Relativistic second-order perturbations of the Einstein–de Sitter universe

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We consider the evolution of relativistic perturbations in the Einstein–de Sitter cosmological model, including second-order effects. The perturbations are considered in two different settings: the widely used synchronous gauge and the Poisson (generalized longitudinal) one. Since, in general, perturbations are gauge dependent, we start by considering gauge transformations at second order. Next, we give the evolution of perturbations in the synchronous gauge, taking into account both scalar and tensor modes in the initial conditions. Using the second-order gauge transformation previously defined, we are then able to transform these perturbations to the Poisson gauge. The most important feature of second-order perturbation theory is mode mixing, which here also means, for instance, that primordial density perturbations act as a source for gravitational waves, while primordial gravitational waves give rise to second-order density fluctuations. Possible applications of our formalism range from the study of the evolution of perturbations in the mildly nonlinear regime to the analysis of secondary anisotropies of the cosmic microwave background. [S0556-2821(98)03314-1]

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I. INTRODUCTION

The study of the evolution of cosmological perturbations is of primary importance for understanding the present properties of the large-scale structure of the Universe and its origin. This study is usually performed with different techniques and approximations, depending on the specific range of scales under analysis. So for scales well within the Hubble radius, the analysis of the gravitational instability of collisionless matter is usually restricted to the Newtonian approximation. As seen in the Eulerian picture, this approximation basically consists in adding a first-order lapse perturbation $2\varphi_g/c^2$ to the line element of a matter-dominated Friedmann-Robertson-Walker (FRW) model, while keeping nonlinear density and velocity perturbations around the background solution. The peculiar gravitational potential φ_g is determined by the dimensionless matter-density contrast δ via the cosmological Poisson equation, $\nabla^2\varphi_g = 4\pi G a^2 \varrho_b \delta$, with ϱ_b the background matter density and a the FRW scale factor. The fluid dynamics is then studied by accounting for mass and momentum conservation, to close the system (see, e.g., Ref. [1]). This procedure is thought to produce accurate results on scales much larger than the Schwarzschild radius of collapsing bodies but much smaller than the Hubble horizon, where φ_g/c^2 stays much less than unity, while the peculiar matter flows never become relativistic. The first-order matter perturbations obtained with this Newtonian treatment can be shown to coincide with the results of linear general relativistic perturbation theory in the so-called longitudinal gauge [2]. To second order, however, the comparison is made nontrivial by the occurrence of nonlinear post-Newtonian (and higher order in $1/c$) terms in the relativistic theory (see also Refs. [3,4]). Some, but not all, of the aspects of the relativistic treatment can be accounted for by adding an extra post-Newtonian perturbation $-2\varphi_g/c^2$ to the conformal spatial metric, an extension that leads to the so-called weak-field approximation (see, e.g., Ref. [5]). This improvement allows, for instance, a rather accurate treatment of photon trajectories in the geometry produced by matter inhomogeneities, as required in the study of gravitational lensing by cosmic structures (see, e.g., Ref. [6]) and other applications. It is worth mentioning that the full post-Newtonian line element in Eulerian coordinates would also include nonvanishing shift components (see, e.g., Ref. [7]). A second-order perturbative expansion starting from this metric would lead to the same result of our Poisson-gauge approach discussed below, with the obvious exception of those terms which are post-post-Newtonian or higher in a $1/c$ expansion.

From the point of view of the Lagrangian frame of the matter, corresponding to our synchronous and comoving gauge below, the Newtonian approach is quite different: the ''Newtonian Lagrangian metric'' can be cast in a simple form, where the spatial metric tensor is written in terms of the Jacobian matrix connecting Lagrangian to Eulerian coordinates. According to this approach, post-Newtonian terms of any order appear as spatial metric perturbations over this ''Newtonian background'' [4]. Without discussing the long list of cosmological approximation schemes which have been proposed to follow the nonlinear dynamics of collisionless

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matter in the Newtonian framework, let us only mention the celebrated Zel'dovich approximation [8], which is strictly related to the Lagrangian Newtonian approach. Various extensions of Zel'dovich theory to the relativistic case have been proposed in the literature; all of them, however, require either global or local symmetries, thereby preventing their application to the cosmological structure formation problem. A relativistic formulation of the Zel'dovich approximation, assuming no limitations on the initial conditions, is instead introduced in Ref. [9].

So far the list basically covers all those methods which have been devised to follow nonlinear structure formation by gravitational instability in the Universe, with the only possible exception of a few relevant exact solutions of Einstein's field equations, such as the Tolman-Bondi one and some of the Bianchi and Szekeres solutions (see, e.g., Ref. [10] and references therein for a review). These exact solutions, however, have only limited application to realistic cosmological problems.

The study of small perturbations giving rise to large-scale temperature anisotropies of the cosmic microwave background is instead usually treated with the full technology of first-order relativistic perturbation theory, either in a gauge-invariant fashion or by specifying a suitable gauge. On small and intermediate angular scales, however, where the description in terms of first-order perturbation theory is no longer accurate, second-order metric perturbations can play a nontrivial role and determine new contributions, such as those leading to the nonlinear Rees-Sciama effect [11]. In such a case a fully general relativistic treatment is needed, such as that recently put forward by Pyne and Carroll [12] and implemented in second-order perturbation theory by Mollerach and Matarrese [13].

The aim of the present paper is to provide a complete account of second-order cosmological perturbations in two gauges: the synchronous and comoving gauge and the so-called Poisson one, a generalization of the longitudinal gauge discussed by Bertschinger [2] and Ma and Bertschinger [14]. The former was chosen here because of the advantages it presents in performing perturbative calculations. The latter, on the other hand, being the closest to the Eulerian Newtonian picture, allows a simpler physical understanding of the various perturbation modes. The link between these gauges is provided by a second-order gauge transformation of all the geometrical and physical variables of the problem. The general problem of nonlinear gauge transformations in a given background spacetime has been recently dealt with by Bruni *et al.* [15] (see also [16]), and will be shortly reviewed in the following section.

The range of applicability of our general relativistic second-order perturbative technique is that of small fluctuations around a FRW background, but with no extra limitations. It basically allows one to describe perturbations down to scales which experience slight departures from a linear behavior, which, in present-day units, would include all scales above about 10 Mpc in any realistic scenario of structure formation. Accounting for second-order effects generally helps to follow the gravitational instability on a longer time scale and to include new nonlinear and nonlocal phenomena. The advantage of such a treatment is precisely that it enables one to treat a large variety of phenomena and scales within the same computational technique.

The literature on relativistic second-order perturbation theory in a cosmological framework is not so vast. The pioneering work in this field is by Tomita [17], who, back in 1967, performed a synchronous-gauge calculation of the second-order terms produced by the mildly nonlinear evolution of scalar perturbations in the Einstein–de Sitter universe. Matarrese, Pantano, and Sáez [3] obtained an equivalent result, but with a different technique, in comoving and synchronous coordinates. Using a tetrad formalism, Russ *et al.* [18] recently extended these calculations to include the second-order terms generated by the mixing of growing and decaying linear scalar modes. Salopek, Stewart, and Croudace [19] applied a gradient expansion technique to the calculation of second-order metric perturbations. The inclusion of vector and tensor modes at the linear level, acting as further seeds for the origin of second-order perturbations of any kind (scalar, vector, and tensor), was, once again, first considered by Tomita [20].

In this paper we study the second-order perturbations of an irrotational collisionless fluid in the Einstein–de Sitter background, including both growing-mode scalar perturbations and gravitational waves at the linear level. The plan of the paper is as follows. In the next section we summarize the results of Ref. [15] regarding nonlinear gauge transformations for perturbations of any given background spacetime. In Sec. III we consider perturbations of a generic flat Robertson-Walker model, and we give the transformations between any two given gauges, up to second order. Section IV is devoted to the study of the evolution of perturbations in the synchronous gauge, in the specific case of irrotational dust in the Einstein–de Sitter background. In Sec. V we apply the formulas obtained in Sec. III to obtain the transformations between the synchronous [21] and Poisson (generalized longitudinal [2]) gauges. Using these transformations, in Sec. VI the results of Sec. IV are transformed to the Poisson gauge. Section VII contains a final discussion. Appendix A reviews some mathematical results obtained in Ref. [15] and used in Sec. II; Appendixes B and C contain useful formulas in the synchronous gauge, used in Sec. IV. Appendix D contains formulas used to obtain some of the Poisson-gauge results in Sec. VI.

II. GAUGE DEPENDENCE AT SECOND AND HIGHER ORDER

The idea underlying the theory of spacetime perturbations is the same that we have in any perturbative formalism: we try to find approximate solutions of some field equations, regarding them as ''small'' deviations from a known exact background solution. The basic difference arising in general relativity, or in other spacetime theories, is that we have to deal with perturbations not only of fields in a given geometry, but of the geometry itself.

The perturbation ΔT in any relevant quantity, say, represented by a tensor field T, is defined as the difference between the value T has in the physical spacetime (the per-

turbed one) and the value T_0 the same quantity has in the given (unperturbed) background spacetime. However, it is a basic fact of differential geometry that, in order to make the comparison of tensors meaningful at all, one has to consider them at the same point. Since T and T_0 are defined in different spacetimes, they can thus be compared only after a prescription for identifying points of these spacetimes is given. A *gauge choice* is precisely this, i.e., a one-to-one correspondence (a map) between the background and the physical spacetime. A change of this map is then a gauge transformation, and the freedom one has in choosing it gives rise to an arbitrariness in the value of the perturbation of T at any given spacetime point, unless T is gauge invariant. This is the essence of the ''gauge problem,'' which has been discussed—mainly in connection with linear perturbations—in many papers [22–26] and review articles [27,28], following different approaches.

In order to discuss in depth higher-order perturbations and gauge transformations, and to define gauge invariance, one needs to formalize the above ideas, giving a precise geometrical description of what perturbations and gauge choices are. In a previous paper [15] (see also [16]) we have considered this problem in great detail, following in the main the approach used in Refs. [23,26,29]. Instead of directly considering perturbations, we have first looked upon the geometry of the problem in full generality, taking an exact (i.e., nonperturbative) point of view; after that, we have expanded all the geometrical quantities in appropriately defined Taylor series, thus going beyond the usual linear treatment.

In this section we shall summarize the main results obtained in Ref. [15]. Appendix A explains in some more detail how to proceed in the calculations. Here and in the following Greek indices $\mu, \nu, \ldots$ take values from 0 to 3 and the Latin ones $i, j, \ldots$ from 1 to 3; units are such that $c=1$.

We finally recall here some basics about the Lie derivative along a vector field ξ, which will be useful in the following. The Lie derivative of any tensor T of type (p,q) (a tensor with p contravariant and q covariant indices, which we omit here and in the following) is also a tensor of the same type (p,q). For a scalar f, a contravariant vector Z and a covariant tensor T of rank 2, the expressions of the Lie derivative along ξ are, respectively,

$$\pounds_\xi f = f_{,\mu}\xi^\mu, \tag{2.1}$$

$$\pounds_\xi Z^\mu = Z^\mu_{,\nu}\xi^\nu - \xi^\mu_{,\nu}Z^\nu, \tag{2.2}$$

$$\pounds_\xi T_{\mu\nu} = T_{\mu\nu,\sigma}\xi^\sigma + \xi^\sigma_{,\mu}T_{\sigma\nu} + \xi^\sigma_{,\nu}T_{\mu\sigma}. \tag{2.3}$$

Expressions for any other tensor can easily be derived from these. A second or higher Lie derivative is easily defined from these formulas; e.g., for a vector we have $\pounds^2_\xi Z = \pounds_\xi(\pounds_\xi Z)$: since one clearly sees from Eq. (2.2) that $\pounds_\xi Z$ is itself a contravariant vector, one needs only to apply Eq. (2.2) two times to obtain the components of $\pounds^2_\xi Z$. Similarly, one derives expressions for the second Lie derivative of any tensor.

A. Gauge transformations: An exact point of view

A basic assumption in perturbation theory is the existence of a parametric family of solutions of the field equations, to which the unperturbed background spacetime belongs [29]. In cosmology and in many other cases in general relativity, one deals with a one-parameter family of models $\mathcal{M}_\lambda$; λ is real, and $\lambda=0$ identifies the background $\mathcal{M}_0$. On each $\mathcal{M}_\lambda$ there are tensor fields T_λ representing the physical and geometrical quantities (e.g., the metric). The parameter λ is used for Taylor expanding these T_λ; the physical spacetime $\mathcal{M}_\lambda$ can eventually be identified by $\lambda=1$. The aim of perturbation theory is to construct an approximated solution to $\mathcal{M}_\lambda$.

Each one-to-one correspondence between points of $\mathcal{M}_0$ and points of $\mathcal{M}_\lambda$ is thus a one-parameter function of λ: we can represent two such ''point identification maps'' [23] as ψ_λ and φ_λ (for a depiction of this and the following, see Figs. 1 and 2 in [15]). Suppose that coordinates x^μ have been assigned on the background $\mathcal{M}_0$, labeling the different points. A one-to-one correspondence, e.g., ψ_λ, carries these coordinates over $\mathcal{M}_\lambda$, and defines a choice of gauge: therefore, it is natural to call the correspondence itself a ''gauge.'' A change in this correspondence, keeping the background coordinates fixed, is a gauge transformation [24]. Thus, let p be any point in $\mathcal{M}_0$, with coordinates $x^\mu(p)$, and let us use the gauge ψ_λ: $O=\psi_\lambda(p)$ is the point in $\mathcal{M}_\lambda$ corresponding to p, to which ψ_λ assigns the same coordinate labels. However, we could as well use a different gauge φ_λ and think of O as the point of $\mathcal{M}_\lambda$ corresponding to *a different point* q in the background, with coordinates $\tilde{x}^\mu$: then $O=\psi_\lambda(p)=\varphi_\lambda(q)$. Thus, the change of the correspondence, i.e., the gauge transformation, may actually be seen as a *one-to-one correspondence between different points in the background*. Since we start from a point p in $\mathcal{M}_0$, we carry it over to $O=\psi_\lambda(p)$ in $\mathcal{M}_\lambda$, and then we may come back to q in $\mathcal{M}_0$ with φ_λ^{-1}, i.e., $q=\varphi_\lambda^{-1}(O)$; the overall gauge transformation is also a function of λ, which we may denote as Φ_λ, and is given by composing φ_λ^{-1} with ψ_λ, so that we can write $q=\Phi_\lambda(p):=\varphi_\lambda^{-1}(\psi_\lambda(p))$. We then have that the coordinates of q, $\tilde{x}^\mu(\lambda,q)=\Phi^\mu_\lambda(x^\alpha(p))$, are one-parameter functions of those of p, $x^\alpha(p)$. Such a transformation, which in one given coordinate system moves each point to another, is often called an ''active coordinate transformation,'' as opposed to passive ones, which change coordinate labels to each point (see Appendix A).

