}$$

where k and ω are constants which describe the string's momentum (see Sec. IV). For simplicity the phase winding number n of the Higgs field is set to 1. The model parameters are rescaled according to $\beta = \lambda_\phi \overline{\beta}$, $\lambda_\sigma = \lambda_\phi \overline{\lambda}_\sigma$, $\eta_\sigma = \eta_\phi \overline{\eta}_\sigma$, $k^2 = \lambda_\phi \eta_\phi^2 \overline{k^2}$, $\omega^2 = \lambda_\phi \eta_\phi^2 \overline{\omega^2}$ and $e^2 = \lambda_\phi \overline{e^2}$, and the radial coordinate is rescaled to $r = \overline{r}/(\sqrt{\lambda_\phi}\eta_\phi)$, so that distances are measured in terms of the string width $\delta = (\sqrt{\lambda_\phi}\eta_\phi)^{-1}$. Finally, for the numerical calculations, we have chosen $\overline{\beta} = 5.0$, $\overline{\lambda}_\sigma = 40.0$ and $\overline{\eta}_\sigma = 0.387$, and for convenience we set $e^2 = \lambda_\phi$ so that $\overline{e} = 1.0$.

The equations of motion are obtained from the Lagrangian (10) by varying $\overline{\phi}$, A^μ and $\overline{\sigma}$, respectively. Substituting the cylindrically symmetric forms of ϕ, A_θ and σ into the equations of motion produces the following set of equations (in the Lorentz gauge):

$$\nabla_{\overline{r}}^2 f - \frac{f a_\theta^2}{\Psi^2} - \frac{1}{2} f(f^2-1) - \overline{\beta} f s^2 = 0 \tag{14a}$$

$$\frac{\Omega a_\theta''}{\Phi^2+\Lambda\Omega} + \left(\frac{\Psi\Omega(\Lambda\Omega' - \Omega\Lambda' + 2\Phi^2 - 2\Phi\Phi') - 2\Omega(\Phi^2+\Lambda\Omega)\Psi'}{2\Psi(\Phi^2+\Lambda\Omega)^2} \right) a_\theta' - 2\overline{e^2} f^2 a_\theta = 0 \tag{14b}$$

$$\nabla_{\overline{r}}^2 s + \frac{\Omega s \overline{\omega^2} - \Lambda s \overline{k^2} + 2\Phi s \overline{k\omega}}{\Phi^2+\Lambda\Omega} - \frac{\overline{\lambda}_\sigma}{2} s(s^2 - \overline{\eta}_\sigma^2) - \overline{\beta} s f^2 = 0, \tag{14c}$$

where a dash indicates differentiation with respect to $\overline{r}$ and $\nabla_{\overline{r}}^2$ is given by

$$\nabla_{\overline{r}}^2 = \frac{\partial^2}{\partial \overline{r}^2} + \left(\frac{2\Phi^2\Psi' + 2\Lambda\Omega\Psi' + 2\Psi\Phi\Phi' + \Psi\Omega\Lambda' + \Psi\Lambda\Omega'}{2\Psi(\Phi^2+\Lambda\Omega)} \right) \frac{\partial}{\partial \overline{r}}. \tag{15}$$

The boundary conditions imposed on the string field require us to accommodate an undefined phase at the center of the string defect, and the vacuum solution in the far field, whence $f(0)=0$, $f(r\to\infty)\to 1$, $a_\theta(0)=1$, $a_\theta(r\to\infty)\to 0$, $s'(0)=0$ and $s(r\to\infty)\to 0$.

To evaluate the metric we calculate the energy-momentum tensor for the superconducting cosmic string,

$$T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}, \tag{16}$$

for which the tensor components are

$$\overline{T}_{tt} = 2s^2\overline{\omega^2} - \Lambda L \tag{17a}$$

$$\overline{T}_{rr} = 2(f')^2 + 2(s')^2 + \overline{e}^{-2}(a_\theta')^2\Psi^{-2} + L \tag{17b}$$

$$\overline{T}_{\theta\theta} = 2f^2 a_\theta^2 + \overline{e}^{-2}(a_\theta')^2 + \Psi^2 L \tag{17c}$$

$$\overline{T}_{zz} = 2s^2\overline{k^2} + \Omega L \tag{17d}$$

$$\overline{T}_{tz} = -4s^2\overline{k\omega} + \Phi L, \tag{17e}$$

where $T_{\mu\nu} = \lambda_\phi \eta_\phi^4 \overline{T}_{\mu\nu}$, $\overline{T}_{tz} = \overline{T}_{zt}$ and

$$L = -(f')^2 - (s')^2 + \frac{\Omega s^2\overline{\omega^2} - \Lambda s^2\overline{k^2} + 2\Phi s^2\overline{k\omega}}{\Phi^2+\Lambda\Omega} - \frac{f^2 a_\theta^2}{\Psi^2} - \frac{(a_\theta')^2}{2\overline{e^2}\Psi^2} - \frac{1}{4}(f^2-1)^2 - \frac{\overline{\lambda}_\sigma}{4}(s^2-\overline{\eta}_\sigma^2)^2 - \overline{\beta} f^2 s^2 + \frac{\overline{\lambda}_\sigma}{4}\overline{\eta}_\sigma^4. \tag{18}$$