Now, consider the tensor fields T_λ on each $\mathcal{M}_\lambda$. With the gauges φ_λ and ψ_λ we can define, in two different manners, a representation on $\mathcal{M}_0$ of each T_λ: we can denote these simply by $T(\lambda)$ and $\tilde{T}(\lambda)$, respectively. These are tensor fields defined on $\mathcal{M}_0$ in such a way that each of them has, in the related gauge, the same components of T_λ. On the other hand, $T(\lambda)$ and $\tilde{T}(\lambda)$ are related by Φ_λ, which gives rise to a relation between their components given by Eq. (A13). Since in each gauge we now have a field representing T_λ on $\mathcal{M}_0$, at each point of the background we can compare these fields with T_0 and define perturbations. In the first gauge the total perturbation is $\Delta T(\lambda):=T(\lambda)-T_0$, and in the second one is $\Delta\tilde{T}(\lambda):=\tilde{T}(\lambda)-T_0$. This nonuniqueness is the gauge

dependence of the perturbations.

It should be noted at this point that we have not so far made any approximation: the definitions given above are exact, and hold in general, to any perturbative order.

B. Gauge transformations: Second-order expansion

In order to proceed and compute at the desired order of accuracy in λ the effects of a gauge transformation, we need Taylor expansions. In this respect, a crucial point is that gauge choices such as ψ_λ and φ_λ form one-parameters groups with respect to λ, while the gauge transformations Φ_λ form a one-parameter family that in general is not a group (see Appendix A and [15,16] for more details). Only in linear theory is the action of Φ_λ approximated by that of the element of a one-parameter group of transformations. A one-parameter group of transformations is associated with a vector field ξ and the congruence it generates, and therefore at first order in λ the effect of Φ_λ on the coordinates $x^\mu(p)$ is approximated by $\tilde{x}^\mu \simeq x^\mu + \lambda \xi^\mu$, whereas for a tensor T we have $\tilde{T} \simeq T + \lambda \pounds_\xi T$, as is well known. However, the fact that Φ_λ does not form a group comes into play with nonlinearity, and one can show that at second order two vector fields $\xi_{(1)}$ and $\xi_{(2)}$ are involved, and so on. That is, the Taylor expansion of a one-parameter family of transformations Φ_λ involves, at a given order n, n vector fields $\xi_{(k)}$, $k=1,\ldots,n$. At second order, the expansion of the transformation $\tilde{x}^\mu(\lambda) = \Phi^\mu_\lambda(x^\alpha)$ between the coordinates of any pair of points of the background mapped into one another by Φ_λ gives

$$\tilde{x}^\mu(\lambda) = x^\mu + \lambda \xi^\mu_{(1)} + \frac{\lambda^2}{2}(\xi^\mu_{(1),\nu}\xi^\nu_{(1)} + \xi^\mu_{(2)}) + \mathcal{O}(\lambda^3). \tag{2.4}$$

This is often called an ''infinitesimal point transformation.'' From this, one can always define (again, see Appendix A) an associated ordinary (passive) coordinate transformation, Eq. (A19). Substitution of this into Eq. (A13) gives—after properly collecting terms—the gauge transformation for a tensor T:

$$\tilde{T}(\lambda) = T(\lambda) + \lambda \pounds_{\xi_{(1)}} T + \frac{\lambda^2}{2}(\pounds^2_{\xi_{(1)}} + \pounds_{\xi_{(2)}})T + \mathcal{O}(\lambda^3). \tag{2.5}$$

If T is a tensor of type (p,q), the components of each term in this formula have p contravariant and q covariant indices, appropriately given by the rules (2.1)–(2.3).

A simple heuristic argument that may help to understand why two vector fields are involved in the second-order gauge transformations (and n vectors at nth order) is the following. In practice, we usually consider the gauge transformation between two given gauges, e.g., in this paper, the synchronous and the Poisson ones. Therefore, having the conditions that fix the two gauges, the unknowns of the problem are the degrees of freedom that allow us to pass from one gauge to another. Then, consider the usual first-order gauge transformation, i.e., the first-order part of Eqs. (2.4), (2.5): it is clear that this fully determines $\xi_{(1)}$ as a field in the background. Going to second order, it should be clear by the very definition of a Taylor expansion that $\xi_{(1)}$ itself cannot depend on λ, and its contribution to second order is built from what we already know at first order; i.e., it can only give a quadratic contribution. On the other hand, since at second order in both gauges we have new degrees of freedom in any quantity (with respect to first order), it is clear that the gauge transformation itself must contain new degrees of freedom: these are given by $\xi_{(2)}$.

Since in the two gauges we can write, respectively [30],

$$T(\lambda) = T_0 + \lambda\,\delta T + \frac{\lambda^2}{2}\delta^2 T + \mathcal{O}(\lambda^3), \tag{2.6}$$

$$\tilde{T}(\lambda) = T_0 + \lambda\,\delta\tilde{T} + \frac{\lambda^2}{2}\delta^2\tilde{T} + \mathcal{O}(\lambda^3), \tag{2.7}$$

we can substitute Eq. (2.6) and Eq. (2.7) into Eq. (2.5) in order to obtain, at first and second order in λ, the required gauge transformations for δT and $\delta^2 T$:

$$\delta\tilde{T} = \delta T + \pounds_{\xi_{(1)}} T_0, \tag{2.8}$$

$$\delta^2\tilde{T} = \delta^2 T + 2\pounds_{\xi_{(1)}}\delta T + \pounds^2_{\xi_{(1)}} T_0 + \pounds_{\xi_{(2)}} T_0. \tag{2.9}$$

Equation (2.8) is the well-known result mentioned above. Equation (2.9) gives the general gauge transformation for second-order perturbations, and shows that this is made up of three parts: the first couples the first-order generator of the transformation $\xi_{(1)}$ with the first-order perturbation δT; the second part couples the zeroth-order background T_0 with terms quadratic in $\xi_{(1)}$; and the last part couples T_0 with the second order generator $\xi_{(2)}$ of the transformation, in the same manner than the term $\pounds_{\xi_{(1)}} T_0$ does at first order, in Eq. (2.8). Equation (2.9) also shows that there are special second-order gauge transformations, purely due to $\xi_{(2)}$, when $\xi_{(1)}=0$; on the other hand, a nonvanishing $\xi_{(1)}$ always affects second-order perturbations (cf. [31]). Finally, in some specific problems in which only effects quadratic in first-order perturbations are important, one can consider $\xi_{(2)}=0$: for instance, this is the case of back-reaction effects (cf. Ref. [32]).

C. Gauge invariance

It is logically possible to establish a condition for gauge invariance to a given perturbative order n even without knowledge of the gauge transformation rules holding at that order; see [15]. However, we shall focus here on gauge invariance to second order, using Eqs. (2.8), (2.9).

We need to state a clear definition of gauge invariance before giving a condition. The most natural definition is that a tensor T is gauge invariant to order n if and only if $\delta^k\tilde{T} = \delta^k T$ for every $k \leqslant n$ (we define $\delta^0 T := T_0$, $\delta T := \delta^1 T$). Thus, a tensor T is gauge invariant to second order if $\delta^2\tilde{T} = \delta^2 T$ *and* $\delta\tilde{T} = \delta T$. Then, from Eqs. (2.8) and (2.9) we see that, since $\xi_{(1)}$ and $\xi_{(2)}$ are arbitrary, this condition implies that $\pounds_\xi T_0 = 0$ and $\pounds_\xi \delta T = 0$, for every vector field ξ in the

background $\mathcal{M}_0$. Therefore, apart from trivial cases—i.e., constant scalars and combinations of Kroneker deltas with constant coefficients—gauge invariance to second order requires that $T_0=0$ and $\delta T=0$ in any gauge. This condition generalizes to second order the results for first-order gauge invariance that can be found in the literature and is easily extended to order n; see Ref. [15].

III. GAUGE TRANSFORMATION IN A FLAT COSMOLOGY UP TO SECOND ORDER

In view of the application that will follow in Sec. V, we shall now use the formulas obtained in the previous section to show how the perturbations on a spatially flat Robertson-Walker background in two different gauges are related, up to second order. This will also introduce the notation used in all the following sections. Here and in the following Latin indices are raised and lowered using δ^{ij} and δ_{ij}, respectively. As discussed before, we set $\lambda=1$ to describe the physical spacetime.

A. Perturbed flat Robertson-Walker universe

We shall first consider the metric perturbations, then those in the energy density and four-velocity of the matter.

The components of a perturbed spatially flat Robertson-Walker metric can be written as

$$g_{00}=-a^2(\tau)\left(1+2\sum_{r=1}^{+\infty}\frac{1}{r!}\psi^{(r)}\right), \tag{3.1}$$

$$g_{0i}=a^2(\tau)\sum_{r=1}^{+\infty}\frac{1}{r!}\omega_i^{(r)}, \tag{3.2}$$

$$g_{ij}=a^2(\tau)\left\{\left[1-2\left(\sum_{r=1}^{+\infty}\frac{1}{r!}\phi^{(r)}\right)\right]\delta_{ij}+\sum_{r=1}^{+\infty}\frac{1}{r!}\chi_{ij}^{(r)}\right\}, \tag{3.3}$$

where $\chi_i^{(r)i}=0$, and τ is the conformal time. The functions $\psi^{(r)}$, $\omega_i^{(r)}$, $\phi^{(r)}$, and $\chi_{ij}^{(r)}$ represent the rth-order perturbation of the metric.

It is standard to use a nonlocal splitting of perturbations into the so-called scalar, vector, and tensor parts, where scalar (or longitudinal) parts are those related to a scalar potential, vector parts are those related to transverse (divergence-free or solenoidal) vector fields, and tensor parts to transverse trace-free tensors. In our case, the shift $\omega_i^{(r)}$ can be decomposed as

$$\omega_i^{(r)}=\partial_i\omega^{(r)\parallel}+\omega_i^{(r)\perp}, \tag{3.4}$$

where $\omega_i^{(r)\perp}$ is a solenoidal vector, i.e., $\partial^i\omega_i^{(r)\perp}=0$. Similarly, the traceless part of the spatial metric can be decomposed at any order as

$$\chi_{ij}^{(r)}=\mathrm{D}_{ij}\chi^{(r)\parallel}+\partial_i\chi_j^{(r)\perp}+\partial_j\chi_i^{(r)\perp}+\chi_{ij}^{(r)\top}, \tag{3.5}$$

where $\chi^{(r)\parallel}$ is a suitable function, $\chi_i^{(r)\perp}$ is a solenoidal vector field, and $\partial^i\chi_{ij}^{(r)\top}=0$; hereafter,

$$\mathrm{D}_{ij}:=\partial_i\partial_j-\frac{1}{3}\delta_{ij}\nabla^2. \tag{3.6}$$

Now, consider the energy density ϱ, or any other scalar that depends only on τ at zero order: this can be written as

$$\varrho=\varrho_{(0)}+\sum_{r=1}^{+\infty}\frac{1}{r!}\delta^r\varrho. \tag{3.7}$$

For the four-velocity u^μ of matter we can write

$$u^\mu=\frac{1}{a}\left(\delta_0^\mu+\sum_{r=1}^{+\infty}\frac{1}{r!}v_{(r)}^\mu\right). \tag{3.8}$$

In addition, u^μ is subject to the normalization condition $u^\mu u^\nu g_{\mu\nu}=-1$; therefore at any order the time component $v_{(r)}^0$ is related to the lapse perturbation $\psi_{(r)}$. For the first- and second-order perturbations we obtain, in any gauge,

$$v_{(1)}^0=-\psi_{(1)}, \tag{3.9}$$

$$v_{(2)}^0=-\psi_{(2)}+3\psi_{(1)}^2+2\omega_i^{(1)}v_{(1)}^i+v_i^{(1)}v_{(1)}^i. \tag{3.10}$$

The velocity perturbation $v_{(r)}^i$ can also be split into a scalar and vector (solenoidal) part:

$$v_{(r)}^i=\partial^i v_{(r)}^{\parallel}+v_{(r)\perp}^i. \tag{3.11}$$

As we have seen in the last section, the gauge transformation is determined by the vectors $\xi_{(r)}$. Splitting their time and space parts, one can write

$$\xi_{(r)}^0=\alpha^{(r)} \tag{3.12}$$

and

$$\xi_{(r)}^i=\partial^i\beta^{(r)}+d^{(r)i}, \tag{3.13}$$

with $\partial_i d^{(r)i}=0$.

B. First-order gauge transformations

We begin by reviewing briefly some well-known results about first-order gauge transformations, as we shall need them in the following. As in Sec. II, we simply denote quantities in the new gauge by a tilde.

From Eq. (2.8), it follows that the first-order perturbations of the metric transform as

$$\delta\tilde{g}_{\mu\nu}=\delta g_{\mu\nu}+\pounds_{\xi_{(1)}}g_{\mu\nu}^{(0)}, \tag{3.14}$$

where $g_{\mu\nu}^{(0)}$ is the background metric. Therefore, using Eq. (2.3), we obtain the following transformations for the first-order quantities appearing in Eqs. (3.1)–(3.3):

$$\tilde{\psi}_{(1)}=\psi_{(1)}+\alpha_{(1)}'+\frac{a'}{a}\alpha_{(1)}, \tag{3.15}$$

$$\tilde{\omega}_i^{(1)}=\omega_i^{(1)}-\alpha_{,i}^{(1)}+\beta_{,i}^{(1)\prime}+d_i^{(1)\prime}, \tag{3.16}$$

$$\tilde{\phi}_{(1)} = \phi_{(1)} - \frac{1}{3}\nabla^2\beta_{(1)} - \frac{a'}{a}\alpha_{(1)}, \tag{3.17}$$

$$\tilde{\chi}^{(1)}_{ij} = \chi^{(1)}_{ij} + 2\mathrm{D}_{ij}\beta^{(1)} + d^{(1)}_{i,j} + d^{(1)}_{j,i}, \tag{3.18}$$

where a prime denotes the derivative with respect to τ.

For a scalar ϱ, from Eqs. (2.8), (3.7), and (2.1) we have

$$\delta\tilde{\varrho} = \delta\varrho + \varrho'_{(0)}\alpha_{(1)}. \tag{3.19}$$

For the four-velocity u^μ, we have, from Eqs. (2.8),

$$\delta\tilde{u}^\mu = \delta u^\mu + \pounds_{\xi_{(1)}} u^\mu_{(0)}. \tag{3.20}$$

Using Eqs. (2.2) and (3.8) this gives

$$\tilde{v}^0_{(1)} = v^0_{(1)} - \frac{a'}{a}\alpha_{(1)} - \alpha'_{(1)}, \tag{3.21}$$

$$\tilde{v}^i_{(1)} = v^i_{(1)} - \beta'^{,i}_{(1)} - d^{i\prime}_{(1)}. \tag{3.22}$$

The four-velocity is, however, subject to the constraint (3.9); therefore Eq. (3.21) reduces to Eq. (3.15).

C. Second-order gauge transformations

We now extend these well-known transformation rules of linear metric perturbations to second order.

Second-order perturbations of the metric transform, according to Eq. (2.9), as

$$\delta^2\tilde{g}_{\mu\nu} = \delta^2 g_{\mu\nu} + 2\pounds_{\xi_{(1)}}\delta g_{\mu\nu} + \pounds^2_{\xi_{(1)}} g^{(0)}_{\mu\nu} + \pounds_{\xi_{(2)}} g^{(0)}_{\mu\nu}. \tag{3.23}$$

This leads to the following transformations in the second-order quantities appearing in Eqs. (3.1)–(3.3.)

Lapse perturbation:

$$\tilde{\psi}^{(2)} = \psi^{(2)} + \alpha^{(1)}\left[2\left(\psi'_{(1)} + 2\frac{a'}{a}\psi_{(1)}\right) + \alpha''_{(1)} + 5\frac{a'}{a}\alpha'_{(1)} + \left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)\alpha_{(1)}\right] + \xi^i_{(1)}\left(2\psi^{(1)}_{,i} + \alpha^{(1)\prime}_{,i} + \frac{a'}{a}\alpha^{(1)}_{,i}\right)$$
$$+ 2\alpha'_{(1)}(2\psi_{(1)} + \alpha'_{(1)}) + \xi^{i\prime}_{(1)}(\alpha^{(1)}_{,i} - \xi^{(1)\prime}_i - 2\omega^{(1)}_i) + \alpha'_{(2)} + \frac{a'}{a}\alpha_{(2)}. \tag{3.24}$$

Shift perturbation:

$$\tilde{\omega}^{(2)}_i = \omega^{(2)}_i - 4\psi^{(1)}\alpha^{(1)}_{,i} + \alpha^{(1)}\left[2\left(\omega^{(1)\prime}_i + 2\frac{a'}{a}\omega^{(1)}_i\right) - \alpha^{(1)\prime}_{,i} + \xi^{(1)\prime\prime}_i - 4\frac{a'}{a}(\alpha^{(1)}_{,i} - \xi^{(1)\prime}_i)\right] + \xi^j_{(1)}(2\omega^{(1)}_{i,j} - \alpha^{(1)}_{,ij} + \xi^{(1)\prime}_{i,j})$$
$$+ \alpha'_{(1)}(2\omega^{(1)}_i - 3\alpha^{(1)}_{,i} + \xi^{(1)\prime}_i) + \xi^{j\prime}_{(1)}(-4\phi^{(1)}\delta_{ij} + 2\chi^{(1)}_{ij} + 2\xi^{(1)}_{j,i} + \xi^{(1)}_{i,j}) + \xi^j_{(1),i}(2\omega^{(1)}_j - \alpha^{(1)}_{,j}) - \alpha^{(2)}_{,i} + \xi^{(2)\prime}_i. \tag{3.25}$$

Spatial metric, trace:

$$\tilde{\phi}^{(2)} = \phi^{(2)} + \alpha^{(1)}\left[2\left(\phi'_{(1)} + 2\frac{a'}{a}\phi_{(1)x}\right) - \left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)\alpha_{(1)} - \frac{a'}{a}\alpha'_{(1)}\right] + \xi^i_{(1)}\left(2\phi^{(1)}_{,i} - \frac{a'}{a}\alpha^{(1)}_{,i}\right)$$
$$- \frac{1}{3}\left(-4\phi_{(1)} + \alpha_{(1)}\partial_0 + \xi^i_{(1)}\partial_i + 4\frac{a'}{a}\alpha_{(1)}\right)\nabla^2\beta_{(1)} - \frac{1}{3}(2\omega^i_{(1)} - \alpha'^i_{(1)} + \xi^{i\prime}_{(1)})\alpha^{(1)}_{,i}$$
$$- \frac{1}{3}(2\chi^{(1)}_{ij} + \xi^{(1)}_{i,j} + \xi^{(1)}_{j,i})\xi^{(1)}_{j,i} - \frac{a'}{a}\alpha_{(2)} - \frac{1}{3}\nabla^2\beta_{(2)}. \tag{3.26}$$

Spatial metric, traceless part:

$$\tilde{\chi}^{(2)}_{ij} = \chi^{(2)}_{ij} + 2\left(\chi^{(1)\prime}_{ij} + 2\frac{a'}{a}\chi^{(1)}_{ij}\right)\alpha_{(1)} + 2\chi^{(1)}_{ij,k}\xi^k_{(1)} + 2\left(-4\phi_{(1)} + \alpha_{(1)}\partial_0 + \xi^k_{(1)}\partial_k + 4\frac{a'}{a}\alpha_{(1)}\right)(d^{(1)}_{(i,j)} + \mathrm{D}_{ij}\beta_{(1)})$$
$$+ 2\left[(2\omega^{(1)}_{(i} - \alpha^{(1)}_{,(i} + \xi^{(1)\prime}_{(i})\alpha^{(1)}_{,j)} - \frac{1}{3}\delta_{ij}(2\omega^k_{(1)} - \alpha^{,k}_{(1)} + \xi^{k\prime}_{(1)})\alpha^{(1)}_{,k}\right]$$
$$+ 2\left[(2\chi^{(1)}_{(i|k|} + \xi^{(1)}_{k,(i} + \xi^{(1)}_{(i,|k|)})\xi^{(1)k}_{,j)} - \frac{1}{3}\delta_{ij}(2\chi^{(1)}_{lk} + \xi^{(1)}_{k,l} + \xi^{(1)}_{l,k})\xi^{k,l}_{(1)}\right] + 2(d^{(2)}_{(i,j)} + \mathrm{D}_{ij}\beta_{(2)}). \tag{3.27}$$

For the energy density ϱ, or any other scalar, we have, from Eq. (2.9),

$$\delta^2\tilde{\varrho}=\delta^2\varrho+(\pounds_{\xi_{(2)}}+\pounds^2_{\xi_{(1)}})\varrho_{(0)}+2\pounds_{\xi_{(1)}}\delta\varrho. \tag{3.28}$$

From this we obtain, using Eq. (2.1),

$$\delta^2\tilde{\varrho}=\delta^2\varrho+\varrho'_{(0)}\alpha_{(2)}+\alpha_{(1)}(\varrho''_{(0)}\alpha_{(1)}+\varrho'_{(0)}\alpha'_{(1)}+2\,\delta\varrho')+\xi^i_{(1)}(\varrho'_{(0)}\alpha^{(1)}_{,i}+2\,\delta\varrho_{,i}). \tag{3.29}$$

.

For the four-velocity u^μ, we have, from Eq. (2.9),

$$\delta^2\tilde{u}^\mu=\delta^2u^\mu+(\pounds_{\xi_{(2)}}+\pounds^2_{\xi_{(1)}})u^\mu_{(0)}+2\pounds_{\xi_{(1)}}\delta u^\mu. \tag{3.30}$$

Using Eqs. (3.8) and (2.2) this gives

$$\begin{aligned}\tilde{v}^0_{(2)}=&v^0_{(2)}-\frac{a'}{a}\alpha_{(2)}-\alpha'_{(2)}+\alpha_{(1)}\left[2\left(v^{0\prime}_{(1)}-\frac{a'}{a}v^0_{(1)}\right)+\left(2\frac{a'^2}{a^2}-\frac{a''}{a}\right)\alpha_{(1)}+\frac{a'}{a}\alpha'_{(1)}-\alpha''_{(1)}\right]\\&+\xi^i_{(1)}\left(2v^0_{(1),i}-\frac{a'}{a}\alpha^{(1)}_{,i}-\alpha^{(1)\prime}_{,i}\right)+\alpha'_{(1)}(\alpha'_{(1)}-2v^0_{(1)})-2\alpha^{(1)}_{,i}v^i_{(1)}+\alpha^{(1)}_{,i}\xi^{i\prime}_{(1)},\end{aligned} \tag{3.31}$$

$$\begin{aligned}\tilde{v}^i_{(2)}=&v^i_{(2)}-\beta'^{,i}_{(2)}-d^{i\prime}_{(2)}+\alpha_{(1)}\left[2\left(v^{i\prime}_{(1)}-\frac{a'}{a}v^i_{(1)}\right)-\left(\xi^{i\prime\prime}_{(1)}-2\frac{a'}{a}\xi^{i\prime}_{(1)}\right)\right]\\&+\xi^j_{(1)}(2v^i_{(1),j}-\xi^{i\prime}_{(1),j})-\xi^i_{(1),j}(2v^j_{(1)}-\xi^{j\prime}_{(1)})+\xi^{i\prime}_{(1)}(2\psi_{(1)}+\alpha'_{(1)}),\end{aligned} \tag{3.32}$$

for the time and the space components, respectively. Again, the four-velocity u^μ is subject to $u^\mu u^\nu g_{\mu\nu}=-1$, which gives Eq. (3.10); therefore Eq. (3.31) reduces to Eq. (3.24).

IV. EVOLUTION IN THE SYNCHRONOUS GAUGE

A. General formalism

In this section we will obtain the second-order perturbations of the Einstein–de Sitter cosmological model in the synchronous gauge, including scalar and tensor modes in the initial conditions. The synchronous gauge, which has been one of the most frequently used in cosmological perturbation theory, is defined by the conditions $g_{00}=-a(\tau)^2$ and $g_{0i}=0$ [21]. In this way the four degrees of freedom associated with the coordinate invariance of the theory are fixed.

We start by writing the Einstein's equations for a perfect fluid of irrotational dust in synchronous and comoving coordinates. The formalism outlined in this subsection is discussed in greater detail in Ref. [4]. With the purpose of studying gravitational instability in the Einstein–de Sitter background, we first factor out the homogeneous and isotropic expansion of the universe.

The line element is written in the form

$$ds^2=a^2(\tau)[-d\tau^2+\gamma_{ij}(\mathbf{x},\tau)dx^idx^j], \tag{4.1}$$

with the spatial coordinates $\mathbf{x}$ representing Lagrangian coordinates for the fluid elements. The scale factor $a(\tau)\propto\tau^2$ is the solution of the Friedmann equations for a perfect fluid of dust in the Einstein–de Sitter universe.

By subtracting the isotropic Hubble flow, one introduces the extrinsic curvature of constant τ hypersurfaces:

$$\vartheta^i{}_j=\frac{1}{2}\gamma^{ik}\gamma'_{kj}, \tag{4.2}$$

with a prime denoting differentiation with respect to the conformal time τ.

One can then write Einstein's equations in a cosmologically convenient form. The energy constraint reads

$$\vartheta^2-\vartheta^i{}_j\vartheta^j{}_i+\frac{8}{\tau}\vartheta+\mathcal{R}=\frac{24}{\tau^2}\delta, \tag{4.3}$$

where $\mathcal{R}^i{}_j(\gamma)$ is the intrinsic curvature of constant time hypersurfaces, i.e., the conformal Ricci curvature of the three-space with metric γ_{ij}, and $\mathcal{R}=\mathcal{R}^i{}_i$. We also introduced the density contrast $\delta\equiv(\varrho-\varrho_{(0)})/\varrho_{(0)}$, with $\varrho(\mathbf{x},\tau)$ the mass density and $\varrho_{(0)}(\tau)=3/2\pi Ga^2(\tau)\tau^2$ its background mean value.

The momentum constraint reads

$$\vartheta^i{}_{j|i}=\vartheta_{,j}, \tag{4.4}$$

where the vertical bar indicates a covariant derivative in the three-space with metric γ_{ij}.

Finally, after replacing the density from the energy constraint and subtracting the background contribution, the evolution equation for the extrinsic curvature reads

$$\vartheta^{i\prime}_j+\frac{4}{\tau}\vartheta^i{}_j+\vartheta\vartheta^i{}_j+\frac{1}{4}(\vartheta^k{}_l\vartheta^l{}_k-\vartheta^2)\delta^i{}_j+\mathcal{R}^i{}_j-\frac{1}{4}\mathcal{R}\delta^i{}_j=0. \tag{4.5}$$

Also useful is the Raychaudhuri equation for the evolution of the peculiar volume expansion scalar ϑ, namely,

$$\vartheta' + \frac{2}{\tau}\vartheta + \vartheta^i{}_j \vartheta^j{}_i + \frac{6}{\tau^2}\delta = 0. \tag{4.6}$$

An advantage of this gauge is that there are only geometric quantities in the equations, namely, the spatial metric tensor with its time and space derivatives. The only remaining variable, the density contrast, can indeed be rewritten in terms of γ_{ij}, by solving the continuity equation. We have

$$\delta(\mathbf{x},\tau) = (1+\delta_0(\mathbf{x}))[\,\gamma(\mathbf{x},\tau)/\gamma_0(\mathbf{x})\,]^{-1/2} - 1, \tag{4.7}$$

with $\gamma \equiv \det\gamma_{ij}$. We denote by a subscript 0 without parentheses the initial condition of the referred quantity.