The Einstein field equations can be written in the form

$$\overline{R}_{\mu\nu} = 4\pi G \eta_\phi^2 (2\overline{T}_{\mu\nu} - g_{\mu\nu}\overline{T}), \tag{19}$$

where $\overline{T} = g^{\mu\nu}\overline{T}_{\mu\nu}$ and $R_{\mu\nu} = \lambda_\phi \eta_\phi^2 \overline{R}_{\mu\nu}$. The field equations (19) become

$$\overline{R}_{tt}=\frac{\Lambda''}{2}+\frac{\Psi'\Lambda'}{2\Psi}+\frac{2\Lambda\Phi'^2+\Lambda\Lambda'\Omega'-2\Phi\Phi'\Lambda'-\Omega\Lambda'^2}{4(\Phi^2+\Lambda\Omega)} \tag{20a}$$

$$\overline{R}_{\theta\theta}=-\Psi\left(\Psi''+\frac{2\Phi\Psi'\Phi'+\Omega\Psi'\Lambda'+\Lambda\Psi'\Omega'}{2(\Phi^2+\Lambda\Omega)}\right) \tag{20b}$$

$$\overline{R}_{zz}=-\frac{\Omega''}{2}-\frac{\Psi'\Omega'}{2\Psi}-\frac{2\Omega\Phi'^2+\Omega\Lambda'\Omega'-2\Phi\Phi'\Omega'-\Lambda\Omega'^2}{4(\Phi^2+\Lambda\Omega)} \tag{20c}$$

$$\overline{R}_{tz}=-\frac{\Phi''}{2}-\frac{\Psi'\Phi'}{2\Psi}+\frac{\Omega\Phi'\Lambda'+\Lambda\Phi'\Omega'-2\Phi\Lambda'\Omega'}{4(\Phi^2+\Lambda\Omega)}, \tag{20d}$$

where $\overline{R}_{tz}=\overline{R}_{zt}$ and $\overline{R}_{rr}$ is redundant [17]. The boundary conditions for the metric components are chosen to ensure that the metric at the center of the string is smooth, whence $\Lambda(0)=1$, $\Lambda'(0)=0$, $\Phi(0)=0$, $\Phi'(0)=0$, $\Psi(0)=0$, $\Psi'(0)=1$, $\Omega(0)=1$ and $\Omega'(0)=0$.

IV. METRIC SOLUTION AND ITS PROPERTIES

Equations (14), (17), (20) and the attendant boundary conditions enable us to calculate the metric for a superconducting cosmic string with constant momentum. Because of the nonlinear nature of the equations, we use numerical techniques to explore the metric properties of realistic cosmic strings. Where appropriate, comparisons are made with approximate analytical solutions.

To emphasize the unusual properties of the metric, we consider a heavy string for which $G\eta_\phi^2=0.02$ ($\eta_\phi\sim10^{18}$ GeV). The momentum per unit length of the string is calculated from the T_{tz}-component (17e), by choosing the values of $\overline{k}$ and $\overline{\omega}$ according to

$$M=-2\pi\int_0^\infty T_{tz}\Psi\,dr=2\pi\lambda_\phi\eta_\phi^4\int_0^\infty(4s^2\overline{k\omega}-\Phi L)\Psi\,dr. \tag{21}$$

A typical vorton is conjectured to have $\overline{k}\sim\overline{\omega}$ [5]; however, it was found that for $\overline{k}=\overline{\omega}$ the spacetime curvature is sufficient to initiate current quenching. For cosmic strings with charged current carriers, the current is prevented from quenching due to charge conservation. In the case of neutral currents there exists a similar mechanism, which can be seen by writing the current field σ in the form

$$\sigma=s(t,r)\exp\{i[\overline{k}z-\kappa(t)]\}. \tag{22}$$

The imaginary part of the equation of motion becomes

$$2\dot{\kappa}\dot{s}+\ddot{\kappa}s=0, \tag{23}$$

where an overdot denotes differentiation with respect to t. Equation (23) has the solution

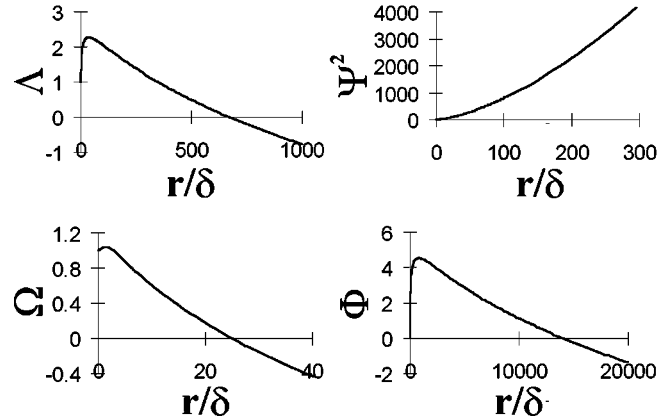

FIG. 1. Plot of the metric components for a superconducting cosmic string with constant momentum ($\overline{k}=0.85$ and $\overline{\omega}=1.15$). The radius is scaled by the string width δ. The mixed metric component Φ allows the t- and z-directions to interchange spacetime roles for negative values of Λ and Ω, respectively. Since Ω becomes negative before Λ, there exists a region which has two timelike metric components.

$$\dot{\kappa}=\frac{S_1}{s^2}, \tag{24}$$

where S_1 is a constant. As the current field quenches ($s\to0$), $\dot{\kappa}$ increases until it offsets the quenching. Typically for the static metric of a heavy string we have $\overline{\omega}>\overline{k}$. For the time independent metric considered in this paper we choose parameters for which $\overline{\omega}>\overline{k}$. To investigate the metric solution, the superconducting string vortex solution was evaluated numerically by relaxation techniques with a fourth-order Runge-Kutta scheme used to calculate the metric components. The coupled vortex and Einstein field equations were solved iteratively until the solution converged.

A. Oscillatory solutions

The metric solution with $\overline{k}=0.85$ and $\overline{\omega}=1.15$ is displayed in Fig. 1. This differs from previous string solutions [11–13] by the appearance of a mixed metric component g_{tz}, which we have shown to be a direct consequence of the string's momentum (Sec. II). The appearance of a nonzero g_{tz}-component allows g_{tt} and g_{zz} to become spacelike and timelike, respectively (i.e., $\Lambda<0$ and $\Omega<0$). This behavior is impossible for strings without momentum, which is seen by setting $\Phi(r)=0$ (i.e., $M=0$) in the field equations (20). For example, writing $\overline{R}_{tt}$ as

$$\overline{R}_{tt}=\Lambda''+\frac{\Psi'\Lambda'}{2\Psi}+\Lambda'\left(\frac{\Omega'}{\Omega}-\frac{\Lambda'}{\Lambda}\right), \tag{25}$$

reveals that far from the string (where $\overline{T}_{tt}=0$), as $\Lambda\to0$ we have $\Lambda''\to+\infty$, and hence the metric component Λ cannot become negative. A similar analysis also applies to Ω.

Extending the domain of the metric solution in r (see Fig. 2) reveals that the metric components, g_{tt}, g_{zz} and g_{tz}, os-

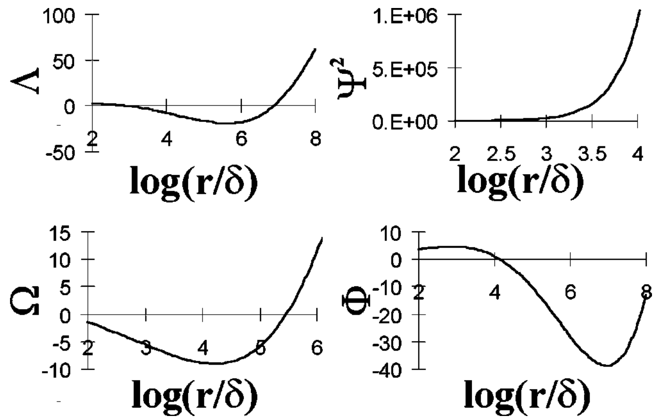

FIG. 2. Extended plot of the metric components in Fig. 1. The metric components are seen to revert to their original spacetime roles. Note that the spatial period and the amplitude of the oscillations increase with r and that the g_{zz} metric component reverts to spacelike before the g_{tt} component becomes timelike. This behavior produces a spatial region with no timelike metric component.

cillate successively between positive and negative values (i.e., between timelike and spacelike). The oscillations in the metric components appear to continue indefinitely, with the amplitude of each oscillation and the distance between successive interchanges quickly becoming large.