B. First-order perturbations

We are now ready to deal with the equations above at the linear level. Let us then write the conformal spatial metric tensor in the form

$$\gamma_{ij} = \delta_{ij} + \gamma^{(1)}_{\mathrm{S}ij}. \tag{4.8}$$

According to our general definitions we then write

$$\gamma^{(1)}_{\mathrm{S}ij} = -2\phi^{(1)}_{\mathrm{S}}\delta_{ij} + \mathrm{D}_{ij}\chi^{(1)\parallel}_{\mathrm{S}} + \partial_i\chi^{(1)\perp}_{\mathrm{S}j} + \partial_j\chi^{(1)\perp}_{\mathrm{S}i} + \chi^{(1)\top}_{ij}, \tag{4.9}$$

with

$$\partial^i\chi^{(1)\perp}_{\mathrm{S}i} = \chi^{(1)\top i}_{\ \ i} = \partial^i\chi^{(1)\top}_{ij} = 0. \tag{4.10}$$

Recall that at first order the tensor modes $\chi^{(1)\top}_{ij}$ are gauge invariant.

As is well known, in linear theory, scalar, vector, and tensor modes are independent. The equation of motion for the tensor modes is obtained by linearizing the traceless part of the $\vartheta^i{}_j$ evolution equation. One has

$$\chi^{(1)\top\prime\prime}_{ij} + \frac{4}{\tau}\chi^{(1)\top\prime}_{ij} - \nabla^2\chi^{(1)\top}_{ij} = 0, \tag{4.11}$$

which is the equation for the free propagation of gravitational waves in the Einstein–de Sitter universe. The general solution of this equation is

$$\chi^{\top(1)}_{ij}(\mathbf{x},\tau) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,\exp(i\mathbf{k}\cdot\mathbf{x})\chi^{(1)}_{\sigma}(\mathbf{k},\tau)\epsilon^{\sigma}_{ij}(\hat{\mathbf{k}}), \tag{4.12}$$

where $\epsilon^{\sigma}_{ij}(\hat{\mathbf{k}})$ is the polarization tensor, with σ ranging over the polarization components $+,\times$, and $\chi^{(1)}_{\sigma}(\mathbf{k},\tau)$ the amplitudes of the two polarization states, whose time evolution can be represented as

$$\chi^{(1)}_{\sigma}(\mathbf{k},\tau) = A(k)a_{\sigma}(\mathbf{k})\left(\frac{3j_1(k\tau)}{k\tau}\right), \tag{4.13}$$

with j_1 the spherical Bessel function of order 1 and $a_{\sigma}(\mathbf{k})$ a zero mean random variable with autocorrelation function $\langle a_{\sigma}(\mathbf{k})a_{\sigma'}(\mathbf{k}')\rangle = (2\pi)^3 k^{-3}\delta^3(\mathbf{k}+\mathbf{k}')\delta_{\sigma\sigma'}$. The spectrum of the gravitational wave background depends on the processes by which it was generated, and, for example, in most inflationary models, $A(k)$ is nearly scale invariant and proportional to the Hubble constant during inflation.

In the irrotational case the linear vector perturbations represent gauge modes which can be set to zero: $\chi^{(1)\perp}_i = 0$.

The two scalar modes are linked together via the momentum constraint, leading to the condition

$$\phi^{(1)}_{\mathrm{S}} + \frac{1}{6}\nabla^2\chi^{(1)\parallel}_{\mathrm{S}} = \phi^{(1)}_{\mathrm{S}0} + \frac{1}{6}\nabla^2\chi^{(1)\parallel}_{\mathrm{S}0}. \tag{4.14}$$

The energy constraint gives

$$\nabla^2\left[\frac{2}{\tau}\chi^{(1)\parallel\prime}_{\mathrm{S}} + \frac{6}{\tau^2}(\chi^{(1)\parallel}_{\mathrm{S}} - \chi^{(1)\parallel}_{\mathrm{S}0}) + 2\phi^{(1)}_{\mathrm{S}0} + \frac{1}{3}\nabla^2\chi^{(1)\parallel}_{\mathrm{S}_0}\right] = \frac{12}{\tau^2}\delta_0, \tag{4.15}$$

having consistently assumed $\delta_0 \ll 1$.

The evolution equation also gives an equation for the scalar modes:

$$\chi^{(1)\parallel\prime\prime}_{\mathrm{S}} + \frac{4}{\tau}\chi^{(1)\parallel\prime}_{\mathrm{S}} + \frac{1}{3}\nabla^2\chi^{(1)\parallel}_{\mathrm{S}} = -2\phi^{(1)}_{\mathrm{S}}. \tag{4.16}$$

An equation only for the scalar mode $\chi^{(1)\parallel}_{\mathrm{S}}$ can be obtained by combining together the evolution equation and the energy constraint:

$$\nabla^2\left[\chi^{(1)\parallel\prime\prime}_{\mathrm{S}} + \frac{2}{\tau}\chi^{(1)\parallel\prime}_{\mathrm{S}} - \frac{6}{\tau^2}(\chi^{(1)\parallel}_{\mathrm{S}} - \chi^{(1)\parallel}_{\mathrm{S}0})\right] = -\frac{12}{\tau^2}\delta_0. \tag{4.17}$$

On the other hand, by linearizing the solution of the continuity equation, we obtain

$$\delta^{(1)}_{\mathrm{S}} = \delta_0 - \frac{1}{2}\nabla^2(\chi^{(1)\parallel}_{\mathrm{S}} - \chi^{(1)\parallel}_{\mathrm{S}0}), \tag{4.18}$$

which replaced in the previous expression gives

$$\delta^{(1)\prime\prime}_{\mathrm{S}} + \frac{2}{\tau}\delta^{(1)\prime}_{\mathrm{S}} - \frac{6}{\tau^2}\delta^{(1)}_{\mathrm{S}} = 0. \tag{4.19}$$

This is the equation for linear density fluctuation (see, e.g., Ref. [1]), whose general solution is straightforward to obtain.

The equations above have been obtained in whole generality; one could have used instead the well-known residual gauge ambiguity of the synchronous coordinates (see, e.g., Refs. [4,18]) to simplify their form. For instance, one could fix $\chi^{(1)\parallel}_0$ so that $\nabla^2\chi^{(1)\parallel}_{\mathrm{S}0} = -2\delta_0$, and thus the $\chi^{(1)\parallel}_{\mathrm{S}}$ evolution equation takes the same form as that for δ. With such a gauge fixing one obtains

$$\chi^{(1)\parallel}_{\mathrm{S}}(\mathbf{x},\tau) = \chi_+(\mathbf{x})\tau^2 + \chi_-(\mathbf{x})\tau^{-3}, \tag{4.20}$$

where $\chi_{\pm}$ set the amplitudes of the growing $(+)$ and decaying $(-)$ modes. In what follows, we shall restrict ourselves to the growing mode. The effect of the decaying mode on second-order perturbations has been considered in Ref. [17] and in Ref. [18] and will not be studied here. The amplitude of the growing mode is related to the initial *peculiar gravitational potential*, through $\chi_+ \equiv -(\frac{1}{3})\varphi$, where, in turn, φ is related to δ_0 through the cosmological Poisson equation $\nabla^2\varphi(\mathbf{x}) = (6/\tau_0^2)\,\delta_0(\mathbf{x})$. Therefore,

$$\mathrm{D}_{ij}\chi_{\mathrm{S}}^{(1)\parallel} = -\frac{\tau^2}{3}\left(\varphi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\varphi\right). \tag{4.21}$$

The remaining scalar mode

$$\phi_{\mathrm{S}}^{(1)}(\mathbf{x},\tau) = \frac{5}{3}\varphi(\mathbf{x}) + \frac{\tau^2}{18}\nabla^2\varphi(\mathbf{x}) \tag{4.22}$$

immediately follows.

The linear metric perturbation therefore reads

$$\gamma_{\mathrm{S}ij}^{(1)} = -\frac{10}{3}\varphi\,\delta_{ij} - \frac{\tau^2}{3}\varphi_{,ij} + \chi_{ij}^{(1)\top}. \tag{4.23}$$

With purely growing-mode initial conditions, the linear density contrast reads

$$\delta_{\mathrm{S}}^{(1)} = \frac{\tau^2}{6}\nabla^2\varphi. \tag{4.24}$$

C. Second-order perturbations

The conformal spatial metric tensor up to second order is expanded as

$$\gamma_{ij} = \delta_{ij} + \gamma_{\mathrm{S}ij}^{(1)} + \frac{1}{2}\gamma_{\mathrm{S}ij}^{(2)}, \tag{4.25}$$

with

$$\gamma_{\mathrm{S}ij}^{(2)} = -2\,\phi_{\mathrm{S}}^{(2)}\delta_{ij} + \chi_{\mathrm{S}ij}^{(2)} \tag{4.26}$$

and $\chi_{\mathrm{S}j}^{(2)i} = 0$.

The technique of second-order perturbation theory is straightforward: with the help of the relations reported in Appendix B, we first substitute the expansion above in our exact fluid-dynamical equations (momentum and energy constraints plus evolution and Raychaudhuri equations), obtaining equations for $\gamma_{\mathrm{S}ij}^{(2)}$ with source terms containing quadratic combinations of $\gamma_{\mathrm{S}ij}^{(1)}$ plus a few more terms involving δ_0. Next, we have to solve these equations for the modes $\phi_{\mathrm{S}}^{(2)}$ and $\chi_{\mathrm{S}ij}^{(2)}$ in terms of the initial peculiar gravitational potential φ and the linear tensor modes $\chi_{ij}^{(1)\top}$.

Let us now give the equations which govern the evolution of the second-order metric perturbations.

Raychaudhuri equation:

$$\begin{aligned}\phi_{\mathrm{S}}^{(2)\prime\prime} + \frac{2}{\tau}\phi_{\mathrm{S}}^{(2)\prime} - \frac{6}{\tau^2}\phi_{\mathrm{S}}^{(2)} = &-\frac{1}{6}\gamma_{\mathrm{S}}^{(1)ij\prime}\left(\gamma_{\mathrm{S}ij}^{(1)\prime} - \frac{4}{\tau}\gamma_{\mathrm{S}ij}^{(1)}\right) + \frac{1}{6}\left[2g_{\mathrm{S}}^{(1)ij}\left(2\gamma_{\mathrm{S}i,kj}^{(1)k} - \nabla^2\gamma_{\mathrm{S}ij}^{(1)} - \gamma_{\mathrm{S}k,ij}^{(1)k}\right) - \gamma_{\mathrm{S}k}^{(1)k}\left(\gamma_{\mathrm{S},ij}^{(1)ij} - \nabla^2\gamma_{\mathrm{S}i}^{(1)i}\right)\right] \\ &- \frac{2}{\tau^2}\left[-\frac{1}{4}\left(\gamma_{\mathrm{S}i}^{(1)i} - \gamma_{\mathrm{S}0i}^{(1)i}\right)^2 - \frac{1}{2}\left(\gamma_{\mathrm{S}}^{(1)ij}\gamma_{\mathrm{S}ij}^{(1)} - \gamma_{\mathrm{S}0}^{(1)ij}\gamma_{\mathrm{S}0ij}^{(1)}\right) + \delta_0\left(\gamma_{\mathrm{S}i}^{(1)i} - \gamma_{\mathrm{S}0i}^{(1)i}\right)\right].\end{aligned} \tag{4.27}$$

Energy constraint:

$$\begin{aligned}\frac{2}{\tau}\phi_{\mathrm{S}}^{(2)\prime} - \frac{1}{3}\nabla^2\phi_{\mathrm{S}}^{(2)} + \frac{6}{\tau^2}\phi_{\mathrm{S}}^{(2)} - \frac{1}{12}\chi_{\mathrm{S},ij}^{(2)ij} = &-\frac{2}{3\tau}\gamma_{\mathrm{S}}^{(1)ij}\gamma_{\mathrm{S}ij}^{(1)\prime} - \frac{1}{24}\left(\gamma_{\mathrm{S}}^{(1)ij\prime}\gamma_{\mathrm{S}ij}^{(1)\prime} - \gamma_{\mathrm{S}i}^{(1)i\prime}\gamma_{\mathrm{S}j}^{(1)j\prime}\right) \\ &+ \frac{1}{6}\left[\gamma_{\mathrm{S}}^{(1)ij}\left(\nabla^2\gamma_{\mathrm{S}ij}^{(1)} + \gamma_{\mathrm{S}k,ij}^{(1)k} - 2\gamma_{\mathrm{S}i,jk}^{(1)k}\right) + \gamma_{\mathrm{S},k}^{(1)ki}\left(\gamma_{\mathrm{S}j,i}^{(1)j} - \gamma_{\mathrm{S}i,j}^{(1)j}\right)\right. \\ &\left. + \frac{3}{4}\gamma_{\mathrm{S}}^{(1)ij}{}_{,k}\gamma_{\mathrm{S}ij}^{(1),k} - \frac{1}{2}\gamma_{\mathrm{S}}^{(1)ij}{}_{,k}\gamma_{\mathrm{S}i,j}^{(1)k} - \frac{1}{4}\gamma_{\mathrm{S}i}^{(1)i,k}\gamma_{\mathrm{S}j}^{(1)j}{}_{,k}\right] \\ &+ \frac{2}{\tau^2}\left[-\frac{1}{4}\left(\gamma_{\mathrm{S}i}^{(1)i} - \gamma_{\mathrm{S}0i}^{(1)i}\right)^2 - \frac{1}{2}\left(\gamma_{\mathrm{S}}^{(1)ij}\gamma_{\mathrm{S}ij}^{(1)} - \gamma_{\mathrm{S}0}^{(1)ij}\gamma_{\mathrm{S}0ij}^{(1)}\right) + \delta_0\left(\gamma_{\mathrm{S}i}^{(1)i} - \gamma_{\mathrm{S}0i}^{(1)i}\right)\right].\end{aligned} \tag{4.28}$$