We can gain insight into the spacetime properties of Λ and Ω by finding analytical exterior solutions to the field equations [i.e., $T_{\mu\nu}(r>\delta)=0$]. To simplify the problem we will assume that $T_{tt}\approx T_{zz}$ everywhere, so that the field equations $R_{tt}=8\pi G T_{tt}$ and $R_{zz}=8\pi G T_{zz}$ are equivalent (i.e., $\Lambda=\Omega$). This assumption is reasonable for strings with small momentum since

$$\lim_{M\to 0}\left(\frac{T_{tt}}{T_{zz}}\right)=1, \tag{26}$$

for which the Einstein field equations become

$$\Psi''+\Psi'\left(\frac{\Phi\Phi'+\Lambda\Lambda'}{\Phi^2+\Lambda^2}\right)=0 \tag{27a}$$

$$\Lambda''+\frac{\Psi'\Lambda'}{\Psi}+\Phi'\left(\frac{\Lambda\Phi'-\Phi\Lambda'}{\Phi^2+\Lambda^2}\right)=0 \tag{27b}$$

$$\Phi''+\frac{\Psi'\Phi'}{\Psi}-\Lambda'\left(\frac{\Lambda\Phi'-\Phi\Lambda'}{\Phi^2+\Lambda^2}\right)=0, \tag{27c}$$

where we have made the additional assumption that $\Psi\neq 0$.

Equation (27a) can be simplified to

$$\Psi''+\frac{1}{2}\Psi'\frac{d\ln(\Phi^2+\Lambda^2)}{dr}=0. \tag{28}$$

We define an arbitrary function $K(r)$ in terms of Λ and Ω, by

$$\Phi^2+\Lambda^2=K(r). \tag{29}$$

We also define the function $C(r)$ according to

$$\Lambda\Phi'-\Phi\Lambda'=C(r)K(r). \tag{30}$$

The field equations (27) then reduce to

$$\Psi''+\frac{1}{2}\Psi'\frac{d\ln[K(r)]}{dr}=0 \tag{31a}$$

$$\Lambda''+\frac{\Psi'\Lambda'}{\Psi}+C(r)\Phi'=0 \tag{31b}$$

$$\Phi''+\frac{\Psi'\Phi'}{\Psi}-C(r)\Lambda'=0. \tag{31c}$$

Equation (31a) can be solved immediately to give

$$\Psi'=\frac{P_1}{\sqrt{K}}, \tag{32}$$

where P_1 is a constant and $K\neq 0$.

After differentiation, Eq. (29) becomes

$$\Phi\Phi'+\Lambda\Lambda'=K'/2. \tag{33}$$

Solving Eq. (33) and Eq. (30) simultaneously yields the differential equations

$$\Lambda'=\frac{K'}{2K}\Lambda-C\Phi \tag{34a}$$

$$\Phi'=\frac{K'}{2K}\Phi+C\Lambda. \tag{34b}$$

Differentiating Eq. (34) and substituting into the field equations (27) for Λ and Φ yields

$$\Phi'\left(\frac{K'}{2K}+\frac{\Psi'}{\Psi}\right)+\frac{\Phi}{2}\frac{d}{dr}\left(\frac{K'}{K}\right)+C'\Lambda=0 \tag{35a}$$

$$\Lambda'\left(\frac{K'}{2K}+\frac{\Psi'}{\Psi}\right)+\frac{\Lambda}{2}\frac{d}{dr}\left(\frac{K'}{K}\right)-C'\Phi=0. \tag{35b}$$

If we now define the functions $h(r)$ and $g(r)$ by

$$h(r)=C'\left(\frac{K'}{2K}+\frac{\Psi'}{\Psi}\right)^{-1} \tag{36a}$$

$$g(r)=\left[\frac{d}{dr}\left(\frac{K'}{2K}\right)\right]\left(\frac{K'}{2K}+\frac{\Psi'}{\Psi}\right)^{-1}, \tag{36b}$$

then Eq. (35) can be written in matrix form as

$$\begin{pmatrix}\Phi'\\ \Lambda'\end{pmatrix}=\begin{pmatrix}-g(r) & -h(r)\\ h(r) & -g(r)\end{pmatrix}\begin{pmatrix}\Phi\\ \Lambda\end{pmatrix}. \tag{37}$$

Equation (37) can be written more compactly as

$$\mathbf{x}'=\mathcal{B}(r)\mathbf{x}, \tag{38}$$

where $\mathbf{x}$ is a vector and $\mathcal{B}$ is a 2×2 matrix. The problem can be simplified further if we choose the solution to Eq. (38) to be of the form

$$\mathbf{x}=e^{z(r)}\mathbf{u}, \tag{39}$$

where $\mathbf{u}$ is a constant vector. Substituting Eq. (39) into Eq. (38) gives

$$\mathcal{B}\mathbf{u}=z'\mathbf{u}. \tag{40}$$

Hence solving the field equations is reduced to an eigenvalue problem. The eigenvalues of $\mathcal{B}$ are

$$z'=-g(r)\pm ih(r), \tag{41}$$

with corresponding eigenvectors

$$\mathbf{u}=\begin{pmatrix}\pm i\\ 1\end{pmatrix}. \tag{42}$$

The appearance of constant eigenvectors is consistent with the assumption that $\mathbf{u}$ is constant. Finally we write the exterior metric solution to Λ and Φ as

$$\Phi=e^{-G(r)}\{-D_1\sin[H(r)]+D_2\cos[H(r)]\} \tag{43a}$$

$$\Lambda=\Omega=e^{-G(r)}\{D_1\cos[H(r)]+D_2\sin[H(r)]\}, \tag{43b}$$

where D_1 and D_2 are constants,

$$H(r)=\int h(r)\;dr+H_1 \tag{44a}$$

$$G(r)=\int g(r)\;dr+G_1, \tag{44b}$$

and H_1 and G_1 are integration constants. Equations (43) describe metric components that oscillate harmonically, with the amplitude and spatial frequency varying as a function of r. The numerical solution for a more realistic energy-momentum tensor also suggests that oscillatory solutions will occur for $T_{tt}\neq T_{zz}$. This behavior leads to exotic space-time properties which are discussed in the following sections.

The equations for $C(r)$ and $K(r)$ are found by substituting the solution (43) into the defining equations (29) and (30), i.e.,

$$K(r)=e^{-2G(r)}(D_1^2+D_2^2) \tag{45a}$$

$$C(r)=-h(r). \tag{45b}$$

Equation (45a) can be differentiated to give

$$\frac{d^2\ln(K)}{dr^2}=-\frac{d\ln(K)}{dr}\left(\frac{1}{2}\frac{d\ln(K)}{dr}+\frac{d\ln(\Psi)}{dr}\right). \tag{46}$$

From Eq. (32) we know that Ψ can be solved in terms of K and hence Eq. (46) is a nonlinear ordinary differential equation. Furthermore, Eq. (45b) can be simplified to

$$C=\frac{C_1}{\Psi\sqrt{K}}, \tag{47}$$

where C_1 is a constant and $K\neq 0$. C is then solved in terms of K. Hence the Einstein field equations have been reduced to a single nonlinear ordinary differential equation (46).

A trivial solution to Eq. (46) is

$$K=A_1 \tag{48}$$

where A_1 is a constant. The solution for K allows us to write Ψ as

$$\Psi=A_2 r, \tag{49}$$

where A_2 is an integration constant and we have chosen the other constant of integration to be consistent with the boundary condition $\Psi(r=0)=0$. This form of Ψ can describe a conical metric, depending on the form of Λ and Φ. The solution to Eq. (47) gives the function $C(r)$ as

$$C=\frac{A_3}{r}, \tag{50}$$

where A_3 is another constant. We now have $h=-A_3/r$ and $g=0$, and hence the solution to the metric (43) may be written as

$$\begin{aligned}\Phi=e^{-G_1}\{&D_1\ \sin[A_3\ln(r)-H_1]\\ &+D_2\ \cos[A_3\ln(r)-H_1]\}\end{aligned} \tag{51a}$$

$$\begin{aligned}\Lambda=\Omega=e^{-G_1}\{&D_1\ \cos[A_3\ln(r)-H_1]\\ &-D_2\ \sin[A_3\ln(r)-H_1]\}.\end{aligned} \tag{51b}$$

We can evaluate the constants by equating Eqs. (51) to the weak field solution (7) at $r=\delta$. Requiring the metric and its first derivative to be continuous gives $(A_2)^2=(1-8GE)$, $e^{-G_1}D_1=1$, $D_2=0$, $H_1=A_3\ln\delta$ and $A_3=8GM\delta^{-1}$. Equations (51) then become

$$\Psi^2=(1-8GE)r^2 \tag{52a}$$

$$\Phi=\sin\{8GM\delta^{-1}[\ln(r)-\ln(\delta)]\} \tag{52b}$$

$$\Lambda=\Omega=\cos\{8GM\delta^{-1}[\ln(r)-\ln(\delta)]\}, \tag{52c}$$

or in terms of the metric

$$\begin{aligned}ds^2=&\cos\{8GM\delta^{-1}[\ln(r)-\ln(\delta)]\}(dt^2-dz^2)\\ &-2\sin\{8GM\delta^{-1}[\ln(r)-\ln(\delta)]\}dtdz\\ &-dr^2-(1-8GE)r^2d\theta^2.\end{aligned} \tag{53}$$

The solutions (52) display an harmonic variation of the g_{tt}, g_{zz} and g_{tz} metric components, which oscillate between timelike and spacelike. The exterior metric described by Eq. (53) is valid for all $r>\delta$ (i.e., where $T_{\mu\nu}=0$). We also note that the spatial frequency of the oscillations is determined by M; for $M=0$, the solution describes a conical metric.