Momentum constraint:

$$2\phi_{\mathrm{S},j}^{(2)\prime} + \frac{1}{2}\chi_{\mathrm{S}j,i}^{(2)i\prime} = \gamma_{\mathrm{S}}^{(1)ik}\left(\gamma_{\mathrm{S}jk,i}^{(1)\prime} - \gamma_{\mathrm{S}ik,j}^{(1)\prime}\right) + \gamma_{\mathrm{S}}^{(1)ik}{}_{,i}\gamma_{\mathrm{S}jk}^{(1)\prime} - \frac{1}{2}\gamma_{\mathrm{S}}^{(1)ik}{}_{,j}\gamma_{\mathrm{S}ik}^{(1)\prime} - \frac{1}{2}\gamma_{\mathrm{S}i}^{(1)i}{}_{,k}\gamma_{\mathrm{S}j}^{(1)k\prime}. \tag{4.29}$$

Evolution equation:

$$
\begin{aligned}
&-\left(\phi_{\rm S}^{(2)\prime\prime}+\frac{4}{\tau}\phi_{\rm S}^{(2)\prime}\right)\delta^i{}_j+\frac{1}{2}\left(\chi_{{\rm S}j}^{(2)i\prime\prime}+\frac{4}{\tau}\chi_{{\rm S}j}^{(2)i\prime}\right)+\phi_{{\rm S},j}^{(2),i}-\frac{1}{4}\chi_{\rm S}^{(2)kl}{}_{,k\not\prime}\delta^i{}_j+\frac{1}{2}\chi_{{\rm S},kj}^{(2)ki}+\frac{1}{2}\chi_{{\rm S}j,k}^{(2)k,i}-\frac{1}{2}\nabla^2\chi_{{\rm S}j}^{(2)i}\\
&\quad=\gamma_{\rm S}^{(1)ik\prime}\gamma_{{\rm S}kj}^{(1)\prime}-\frac{1}{2}\gamma_{{\rm S}k}^{(1)k\prime}\gamma_{{\rm S}j}^{(1)i\prime}+\frac{1}{8}[(\gamma_{{\rm S}k}^{(1)k\prime})^2-\gamma_{{\rm S}\not\prime}^{(1)k\prime}\gamma_{{\rm S}k}^{(1)l\prime}]\delta^i{}_j-\frac{1}{2}\Big[-\gamma_{{\rm S}j}^{(1)i}(\gamma_{{\rm S}\not\prime,k}^{(1)k,l}-\nabla^2\gamma_{{\rm S}k}^{(1)k})\\
&\qquad+2\gamma_{\rm S}^{(1)kl}(\gamma_{{\rm S}j,kl}^{(1)i}+\gamma_{{\rm S}kl,j}^{(1),i}-\gamma_{{\rm S}l,jk}^{(1)i}-\gamma_{{\rm S}lj,k}^{(1),i})+2\gamma_{{\rm S},k}^{(1)kl}(\gamma_{{\rm S}j,l}^{(1)i}-\gamma_{{\rm S}l,j}^{(1)i}-\gamma_{{\rm S}jl}^{(1),i})\\
&\qquad+2\gamma_{\rm S}^{(1)ki}{}_{,l}\gamma_{{\rm S}jk}^{(1),l}-2\gamma_{\rm S}^{(1)ki}{}_{,l}\gamma_{{\rm S}j,k}^{(1)l}+\gamma_{\rm S}^{(1)kl}{}_{,j}\gamma_{{\rm S}kl}^{(1),i}\\
&\qquad+\gamma_{{\rm S}l,k}^{(1)l}(\gamma_{{\rm S},j}^{(1)ki}+\gamma_{{\rm S}j}^{(1)k,i}-\gamma_{{\rm S}j}^{(1)i,k})-\gamma_{\rm S}^{(1)kl}(\nabla^2\gamma_{{\rm S}kl}^{(1)}+\gamma_{{\rm S}m,kl}^{(1)m}-2\gamma_{{\rm S}k,ml}^{(1)m})\delta^i{}_j-\gamma_{{\rm S},l}^{(1)lk}(\gamma_{{\rm S}m,k}^{(1)m}-\gamma_{{\rm S}k,m}^{(1)m})\delta^i{}_j\\
&\qquad-\frac{3}{4}\gamma_{\rm S}^{(1)kl}{}_{,m}\gamma_{{\rm S}kl}^{(1),m}\delta^i{}_j+\frac{1}{2}\gamma_{\rm S}^{(1)kl}{}_{,m}\gamma_{\rm Sk}^{(1)m}{}_{,l}\delta^i{}_j+\frac{1}{4}\gamma_{{\rm S}k}^{(1)k,m}\gamma_{\rm S}^{(1)l}{}_{,m}\delta^i{}_j\,. \qquad (4.30)
\end{aligned}
$$

The next step is to solve these equations. In these calculations, we can make the simplifying assumption that the initial conditions are taken at conformal time $\tau_0=0$ (implying also $\delta_0=0$). One can start from the Raychaudhuri equation, to obtain the trace of the second-order metric tensor. (Actually, in order to obtain the subleading mode generated by linear scalar modes, we also need the energy constraint.) The resulting expression for $\phi_{\rm S}^{(2)}$ is

$$
\phi_{\rm S}^{(2)}=\frac{\tau^4}{252}\left(-\frac{10}{3}\varphi^{,ki}\varphi_{,ki}+(\nabla^2\varphi)^2\right)+\frac{5\tau^2}{18}\left(\varphi^{,k}\varphi_{,k}+\frac{4}{3}\varphi\nabla^2\varphi\right)+\phi_{\rm S(t)}^{(2)}, \qquad (4.31)
$$

where $\phi_{\rm S(t)}^{(2)}$, which is the part of $\phi_{\rm S}^{(2)}$ generated by the presence of tensor modes at the linear level, reads

$$
\phi_{\rm S(t)}^{(2)}=\frac{\tau^2}{5}\int_0^\tau\frac{d\tau'}{\tau'}\mathcal{Q}(\tau')-\frac{1}{5\tau^3}\int_0^\tau d\tau'\,\tau'^4\mathcal{Q}(\tau'), \qquad (4.32)
$$

with $\mathcal{Q}(\mathbf{x},\tau)$ a source term whose explicit form is reported in Appendix C.

The expression for $\chi_{{\rm S}ij}^{(2)}$ is obtained by first replacing $\phi_{\rm S}^{(2)}$ into the remaining equations and solving them in the following order: energy constraint $\rightarrow$ momentum constraint $\rightarrow$ (traceless part of the) evolution equation. We obtain

$$
\begin{aligned}
\chi_{{\rm S}ij}^{(2)}=&\frac{\tau^4}{126}\left(19\varphi^{,k}_{\ ,i}\varphi_{,kj}-12\varphi_{,ij}\nabla^2\varphi+4(\nabla^2\varphi)^2\delta_{ij}-\frac{19}{3}\varphi^{,kl}\varphi_{,kl}\delta_{ij}\right)\\
&+\frac{5\tau^2}{9}\left(-6\varphi_{,i}\varphi_{,j}-4\varphi\varphi_{,ij}+2\varphi^{,k}\varphi_{,k}\delta_{ij}+\frac{4}{3}\varphi\nabla^2\varphi\,\delta_{ij}\right)+\pi_{{\rm S}ij}+\chi_{{\rm S(t)}ij}^{(2)}, \qquad (4.33)
\end{aligned}
$$

where $\chi_{{\rm S(t)}ij}^{(2)}$ is the part of the traceless tensor $\chi_{{\rm S}ij}^{(2)}$ generated by the presence of tensor modes at the linear level and includes the effects of scalar-tensor and tensor-tensor couplings; its expression can be derived from the equations given in Appendix C. The transverse and traceless contribution $\pi_{{\rm S}ij}$, which represents the second-order tensor mode generated by scalar initial perturbations, is determined by the inhomogeneous wave equation

$$
\pi_{{\rm S}ij}''+\frac{4}{\tau}\pi_{{\rm S}ij}'-\nabla^2\pi_{{\rm S}ij}=-\frac{\tau^4}{21}\nabla^2\mathcal{S}_{ij}, \qquad (4.34)
$$

with

$$
\mathcal{S}_{ij}=\nabla^2\Psi_0\delta_{ij}+\Psi_{0,ij}+2(\varphi_{,ij}\nabla^2\varphi-\varphi_{,ik}\varphi^{,k}_{\ ,j}), \qquad (4.35)
$$

where

$$
\nabla^2\Psi_0=-\frac{1}{2}[(\nabla^2\varphi)^2-\varphi_{,ik}\varphi^{,ik}]. \qquad (4.36)
$$

This equation can be solved using the Green method; we obtain for $\pi_{{\rm S}ij}$ that

$$
\pi_{ij}(\mathbf{x},\tau)=\frac{\tau^4}{21}\mathcal{S}_{ij}(\mathbf{x})+\frac{4\tau^2}{3}\mathcal{T}_{ij}(\mathbf{x})+\tilde{\pi}_{ij}(\mathbf{x},\tau), \qquad (4.37)
$$

where $\nabla^2\mathcal{T}_{ij}=\mathcal{S}_{ij}$ and the remaining piece $\tilde{\pi}_{ij}$, containing a term that is constant in time and another one that oscillates with decreasing amplitude, can be written as

$$
\tilde{\pi}_{ij}(\mathbf{x},\tau)=\frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,\exp(i\mathbf{k}\cdot\mathbf{x})\frac{40}{k^4}\mathcal{S}_{ij}(\mathbf{k})\left(\frac{1}{3}-\frac{j_1(k\tau)}{k\tau}\right), \qquad (4.38)
$$

with $\mathcal{S}_{ij}(\mathbf{k})=\int d^3\mathbf{x}\,\exp(-i\mathbf{k}\cdot\mathbf{x})\mathcal{S}_{ij}(\mathbf{x})$.

The second-order density contrast reads

$$\delta_{\rm S}^{(2)}=\frac{\tau^4}{252}[5(\nabla^2\varphi)^2+2\varphi^{,ij}\varphi_{,ij}]+\frac{\tau^2}{36}(15\varphi^{,i}\varphi_{,i}+40\varphi\nabla^2\varphi-6\varphi^{,ij}\chi_{ij}^{(1)\top})+\frac{1}{4}(\chi^{(1)\top ij}\chi_{ij}^{(1)\top}-\chi_0^{(1)\top ij}\chi_{0ij}^{(1)\top})+\frac{3}{2}\phi_{\rm S(t)}^{(2)}. \tag{4.39}$$

An important aspect of our results is that linear tensor modes (gravitational waves) can generate second-order perturbations of any kind (scalars, vectors, and tensors). This interesting fact, which was first noticed by Tomita [20], is nicely displayed by the above formula for the mass-density contrast, which even in the absence of initial density fluctuations takes a contribution from primordial gravitational waves. More in general, we should stress that our expressions completely determine the rate of growth of perturbations up to second order.

V. FROM THE SYNCHRONOUS TO THE POISSON GAUGE

In this section we are going to obtain the metric perturbations in the Poisson gauge by transforming the results obtained in the synchronous gauge in the previous section. The Poisson gauge, recently discussed by Bertschinger [2] and Ma and Bertschinger [14], is defined by $\omega_i^{(r),i}=\chi_{ij}^{(r),j}=0$. Then, one scalar degree of freedom is eliminated from g_{0i} ($\omega^{(r)\|}=0$) and one scalar and two vector degrees of freedom from g_{ij} ($\chi^{(r)\|}=\chi_i^{(r)\perp}=0$). This gauge generalizes the well-known *longitudinal gauge* to include vector and tensor modes. The latter gauge, in which $\omega_i^{(r)}=\chi_{ij}^{(r)}=0$, has been widely used in the literature to investigate the evolution of scalar perturbations [28]. Since the vector and tensor modes are set to zero by hand, the longitudinal gauge cannot be used to study perturbations beyond the linear regime, because in the nonlinear case the scalar, vector, and tensor modes are dynamically coupled. In other words, even if one starts with purely scalar linear perturbations as initial conditions for the second-order theory, vector and tensor modes are dynamically generated [3].

A. First-order transformations

Given the perturbation of the metric in one gauge, it is easy to obtain, from Eqs. (3.15)–(3.18), the gauge transformation to the other one, hence the perturbations in the new gauge. In the particular case of the synchronous and Poisson gauges, we have

$$\psi_{\rm P}^{(1)}=\alpha^{(1)\prime}+\frac{a'}{a}\alpha^{(1)}, \tag{5.1}$$

$$\alpha^{(1)}=\beta^{(1)\prime}, \tag{5.2}$$

$$\omega_{{\rm P}\,i}^{(1)}=d_i^{(1)\prime}, \tag{5.3}$$

$$\phi_{\rm P}^{(1)}=\phi_{\rm S}^{(1)}-\frac{1}{3}\nabla^2\beta^{(1)}-\frac{a'}{a}\alpha^{(1)}, \tag{5.4}$$

$$\mathrm{D}_{ij}(\chi_{\rm S}^{(1)\|}+2\beta^{(1)})=0, \tag{5.5}$$

$$\chi_{{\rm S}\,(i,j)}^{(1)\perp}+d_{(i,j)}^{(1)}=0, \tag{5.6}$$

$$\chi_{{\rm P}\,ij}^{(1)\top}=\chi_{{\rm S}\,ij}^{(1)\top}. \tag{5.7}$$

The parameters $\beta^{(1)}$, $\alpha^{(1)}$, and $d_i^{(1)}$ of the gauge transformation can be obtained from Eqs. (5.5), (5.2), and (5.6), respectively, while the transformed metric perturbations follow from Eqs. (5.1), (5.3), (5.4), and (5.7).