There are, however, some notable differences between the numerical solution and the approximate analytical solution. Since $T_{tt}\neq T_{zz}$ everywhere, the numerical solutions to Λ and Ω are of a different oscillatory form. The oscillations in the analytical solution have a constant amplitude, whereas the amplitude of the oscillations determined from the numerical solution increases with increasing r. However, the spatial frequency predicted by both the numerical and analytical solutions decreases with increasing r (i.e., according to $\ln r$ which grows more slowly than r). The oscillations for the numerical solution appear to continue indefinitely, which is supported by the indefinite oscillations displayed by the analytical solution. Hence, the metric components never limit to a flat spacetime. The indefinite oscillation of the metric components is discussed in Sec. V.

B. Timelike directions

From Fig. 1, we observe an unusual situation in which g_{zz} becomes timelike before g_{tt} becomes spacelike (i.e., where $\Omega<0$ and $\Lambda>0$); consequently, there is a region where two metric components are timelike. The appearance of two timelike metric components is indicative of a poor choice of coordinate system. To ameliorate this situation, we make the coordinate transformation, $z=\chi-\xi$ and $t=\chi+\xi$. The metric (12) now becomes

$$ds^2=(\Lambda+2\Phi-\Omega)d\xi^2+2(\Lambda+\Omega)d\xi d\chi + (\Lambda-2\Phi-\Omega)d\chi^2-dr^2-\Psi^2 d\theta^2. \tag{54}$$

Since $\Phi>0$ in the region where $\Lambda>0$ and $\Omega<0$, $g_{\xi\xi}$ is timelike and $g_{\chi\chi}$ is spacelike and the transformed metric (54) has only one timelike component. In this region all timelike vectors must have a component in the ξ-direction. Extending the domain of the Λ and Ω solutions radially, we find that Φ becomes negative and the metric components g_{tt} and g_{zz} revert to their original roles. Since g_{zz} becomes spacelike before g_{tt} reverts to timelike, there exists a region where the metric (12) has no timelike component. Once again this indicates a poor choice of coordinate system. From Eq. (54), $\Phi<0$ implies $g_{\xi\xi}$ is spacelike and $g_{\chi\chi}$ is the only timelike metric component. Timelike vectors in this region must have a component in the χ-direction. We conclude that, as one moves out radially from the string core, a timelike vector must rotate periodically through the directions t, ξ, z, χ and back to t.

C. Isolated cosmic strings

To understand the physical significance of the string spacetime, consider a photon moving outwards from the string core. The photon geodesics are obtained by varying the metric with respect to proper time. For $\theta=$const. the geodesics are given by

$$\dot{t}=\frac{A\Omega-B\Phi}{\Lambda\Omega+\Phi^2} \tag{55a}$$

$$\dot{r}^2=\frac{\Lambda(A\Omega-B\Phi)^2+2\Phi(A\Omega-B\Phi)(A\Phi+B\Lambda)-\Omega(A\Phi+B\Lambda)^2}{(\Lambda\Omega+\Phi^2)^2} \tag{55b}$$

$$\dot{z}=-\frac{A\Phi+B\Lambda}{\Lambda\Omega+\Phi^2}, \tag{55c}$$

where A and B are constants, and an overdot denotes differentiation with respect to an affine parameter τ. We choose to set $\tau=t$ at $r=0$, for which $A=1$. Equations (55) describe a family of photon trajectories governed by B, where $B=-\dot{z}$ at $r=0$, which implies $|B|\leqslant 1$ for a photon originating at the string core. Consider the case where $B=0$ for which $\dot{r}$ becomes

$$\dot{r}^2=\frac{\Omega}{\Lambda\Omega+\Phi^2}. \tag{56}$$

From Eq. (56) it is evident that there is a distance r_n at which $\Omega(r_n)=0$, beyond which $\dot{r}$ becomes imaginary ($\Omega<0$). A photon which originates at the string core cannot travel beyond r_n. Since $\dot{z}<0$ for $r_n\geqslant r>0$, photons are deflected in the negative z-direction until they move parallel to the cosmic string as $r\to r_n$. This behavior is a consequence of $\Phi\neq 0$ and exemplifies how the mixed metric components affect photon geodesics.

Regardless of the choice of B, $\dot{r}$ is always imaginary at some distance, $r_n(B)$, from the core, and hence no photon originating at the core can escape to the outside Universe. This is seen from calculating $\dot{r}$ for $B=1$ and $B=-1$:

$$\dot{r}(B=1) = -\frac{\Lambda+2\Phi-\Omega}{\Lambda\Omega+\Phi^2} \tag{57a}$$

$$\dot{r}(B=-1) = -\frac{\Lambda-2\Phi-\Omega}{\Lambda\Omega+\Phi^2}. \tag{57b}$$

At distances where $\Lambda=\Omega$, if Φ is positive, then the $B=1$ geodesic becomes imaginary and if Φ is negative, then the $B=-1$ geodesic is imaginary. In terms of the rotation of a timelike vector, the $B=1$ geodesic is imaginary when $g_{\xi\xi}$ is timelike and the $B=-1$ geodesic is imaginary when $g_{\chi\chi}$ is timelike. As both these situations are exhibited by the numerical and analytical solutions to the metric, we conclude that photons at the string core cannot escape to the outside Universe, and conversely, no photons can reach the center of the string. Photons incident on the string will be deflected increasingly in the z-direction, until at a distance $r_n(B)$ they are moving parallel to the string. As $\Phi>0$ near the string core, photon geodesics described by $B>0$ are deflected at smaller distances than geodesics described by $B<0$ [i.e., $r_n(B>0)<r_n(B=0)<r_n(B<0)$].

The analytical and numerical solutions suggests that the metric components will oscillate indefinitely. As a result, there are multiple regions where $\dot{r}$ is imaginary for a given photon geodesic. Photons on a given geodesic cannot pass through these multiple regions, which form a series of ''horizons.'' Since photons are deflected, we can infer that uncharged particles are similarly deflected. Moreover, as the string's electromagnetic field cannot extend beyond $r_n(B=-1)$, there are no magnetic fields generated by the string beyond this distance. As such, charged particles beyond this distance will not be subject to electromagnetic fields and will be deflected in the same way as uncharged particles. Therefore, no particles from the outside Universe can reach the string core.

D. Behavior of clocks

We now consider the behavior of photon geodesics in terms of coordinate time t. The value of $\dot{r}^2$, when $\dot{t}=0$, for an arbitrary choice of B $(A=1)$ is given by

$$\dot{r}^2 = -B/\Phi. \tag{58}$$

The analytical and numerical results indicate that for all photon geodesics which originate at the string core Φ is positive (i.e., particles cannot escape into a region where $\Phi<0$). The choice of B then determines the behavior of coordinate time t at different distances from the string. For $B<0$, outgoing photons will reach a distance at which $\dot{t}<0$, since Eq. (58) indicates that $\dot{r}$ is real when $\dot{t}=0$. As the photon continues to move out from the string, coordinate time runs backwards. However, coordinate time is only physically meaningful at $r=0$ (where $t=\tau$) and measures the history of a photon according to a stationary observer at the string core. To illustrate the consequences of $\dot{t}<0$, consider the scenario depicted in Fig. 3, where we have plotted the path of a photon for $B=-0.4$. A photon originating at the string core is reflected back to the string by a mirror located at $\bar{r}=150$. In Fig. 3(a), the photon appears (in coordinate time) to be moving back to the string before reaching the mirror, which allows the photon on its return path B to meet itself on the outward path A. This causal paradox is resolved in Fig. 3(b), where the photon has been deflected along the negative z-direction as a consequence of the g_{tz} metric component. Hence the outgoing photon never meets the returning photon, since they are spatially separated at the string core (i.e., events which would be timelike separated in Minkowski spacetime are spacelike separated in the string spacetime). Causal paradoxes do not arise for $B\geqslant 0$, since $\dot{r}$ becomes imaginary before $\dot{t}<0$.

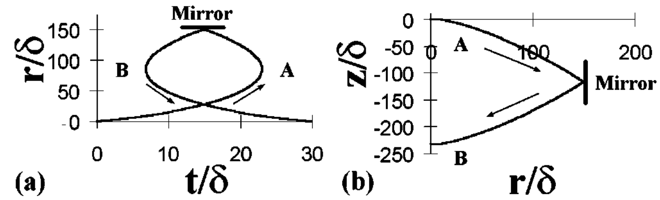

FIG. 3. (a) Photon geodesic, measured in coordinate time t, is plotted for $B=-0.4$. Arrows indicate the direction of proper time. The photon is reflected back to the string by a mirror located at $r/\delta=150$. According to t measured at the string core, the photon on path B appears to be moving back to the string before arriving at the mirror on its outward path A. (b) The causal paradox is avoided since paths A and B are spatially separated. The displacement of the photon in the z-direction is a direct consequence of the g_{tz} metric component as discussed in the text.