Once these parameters are known, the transformation rules for the energy density ϱ or any other scalar, and those for the four-velocity u^μ, follow trivially from Eqs. (3.19), (3.21), and (3.22). In the irrotational case studied in the last section $\chi_{{\rm S}\,ij}^{(1)\perp}=v_{(1)\perp}^{i}=0$ and thus $d_i^{(1)}=\omega_{{\rm P}\,i}^{(1)}=\chi_{{\rm P}\,ij}^{(1)\perp}=0$.

B. Second-order transformations

The more general transformation expressions follow straightforwardly from Eqs. (3.24)–(3.27), (3.29), and (3.32).

Transforming from the synchronous to the Poisson gauge, the expression for $\psi_{\rm P}^{(2)}$ can be easily obtained from Eq. (3.24), using Eq. (5.2) and the condition $d_i^{(1)}=0$ to express all the first-order quantities in terms of $\beta^{(1)}$:

$$\psi_{\rm P}^{(2)}=\beta_{(1)}'\left[\beta_{(1)}'''+5\frac{a'}{a}\beta_{(1)}''+\left(\frac{a''}{a}+\frac{a'^2}{a^2}\right)\beta_{(1)}'\right]+\beta_{(1)}^{,i}\left(\beta_{,i}^{(1)\prime\prime}+\frac{a'}{a}\beta_{,i}^{(1)\prime}\right)+2\beta_{(1)}''^2+\alpha^{(2)\prime}+\frac{a'}{a}\alpha^{(2)}. \tag{5.8}$$

For $\omega_{{\rm P}\,i}^{(2)}$ and $\phi_{\rm P}^{(2)}$ we get

$$\omega_{{\rm P}\,i}^{(2)}=-2\left(2\phi_{\rm S}^{(1)}+\beta_{(1)}''-\frac{2}{3}\nabla^2\beta_{(1)}\right)\beta_{,i}^{(1)\prime}-2\beta_{,j}^{(1)\prime}\beta_{,i}^{(1),j}+2\chi_{ij}^{(1)\top}\beta_{(1)}'^{,i}-\alpha_{,i}^{(2)}+\beta_{,i}^{(2)\prime}+d_i^{(2)\prime}, \tag{5.9}$$

$$\phi_{\rm P}^{(2)}=\phi_{\rm S}^{(2)}+\beta'_{(1)}\left[2\left(\phi_{\rm S}^{(1)\prime}+2\frac{a'}{a}\phi_{\rm S}^{(1)}\right)-\left(\frac{a''}{a}+\frac{a'^2}{a^2}\right)\beta'_{(1)}-\frac{a'}{a}\beta''_{(1)}\right]$$
$$-\frac{1}{3}\left(-4\phi_{\rm S}^{(1)}+\beta'_{(1)}\partial_0+\beta^{,i}_{(1)}\partial_i+4\frac{a'}{a}\beta'_{(1)}+\frac{4}{3}\nabla^2\beta_{(1)}\right)\nabla^2\beta_{(1)}$$
$$+\beta^{,i}_{(1)}\left(2\phi_{{\rm S},i}^{(1)}-\frac{a'}{a}\beta_{,i}^{(1)\prime}\right)+\frac{2}{3}\beta^{(1)}_{,ij}\beta^{,ij}_{(1)}-\frac{2}{3}\chi_{ij}^{(1)\top}\beta^{,ij}_{(1)}-\frac{a'}{a}\alpha_{(2)}-\frac{1}{3}\nabla^2\beta_{(2)}. \tag{5.10}$$

For $\chi_{{\rm P}\,ij}^{(2)}$ we obtain

$$\chi_{{\rm P}\,ij}^{(2)}=\chi_{{\rm S}\,ij}^{(2)}+2\left(\frac{4}{3}\nabla^2\beta_{(1)}-4\phi_{\rm S}^{(1)}-\beta'_{(1)}\partial_0-\beta^{,k}_{(1)}\partial_k\right){\rm D}_{ij}\beta_{(1)}-4\left(\beta^{(1)}_{,ik}\beta^{,k}_{(1),j}-\frac{1}{3}\delta_{ij}\beta^{(1)}_{,lk}\beta^{,lk}_{(1)}\right)$$
$$+2\left(\chi_{ij}^{(1)\top\prime}+2\frac{a'}{a}\chi_{ij}^{(1)\top}\right)\beta^{(1)\prime}+2\chi^{(1)\top}_{ij,k}\beta^{(1),k}+2\chi^{(1)\top}_{ik}\beta^{(1),k}_{,j}+2\chi^{(1)\top}_{jk}\beta^{(1),k}_{,i}-\frac{4}{3}\delta_{ij}\chi^{(1)\top}_{lk}\beta^{(1),lk}+2(d^{(2)}_{(i,j)}+{\rm D}_{ij}\beta^{(2)}). \tag{5.11}$$

Given the metric perturbations in the synchronous gauge, these constitute a set of coupled equations for the second-order parameters of the transformation, $\alpha^{(2)}$, $\beta^{(2)}$, and $d_i^{(2)}$ and the second-order metric perturbations in the Poisson gauge, $\psi_{\rm P}^{(2)}$, $\omega_{{\rm P}\,i}^{(2)}$, $\phi_{\rm P}^{(2)}$, and $\chi_{{\rm P}\,ij}^{(2)}$. This system can be solved in the following way. Since in the Poisson gauge $\partial^i\chi_{{\rm P}\,ij}^{(2)}=0$, we can use the fact that $\partial^i\partial^j\chi_{{\rm P}\,ij}^{(2)}=0$ and the property $\partial^i d_i^{(1)}=0$, together with Eq. (5.11), to obtain an expression for $\nabla^2\nabla^2\beta^{(2)}$, from which $\beta^{(2)}$ can be computed:

$$\nabla^2\nabla^2\beta^{(2)}=-\frac{3}{4}\chi^{(2),ij}_{{\rm S}\,ij}+6\phi_{\rm S}^{(1),ij}\beta^{(1)}_{,ij}-2\nabla^2\phi_{\rm S}^{(1)}\nabla^2\beta_{(1)}+8\phi_{\rm S}^{(1),i}\nabla^2\beta^{(1)}_{,i}+4\phi_{\rm S}^{(1)}\nabla^2\nabla^2\beta_{(1)}+4\nabla^2\beta^{(1)}_{,ij}\beta^{,ij}_{(1)}$$
$$-\frac{1}{6}\nabla^2\beta^{,i}_{(1)}\nabla^2\beta^{(1)}_{,i}+\frac{5}{2}\beta^{,ijk}_{(1)}\beta^{(1)}_{,ijk}-\frac{2}{3}\nabla^2\beta_{(1)}\nabla^2\nabla^2\beta_{(1)}+\frac{3}{2}\beta^{,ij}_{(1)\prime}\beta^{(1)\prime}_{,ij}$$
$$-\frac{1}{2}\nabla^2\beta'_{(1)}\nabla^2\beta'_{(1)}+2\beta^{,i\prime}_{(1)}\nabla^2\beta^{(1)\prime}_{,i}+\beta'_{(1)}\nabla^2\nabla^2\beta'_{(1)}$$
$$+\beta^{,i}_{(1)}\nabla^2\nabla^2\beta^{(1)}_{,i}-\frac{3}{2}\left(\chi_{ij}^{(1)\top\prime}+2\frac{a'}{a}\chi_{ij}^{(1)\top}\right)\beta^{(1)\prime,ij}-\frac{5}{2}\chi^{(1)\top}_{ij,k}\beta^{(1),ijk}-2\chi_{ij}^{(1)\top}\nabla^2\beta^{,ij}_{(1)}+\nabla^2\chi_{ij}^{(1)\top}\beta^{,ij}_{(1)}. \tag{5.12}$$

Then, using the condition $\partial^i\chi_{{\rm P}\,ij}^{(2)}=0$ and substituting $\beta^{(2)}$ we obtain an equation for $d_i^{(2)}$:

$$\nabla^2 d_i^{(2)}=-\frac{4}{3}\nabla^2\beta^{(2)}_{,i}-\chi^{(2),j}_{{\rm S}\,ij}+8\phi_{\rm S}^{(1),j}{\rm D}_{ij}\beta_{(1)}+\frac{16}{3}\phi_{\rm S}^{(1)}\nabla^2\beta^{(1)}_{,i}+\frac{2}{3}\nabla^2\beta^{,j}_{(1)}\beta^{(1)}_{,ij}+\frac{10}{3}\beta^{,jk}_{(1)}\beta^{(1)}_{,ijk}$$
$$-\frac{8}{9}\nabla^2\beta_{(1)}\nabla^2\beta^{(1)}_{,i}+2\beta^{,j\prime}_{(1)}{\rm D}_{ij}\beta'_{(1)}+\frac{4}{3}\beta'_{(1)}\nabla^2\beta^{(1)\prime}_{,i}+\frac{4}{3}\beta^{,j}_{(1)}\nabla^2\beta^{(1)}_{,ij}-4\chi^{(1)\top}_{ij,k}\beta^{(1),jk}$$
$$-2\left(\chi_{ij}^{(1)\top\prime}+2\frac{a'}{a}\chi_{ij}^{(1)\top}\right)\beta^{(1)\prime,j}-2\chi_{ij}^{(1)\top}\nabla^2\beta^{,j}_{(1)}-\frac{2}{3}\chi^{(1)\top}_{ik}\beta^{(1),jk}_{,i}+\frac{4}{3}\chi^{(1)\top}_{jk,i}\beta^{(1),jk}. \tag{5.13}$$

Finally, using $\partial^i\omega_{{\rm P}i}^{(2)}=0$ and substituting $\beta^{(2)}$, we get an equation for $\alpha^{(2)}$:

$$\nabla^2\alpha^{(2)}=\nabla^2\beta'_{(2)}-2\left(2\phi_{\rm S}^{(1),i}+\beta''^{,i}_{(1)}+\frac{1}{3}\nabla^2\beta^{,i}_{(1)}\right)\beta^{(1)\prime}_{,i}-2\beta^{,ij}_{(1)}\beta^{(1)\prime}_{,ij}-2\left(2\phi_{\rm S}^{(1)}+\beta''_{(1)}-\frac{2}{3}\nabla^2\beta_{(1)}\right)\nabla^2\beta'_{(1)}+2\chi_{ij}^{(1)\top}\beta^{(1)\prime,ij}. \tag{5.14}$$

Having obtained, at least implicitly, all the parameters of the gauge transformation to second order, one can in principle compute the metric perturbations in the Poisson gauge from Eqs. (5.8)–(5.11).

Similarly, once the parameters are known, the perturbations in any scalar and four-vector, and in particular those in the energy density and in the four-velocity of matter, follow trivially from Eqs. (3.29)–(3.32).

VI. EVOLUTION IN THE POISSON GAUGE

We have obtained in the previous section the general gauge transformation to go from the synchronous to the Poisson gauge up to second order in metric perturbations. We can now apply it to the case of cosmological perturbations in a dust universe and compute the perturbed metric in the Poisson gauge from the solutions obtained in Sec. IV using the synchronous gauge.

A. First-order perturbations

For first order, replacing Eq. (4.21) in Eq. (5.5) and using Eq. (5.2), we obtain that the parameters of the transformation are

$$\alpha^{(1)}=\frac{\tau}{3}\varphi,\qquad \beta^{(1)}=\frac{\tau^2}{6}\varphi, \tag{6.1}$$

and $d^{(1)i}=0$, in the absence of vector modes in the initial conditions.

For the metric perturbations we obtain, from Eqs. (5.1), (5.3), (5.4), and (5.7),

$$\psi_{\rm P}^{(1)}=\phi_{\rm P}^{(1)}=\varphi,\qquad \chi_{{\rm P}ij}^{(1)}=\chi_{ij}^{\top(1)}. \tag{6.2}$$

These equations show the well-known result for scalar perturbations in the longitudinal gauge and the gauge invariance for tensor modes at the linear level.

The linear density contrast reads

$$\delta_{\rm P}^{(1)}=-2\varphi+\frac{\tau^2}{6}\nabla^2\varphi, \tag{6.3}$$

while the first-order four-velocity perturbation has components

$$v_{\rm P}^{(1)0}=-\varphi, \tag{6.4}$$

$$v_{\rm P}^{(1)i}=-\frac{\tau}{3}\varphi^{,i}. \tag{6.5}$$

B. Second-order perturbations

For the second-order parameters of the gauge transformation, replacing the second-order perturbed metric obtained in Sec. IV in Eqs. (5.12)–(5.14) we obtain

$$\alpha^{(2)}=-\frac{2}{21}\tau^3\Psi_0+\tau\left(\frac{10}{9}\varphi^2+4\Theta_0\right)+\alpha_{\rm (t)}^{(2)},$$

$$\beta^{(2)}=\frac{\tau^4}{6}\left(\frac{1}{12}\varphi^{,i}\varphi_{,i}-\frac{1}{7}\Psi_0\right)+\frac{\tau^2}{3}\left(\frac{7}{2}\varphi^2+6\Theta_0\right)+\beta_{\rm (t)}^{(2)}, \tag{6.6}$$

with $\nabla^2\Theta_0=\Psi_0-\frac{1}{3}\varphi^{,i}\varphi_{,i}$ and

$$\nabla^2 d_j^{(2)}=\frac{4\tau^2}{3}(-\varphi_{,j}\nabla^2\varphi+\varphi^{,i}\varphi_{,ij}-2\Psi_{0,j})+\nabla^2 d_{{\rm (t)}j}^{(2)}, \tag{6.7}$$

where the quantities indicated by the subscript (t) stand for the contributions arising from the presence of tensor modes at the linear level and are discussed in greater detail in Appendix D.