V. CONCLUSION

Numerical and analytical solutions to the superconducting cosmic string spacetime have been presented (Sec. IV A) which indicate that the metric components oscillate indefinitely. However, the oscillating metric components are a consequence of nonzero momentum and cylindrical symmetry. As a result, the metric solution is only valid for an infinitely long straight string and would be invalid at distances where the string's curvature becomes apparent. In the case of string loops, toroidal symmetry coupled with the requirement that the metric must be smooth at the center of the loop may not give rise to an infinite number of oscillations. Superconducting cosmic string loops with momentum (e.g., vortons) are therefore expected to have an oscillating metric solution near the core of the string loop, but these oscillations will decay at a scale where the loop's curvature becomes significant.

The analytical solution to the metric components indicates that the metric will be oscillatory for any nonzero value of momentum per unit length, M. Hence the results should be applicable to GUT strings (for which $\eta_\phi \sim 10^{16}$ GeV). However, Eq. (53) indicates that the scale at which the first oscillation of the metric becomes apparent [i.e., $r_n(B=0)$] is very sensitive to M (e.g., $8GM\delta^{-1}\ln[r_n(B=0)]=\pi$). As a result, for GUT strings, the oscillations will only be apparent where string curvature is significant. At these distances the metric solution is invalid and hence GUT vortons are unlikely to exhibit the spacetime properties discussed in this paper. The exotic spacetime properties discussed above will only be found in heavy superconducting cosmic strings (i.e., $\eta_\phi \sim 10^{18}$ GeV). Such objects could arise in phase transitions occurring before the GUT phase transition, as predicted by superstring theory [2].

We have also examined GUT strings with $\overline{k}(=\overline{\omega})>10$ and found that the spacetime curvature is sufficient to initiate current quenching. To investigate GUT strings with large momentum will require a detailed study of the way $\overline{\omega}$ increases to offset quenching; unfortunately, this cannot be described in terms of the static metric (12). Such an investigation would be important, because if the momentum increases significantly (as a consequence of $\overline{\omega}$ counteracting gravitationally induced current quenching), then GUT scale superconducting cosmic strings may exhibit the oscillatory metric properties described in this paper.

There are several important cosmological consequences which follow from the new metric solution. For example, a constraint on the abundance of vortons is predicated on their energy density not exceeding that of photons during the radiation era [9]. If this constraint was not satisfied, then nucleosynthesis could not take place, since the Universe would not be radiation dominated. However, the present work shows that the spacetime properties of heavy vortons may isolate them from particles in the outside Universe and hence heavy vortons may not take part in nucleosynthesis. This in turn would relax the constraint on nucleosynthesis since heavy vortons would no longer adversely effect primordial nucleosynthesis. Another important ramification of the present work relates to the anisotropy in the cosmic microwave background. The conical metric for nonsuperconducting cosmic strings has been shown to induce fluctuations in the cosmic background radiation [18], providing an experimental test of the presence of cosmic strings during the early Universe. However, the fluctuations predicted by earlier models will be altered radically when superconducting strings with nonzero momentum are taken into account. The magnitude and power spectrum of the fluctuations may enable superconducting cosmic strings to be distinguished experimentally from nonsuperconducting cosmic strings.

In this paper we have examined the cylindrically symmetric spacetime of a straight cosmic string with nonzero constant momentum. We have found that momentum induces mixed metric components, which gives rise to oscillatory solutions to the field equations. The oscillations of the metric components led to exotic spacetime behavior, where timelike vectors rotate with r. The rotation of timelike vectors also results in particle geodesics that only extend a finite distance in r and hence isolates the string from the outside Universe. Although the present work considers cosmic strings as a physical source for the energy-momentum tensor, the results are also of interest in themselves as they suggest the existence of a new class of solutions to the Einstein field equations. The development of new metrics may give insight into the structure of cylindrically symmetric field equations. Future work will be directed at examining the spacetime metric of a string loop with momentum to ascertain whether the metric oscillations decay. The behavior of $\overline{\omega}$ during gravitational quenching, in a time-dependent metric, also needs to be studied to determine if this process will significantly increase the momentum of the string. Finally, there are numerous cosmological consequences whose ramifications require careful investigation, and may be subject to experimental tests.

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