For the perturbed metric we obtain

$$\psi_{\rm P}^{(2)}=\tau^2\left(\frac{1}{6}\varphi^{,i}\varphi_{,i}-\frac{10}{21}\Psi_0\right)+\frac{16}{3}\varphi^2+12\Theta_0,+\psi_{\rm P(t)}^{(2)},$$

$$\phi_{\rm P}^{(2)}=\tau^2\left(\frac{1}{6}\varphi^{,i}\varphi_{,i}-\frac{10}{21}\Psi_0\right)+\frac{4}{3}\varphi^2-8\Theta_0,+\phi_{\rm P(t)}^{(2)},$$

$$\nabla^2\omega_{\rm P}^{(2)i}=-\frac{8}{3}\tau(\varphi^{,i}\nabla^2\varphi-\varphi^{,ij}\varphi_{,j}+2\Psi_0^{,i})+\nabla^2\omega_{\rm P(t)}^{(2)i}, \tag{6.8}$$

$$\chi_{{\rm P}ij}^{(2)}=\tilde{\pi}_{ij}+\chi_{{\rm P(t)}ij}^{(2)}.$$

The equations determining the contribution from linear tensor modes are given in Appendix D. Note that the contribution to $\psi_{\rm P}^{(2)}$ and $\phi_{\rm P}^{(2)}$ from linear scalar modes can be recovered, except for the subleading time-independent terms, by taking the weak-field limit of Einstein's theory (see, e.g., Ref. [5]) and then expanding in powers of the perturbation amplitude.

Also interesting is the way in which the second-order tensor modes, generated by the nonlinear growth of scalar perturbations, appear in this gauge: the transformation from the synchronous to the Poisson gauge has in fact dropped the Newtonian and post-Newtonian contributions, whose physical interpretation in terms of gravitational waves is highly nontrivial (see the discussion in Ref. [4]); what remains is the tensor $\tilde{\pi}_{ij}$, whose evolution is governed by the equation

$$\tilde{\pi}_{ij}''+\frac{4}{\tau}\tilde{\pi}_{ij}'-\nabla^2\tilde{\pi}_{ij}=-\frac{40}{3}\mathcal{T}_{ij}. \tag{6.9}$$

Its solution, Eq. (4.38), contains a constant term, deriving from the vanishing initial conditions, plus a wavelike piece, having exactly the same form as linear tensor modes [cf. Eq. (4.13)], whose amplitude is fixed by the source term $\mathcal{T}_{ij}$ (a quadratic combination of linear scalar modes). A more extended discussion of these tensor modes is given in Ref. [33].

Finally, let us give the Poisson-gauge expressions for the second-order density and four-velocity perturbations. One has

$$\delta_{\rm P}^{(2)}=\frac{\tau^4}{252}(5(\nabla^2\varphi)^2+2\varphi^{,ij}\varphi_{,ij}+14\varphi^{,i}\nabla^2\varphi_{,i})$$
$$+\frac{\tau^2}{36}\left(-21\varphi^{,i}\varphi_{,i}+24\varphi\nabla^2\varphi+\frac{144}{7}\Psi_0-6\varphi^{,ij}\chi_{ij}^{(1)\top}\right)$$
$$+\frac{1}{4}(\chi^{(1)\top ij}\chi_{ij}^{(1)\top}-\chi_0^{(1)\top ij}\chi_{0ij}^{(1)\top})$$
$$-\frac{8}{3}\varphi^2-24\Theta_0+\frac{3}{2}\phi_{\rm S(t)}^{(2)}-\frac{6}{\tau}\alpha_{\rm (t)}^{(2)} \tag{6.10}$$

and

$$v_{\rm P}^{(2)0}=\frac{\tau^2}{3}\left(-\frac{1}{6}\varphi^{,i}\varphi_{,i}+\frac{10}{7}\Psi_0\right)-\frac{7}{3}\varphi^2-12\Theta_0-\psi_{\rm P(t)}^{(2)}, \tag{6.11}$$

$$v_{\rm P}^{(2)i}=\frac{\tau^3}{9}\left(-\varphi^{,ij}\varphi_{,j}+\frac{6}{7}\Psi_0^{,i}\right)$$
$$-2\tau\left(\frac{16}{9}\varphi\varphi^{,i}+2\Theta_0^{,i}\right)-d^{(2)i\prime}-\beta_{\rm (t)}^{(2)\prime,i}, \tag{6.12}$$

with the vectors $d^{(2)i}$ defined in Eq. (6.7).

In concluding this section, let us emphasize that all the second-order Poisson-gauge expressions obtained here are new. Only a few terms in these expressions were already known in the literature, based on the weak-field limit of general relativity (e.g., Ref. [5]).

VII. CONCLUSIONS

In this paper we considered relativistic perturbations of a collisionless and irrotational fluid up to second order around the Einstein–de Sitter cosmological model. The most important phenomenon of second-order perturbation theory is mode mixing. An interesting consequence of this phenomenon is that primordial density fluctuations act as seeds for second-order gravitational waves. The specific form of these waves is gauge dependent, as tensor modes are no longer gauge invariant beyond the linear level. A second interesting effect is the generation of density fluctuations from primordial tensor modes. One can even figure out a scenario in which no scalar perturbations were initially present, but they were later generated, as a second-order effect, by the nonlinear evolution of a primordial gravitational-wave background.

The first effect, which is discussed in some detail in Ref. [33], in the synchronous and comoving gauge also contains a term growing like τ^4 and a second one growing like τ^2: the first accounts for the Newtonian tidal induction of the environment on the nonlinear evolution of fluid elements; the second is a post-Newtonian tensor mode induced by the growth of the shear field. The remaining parts of this second-order tensor mode (excluding a constant term required by the vanishing initial conditions) oscillate with decaying amplitude inside the horizon and describe true gravitational *waves*. Quite interesting is the fact that these are the only parts of these second-order tensor modes which survive to the transformation leading to the Poisson gauge.

The second effect is less known, and was only previously considered by Tomita back in the early 1970s [20].

One may naturally wonder whether there is any hope to detect the cosmological stochastic gravitational-wave background produced at second order by scalar fluctuations. It is, of course, the oscillating part of π_{ij} which is relevant for Earth or space detectors. The problem for these wavelike modes is that their energy density suffers the usual a^{-4} dilution caused by free-streaming inside the Hubble radius, while at the horizon crossing their closure density is already extremely small, $\Omega_{gw}\sim\delta_H^4$ (where δ_H is the rms density contrast at the horizon crossing), because of their secondary origin. More promising is the possibility that a non-negligible amount of gravitational radiation can be produced during the strongly nonlinear stages of the collapse of protostructures, an issue which would, however, require a fully nonperturbative approach.

It should be stressed that, while many of our second-order terms had already been computed in the synchronous gauge, all our second-order Poisson-gauge expressions are new. This is a relevant result, as the latter gauge is the one which allows the easiest interpretation of the various physical effects. In particular, the second-order metric perturbations obtained by our method allow one to compute self-consistently gravity-induced secondary anisotropies of the cosmic microwave background. This calculation has been recently performed by Mollerach and Matarrese [13], implementing a general scheme introduced by Pyne and Carroll [12].

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APPENDIX A: TAYLOR EXPANSION OF TENSOR FIELDS

In this appendix we present some mathematical results used in Sec. II, concerning Taylor expansions of tensor fields on a manifold. These results have already been presented in [15], where analyticity of all relevant fields was assumed; they have been generalized in [16] to the case of C^m fields. The theorems obtained in [15,16] are very general, concerning perturbation theory at an arbitrary order n. In order to achieve these general results it is very useful, or perhaps mandatory, to use a fully geometrical approach. However, for our purposes, it is useful to summarize them in terms of coordinates and tensor components, as we shall do in the following. We assume that all quantities are as smooth as necessary.

1. One-parameter groups of transformations

As disccused in Sec. II, gauge choices for perturbations entail the comparison of the tensor field representing a certain physical and/or geometrical quantity in the perturbed

spacetime with the tensor field representing the same quantity in the background spacetime. Consequently, gauge transformations entail the comparison of tensors at different points in the background spacetime. A smallness parameter λ is involved, so that these comparisons are always carried out at the required order of accuracy in λ, using Taylor expansions [34]. Differential geometry tells us that the comparison of tensors is meaningful only when we consider them at the same point. Therefore, supposing we want to compare a tensor field T at points p and q, we need to define a transport law from q to p. This gives us two tensors at p, T, itself, and the transported one, which can be directly compared.

The simplest transport law we need to consider is the Lie dragging by a vector field, which allows us to compare T with its pullback $\tilde{T}(\lambda)$ (the new tensor defined by this transport). To fix ideas, let us first consider, on a manifold $\mathcal{M}$, the comparison of tensors at first order in λ (which we shall define shortly). Suppose a coordinate system x^μ has been given on (an open set of) $\mathcal{M}$, together with a vector field ξ. From $dx^\mu/d\lambda=\xi^\mu$, ξ generates on $\mathcal{M}$ a congruence of curves $x^\mu(\lambda)$: thus λ is the parameter along the congruence. Given a point p, this will always lie on one of these curves, and we can always take p to correspond to $\lambda=0$ on this. The coordinates of a second point q at a parameter distance λ from p on the same curve will be given by

$$\tilde{x}^\mu(\lambda)=x^\mu+\lambda\xi^\mu+\cdots, \tag{A1}$$

where the x^μ are the coordinates of p and the $\tilde{x}^\mu$ are those of q, approximated here at first order in λ. Equation (A1) is usually called an ''infinitesimal point transformation,'' or an ''active coordinate transformation'' (see, e.g., Ref. [35], p. 70; Ref. [36], p. 49; cf. also Ref. [37], p. 291, and [29], Appendix C). At the same time, we may think that a new coordinate system y^μ has been introduced on $\mathcal{M}$, *defined* in such a way that the y coordinates of the point q coincide with the x coordinates of the point p; using Eq. (A1) it then follows from this definition that

$$\begin{aligned} y^\mu(q):=x^\mu(p)&=x^\mu(q)-\lambda\xi^\mu(x(p))+\cdots \\ &\simeq x^\mu(q)-\lambda\xi^\mu(x(q))+\cdots. \end{aligned} \tag{A2}$$

In practice, we have in this way defined at every point a ''passive coordinate transformation'' (i.e., just an ordinary relabeling of point's names), which at first order reads

$$y^\mu(\lambda)=x^\mu-\lambda\xi^\mu+\cdots. \tag{A3}$$

Suppose now that a tensor field has been given on $\mathcal{M}$; e.g., to fix ideas, consider the vector field Z with components Z^μ in the x-coordinate system. In the same way that we defined a new coordinate system y^μ once a relation between points was assigned through Eq. (A1) by the action of ξ, so we can now define a new vector field $\tilde{Z}$, with components $\tilde{Z}^\mu$ in the x coordinates, such that these components at the coordinate point $x^\mu(p)$ are equal to the components Z'^μ the old vector Z has in the y coordinates at the coordinate point $y(q)$:

$$\tilde{Z}^\mu(x(p)):=Z'^\mu(y(q))=\left[\frac{\partial y^\mu}{\partial x^\nu}\right]_{x(q)}Z^\nu(x(q)). \tag{A4}$$

The last equality in this equation is just the ordinary (passive) transformation between the components of Z in the two coordinate systems: we need it in order to relate $\tilde{Z}$ and Z in a single system (the x frame here), thus eventually obtaining a covariant relation. Indeed, substitution of Eq. (A3) into Eq. (A4) and a first-order expansion in λ about $x(p)$ in the right-hand side (RHS) gives

$$\tilde{Z}^\mu(\lambda)=Z^\mu+\lambda\pounds_\xi Z^\mu+\cdots, \tag{A5}$$

$$\pounds_\xi Z^\mu:=Z^\mu_{,\nu}\xi^\nu-\xi^\mu_{,\nu}Z^\nu, \tag{A6}$$

where, given that the point p is arbitrary, the dependence of all terms on $x(p)$ has been omitted. The vector field $\tilde{Z}$ is called the pullback of Z, because it is defined by dragging Z back from q to p, an operation that gives at p a new vector with components $\tilde{Z}^\mu$, given by Eq. (A4). In the particular case of the transformation (A3) this is the Lie dragging. Now, having at the same point two vectors, these can be directly compared: at first order, $\tilde{Z}(\lambda)$ and Z are related by Eqs. (A5), (A6). In fact, in the limit $\lambda\to0$, it is this comparison that allows us to define the Lie derivative, with components (A6); Eq. (A15) below generalizes this to a generic tensor T.

Although the story so far is a textbook one (cf. [29,35–37]), recalling it in some detail allows us to easily extend it to higher order. First, one has to realize that Eq. (A1) is just the first-order Taylor expansion about $x(p)$ of the solution of the ordinary differential equation $dx^\mu/d\lambda=\xi^\mu$ defining the congruence $x^\mu(\lambda)$ associated with ξ. The exact solution of this equation is the Taylor series (cf., e.g., [38], p. 43)

$$x^\mu(q)=x^\mu(p)+\lambda\xi^\mu(x(p))+\frac{\lambda^2}{2}\xi^\mu{}_{,\nu}\xi^\nu(x(p))+\cdots, \tag{A7}$$

on using $dx^\mu/d\lambda=\xi^\mu$, $d^2x^\mu/d\lambda^2=\xi^\mu_{,\nu}\xi^\nu$, etc. In practice, since p and q are arbitrary, we may simply write

$$\tilde{x}^\mu(\lambda)=x^\mu+\lambda\xi^\mu+\frac{\lambda^2}{2}\xi^\mu{}_{,\nu}\xi^\nu+\cdots, \tag{A8}$$

$$=\exp[\lambda\pounds_\xi]x^\mu. \tag{A9}$$

The latter exponential notation is useful, in that it allows us to see the coordinate functions $\tilde{x}^\mu$ as the pullbacks of the functions x^μ given by the exponential pullback operator $\exp[\lambda\pounds_\xi]$. Furthermore, it is clearly seen by $\exp[(\lambda_1+\lambda_2)\pounds_\xi]=\exp[\lambda_1\pounds_\xi]\exp[\lambda_2\pounds_\xi]$ that the point transformations (A8) form a one-parameter group of transformations. Using again the definition $y^\mu(q):=x^\mu(p)$ for the y coordinates, we get, from Eq. (A7),

$$y^\mu(\lambda)=x^\mu-\lambda\xi^\mu+\frac{\lambda^2}{2}\xi^\mu{}_{,\nu}\xi^\nu+\cdots, \tag{A10}$$

on expanding all terms about $x(q)$, eventually omitting again the $x(q)$ dependence, since q is arbitrary. Finally, using Eq. (A10) into Eq. (A4) and expanding all the terms about $x(p)$, we get the x components $\tilde{Z}^\mu(\lambda)$ of the pullback $\tilde{Z}(\lambda)$, which reads [39]

$$\tilde{Z}^\mu(\lambda) = [\exp[\lambda \pounds_\xi] Z]^\mu \tag{A11}$$

$$= Z^\mu + \lambda \pounds_\xi Z^\mu + \frac{\lambda^2}{2} \pounds_\xi^2 Z^\mu + \cdots . \tag{A12}$$

Equation (A4) is readily generalized to more general tensors than Z: we simply have to add to the RHS of Eq. (A4) the right number of transformation matrices. Thus, the pullback $\tilde{T}$ of a tensor field T of type (p,q) is defined by having x components given by

$$\begin{aligned} &\tilde{T}^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}(x(p)) \\ &:= T'^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}(y(q)) \\ &= \left[\frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{\mu_p}}{\partial x^{\alpha_p}} \frac{\partial x^{\beta_1}}{\partial y^{\nu_1}} \cdots \frac{\partial x^{\beta_q}}{\partial y^{\nu_q}} \right]_{x(q)} \\ &\quad \times T^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q}(x(q)). \end{aligned} \tag{A13}$$

Using Eq. (A10) as above then gives, omitting indices for brevity,

$$\tilde{T}(\lambda) = T + \lambda \pounds_\xi T + \frac{\lambda^2}{2} \pounds_\xi^2 T + \cdots . \tag{A14}$$

To summarize, each of the diffeomorphisms forming a one-parameter group, as mathematicians call the transformations generated by a vector field ξ and represented in coordinates by Eq. (A9), gives rise to a new field, the pullback $\tilde{T}(\lambda)$, from any given tensor field T and for any given value of λ. Thus $\tilde{T}(\lambda)$ and T may be compared at every point, which allows one to define the Lie derivative along ξ as the limit $\lambda \to 0$ of the difference $\tilde{T}(\lambda) - T$:

$$\pounds_\xi T := \left[\frac{d}{d\lambda} \right]_{\lambda=0} \tilde{T}(\lambda) = \lim_{\lambda \to 0} \frac{1}{\lambda} [\tilde{T}(\lambda) - T]. \tag{A15}$$

At higher order we have

$$\pounds_\xi^k T := \left[\frac{d^k}{d\lambda^k} \right]_{\lambda=0} \tilde{T}(\lambda). \tag{A16}$$

On the other hand, the relation at each point between any tensor field T and its pullback $\tilde{T}(\lambda)$ is expressed at the required order of accuracy by the Taylor expansion (A14).

2. One-parameter families of transformations

In order to proceed, considering more general point transformations than Eq. (A8) and more general Taylor expansions than Eq. (A14), some general remarks are in order. First, it should be noticed that the definition $y^\mu(q) := x^\mu(p)$ for the y-coordinate system is completely general, given a first coordinate system (the x frame here) and any suitable association between pairs of points (more precisely, any diffeomorphism), of which the one-parameter group of transformations (A8) is a particular example. Second, the same generality is present in the definition of the pullback, Eq. (A13), which is also independent from the specific type of transformation chosen.

As we said in Sec. II, exact gauge transformations do not form a one-parameter group, but a one-parameter family [15,16]. However, the consequences of this fact show up only with nonlinearity, which is why at first order gauge transformations are approximated by Eq. (A1), (A3) (cf. [29,35–37]). Therefore, having in mind a second-order treatment of perturbations, the question we now have to deal with is twofold: (i) which is the general form of families of transformations that depend on one parameter (one-parameter families of diffeomorphisms), but do not form a group; (ii) which is the form of the Taylor expansion of the pullback $\tilde{T}(\lambda)$ of a tensor T generated by one such one-parameter family of transformations.

In [15,16] (cf. also [40]) we have shown that the action of any given one-parameter family of transformations can be represented by the successive action of one-parameter groups, in a fashion that, to order λ^2, reminds us the motion of the knight on the chessboard:

$$\tilde{x}^\mu(\lambda) = x^\mu + \lambda \xi^\mu_{(1)} + \frac{\lambda^2}{2} (\xi^\mu_{(1),\nu} \xi^\nu_{(1)} + \xi^\mu_{(2)}) + \cdots . \tag{A17}$$

A vector field $\xi_{(k)}$ is associated with the kth one-parameter group of transformations, with parameter λ_k (we denote $\lambda_1 = \lambda$). Similarly to the knight, the action of the transformation (A17) first moves from point p (with coordinates x^μ) by an amount λ along the integral curve of $\xi_{(1)}$ [i.e., according to Eq. (A8)]; then, it moves along the integral curve of $\xi_{(2)}$ by an amount $\lambda_2 = \lambda^2/2$. At each kth higher order, a new vector field $\xi_{(k)}$ is involved, generating a motion by $\lambda_k = \lambda^k/k!$. Thus, the action of a one-parameter family of transformations is approximated, at order k, by a ''knight transformation'' of order k (see [15], Theorem 2), of which Eq. (A17) is the second-order example.

Given the ''knight transformation'' (A17), we can now use it to define the y coordinates, which will be given by

$$\begin{aligned} y^\mu(q) := x^\mu(p) = x^\mu(q) &- \lambda \xi^\mu_{(1)}(x(p)) \\ &- \frac{\lambda^2}{2} [\xi^\mu_{(1),\nu}(x(p)) \xi^\nu_{(1)}(x(p)) + \xi^\mu_{(2)}(x(p))] \\ &+ \cdots . \end{aligned} \tag{A18}$$

Expanding the various quantities on the RHS around q, and omitting the $x(q)$ dependence, Eq. (A18) becomes, finally,

$$y^\mu(\lambda)=x^\mu-\lambda\xi^\mu_{(1)}+\frac{\lambda^2}{2}(\xi^\mu_{(1),\nu}\xi^\nu_{(1)}-\xi^\mu_{(2)})+\cdots. \tag{A19}$$

Using again the case of the vector field Z as our paradigmatic example, we can now derive the pullback $\tilde{Z}(\lambda)$ generated by a one-parameter family of transformations. Substituting Eq. (A19) into Eq. (A4), and expanding again every term about $x(p)$, we obtain the x components $\tilde{Z}^\mu(\lambda)$ of $\tilde{Z}(\lambda)$, which (after properly collecting terms) at second order read

$$\tilde{Z}^\mu(\lambda)=Z^\mu+\lambda\pounds_{\xi_{(1)}}Z^\mu+\frac{\lambda^2}{2}(\pounds^2_{\xi_{(1)}}+\pounds_{\xi_{(2)}})Z^\mu+\cdots. \tag{A20}$$

For a generic tensor T, again omitting indices for brevity, use of Eq. (A19) in Eq. (A13) obviously gives

$$\tilde{T}=T+\lambda\pounds_{\xi_{(1)}}T+\frac{\lambda^2}{2}(\pounds^2_{\xi_{(1)}}+\pounds_{\xi_{(2)}})T+\cdots. \tag{A21}$$

APPENDIX B: SECOND-ORDER PERTURBATIONS OF USEFUL QUANTITIES IN THE SYNCHRONOUS GAUGE

In this appendix we report the expansions up to second order of a number of tensors, which have been used in deriving the results of Sec. IV. All calculations are performed in the synchronous and comoving gauge, assuming an Einstein–de Sitter background. No subscripts S on synchronous-gauge quantities will be used in this appendix.

The covariant conformal spatial metric tensor is expanded as follows:

$$\gamma_{ij}=\delta_{ij}+\gamma^{(1)}_{ij}+\frac{1}{2}\gamma^{(2)}_{ij}. \tag{B1}$$

The corresponding contravariant metric takes the form

$$\gamma^{ij}=\delta^{ij}-\gamma^{(1)ij}-\frac{1}{2}\gamma^{(2)ij}+\gamma^{(1)ik}\gamma^{(1)j}_k, \tag{B2}$$

where the indices of the perturbations $\gamma^{(1,2)}_{ij}$ are raised by δ^{ij}.

The extrinsic curvature tensor $\vartheta^i_{\ j}$ up to second order reads

$$\vartheta^i_{\ j}=\frac{1}{2}\left(g^{(1)i\prime}_j+\frac{1}{2}\gamma^{(2)i\prime}_j-\gamma^{(1)ik}\gamma^{(1)\prime}_{kj}\right). \tag{B3}$$

The square root of the metric determinant is

$$\gamma^{1/2}=1+\frac{1}{2}\gamma^{(1)i}_i+\frac{1}{4}\gamma^{(2)i}_i+\frac{1}{8}(\gamma^{(1)i}_i)^2-\frac{1}{4}\gamma^{(1)ij}\gamma^{(1)}_{ij}, \tag{B4}$$

with inverse

$$\gamma^{-1/2}=1-\frac{1}{2}\gamma^{(1)i}_i-\frac{1}{4}\gamma^{(2)i}_i+\frac{1}{8}(\gamma^{(1)i}_i)^2+\frac{1}{4}\gamma^{(1)ij}\gamma^{(1)}_{ij}. \tag{B5}$$

From these quantities we can easily get the density contrast

$$\delta=-\frac{1}{2}\gamma^{(1)i}_i+\frac{1}{2}\gamma^{(1)i}_{0i}+\delta_0-\frac{1}{4}\gamma^{(2)i}_i+\frac{1}{8}(\gamma^{(1)i}_i)^2+\frac{1}{8}(\gamma^{(1)i}_{0i})^2-\frac{1}{4}\gamma^{(1)i}_i\gamma^{(1)j}_{0j}+\frac{1}{4}\gamma^{(1)ij}\gamma^{(1)}_{ij}-\frac{1}{4}\gamma^{(1)ij}_0\gamma^{(1)}_{0ij}-\frac{1}{2}\gamma^{(1)i}_i\delta_0+\frac{1}{2}\gamma^{(1)i}_{0i}\delta_0, \tag{B6}$$

having assumed as initial conditions $\gamma^{(2)}_{0ij}=0$ and $\delta^{(2)}_0=0$ (i.e., $\delta_0=\delta^{(1)}_0$).

The Christoffel symbols up to second order read

$$\Gamma^i_{jk}=\frac{1}{2}(\gamma^{(1)i}_{j,k}+\gamma^{(1)i}_{k,j}-\gamma^{(1),i}_{jk})+\frac{1}{4}(\gamma^{(2)i}_{j,k}+\gamma^{(2)i}_{k,j}-\gamma^{(2),i}_{jk})-\frac{1}{2}\gamma^{(1)i}_l(\gamma^{(1)l}_{j,k}+\gamma^{(1)l}_{k,j}-\gamma^{(1),l}_{jk}), \tag{B7}$$

from which, after a lengthy but straightforward calculation, the conformal Ricci tensor of the spatial hypersurface,

$$\begin{aligned}\mathcal{R}^i_{\ j}=&\frac{1}{2}(\gamma^{(1)ik}_{,jk}+\gamma^{(1)k,i}_{j,k}-\nabla^2\gamma^{(1)i}_j-\gamma^{(1)k,i}_{k,j})+\frac{1}{4}(\gamma^{(2)ik}_{,jk}+\gamma^{(2)k,i}_{j,k}-\nabla^2\gamma^{(2)i}_j-\gamma^{(2)k,i}_{k,j})\\&+\frac{1}{2}\Big[\gamma^{(1)ik}(\nabla^2\gamma^{(1)}_{kj}+\gamma^{(1)l}_{l,jk}-\gamma^{(1)l}_{k,lj}-\gamma^{(1)l}_{j,lk})+\gamma^{(1)lk}(\gamma^{(1)i}_{j,kl}+\gamma^{(1),i}_{lk,j}-\gamma^{(1)i}_{k,jl}-\gamma^{(1),i}_{jl,k})\\&+\gamma^{(1)lk}_{,l}(\gamma^{(1)i}_{j,k}+\gamma^{(1)i}_{k,j}-\gamma^{(1),i}_{jk})+\gamma^{(1)li}{}_{,k}\gamma^{(1),k}_{lj}-\gamma^{(1)li}{}_{,k}\gamma^{(1)k}_{j,l}+\frac{1}{2}\gamma^{(1)lk}{}_{,j}\gamma^{(1),i}_{lk}+\frac{1}{2}\gamma^{(1)m}_{m,l}(\gamma^{(1)li}_{,j}+\gamma^{(1)l,i}_j-\gamma^{(1)i,l}_j)\Big],\end{aligned} \tag{B8}$$

and its trace

$$\mathcal{R}=\gamma^{(1)lk}_{,lk}-\gamma^{(1)k,l}_{k,l}+\frac{1}{2}(\gamma^{(2)lk}_{,lk}-\gamma^{(2)k,l}_{k,l})+\gamma^{(1)jk}(\nabla^2\gamma^{(1)}_{jk}+\gamma^{(1)l}_{l,jk}-2\gamma^{(1)l}_{j,lk})+\gamma^{(1)lk}_{,l}(\gamma^{(1)j}_{j,k}-\gamma^{(1),j}_{jk})$$

$$+\frac{3}{4}\gamma^{(1)lj}{}_{,k}\gamma^{(1),k}_{lj}-\frac{1}{2}\gamma^{(1)lj}{}_{,k}\gamma^{(1)k}_{j,l}-\frac{1}{4}\gamma^{(1)j,l}_{j}\gamma^{(1)k}_{k,l} \tag{B9}$$

follow.

APPENDIX C: SECOND-ORDER PERTURBATIONS GENERATED BY LINEAR TENSOR MODES

In this appendix we report the equations which allow one to determine the second-order perturbations which arise due to the presence of tensor modes at the linear level: these are originated both by the coupling of scalar and tensor modes and by tensor-tensor mode couplings. All calculations are performed in the synchronous and comoving gauge, assuming an Einstein–de Sitter background.

The equations which follow refer only to those parts of the second-order metric perturbations which involve tensor modes in the source terms [hence the subscript (t)].

Raychaudhuri equation:

$$\phi^{(2)\prime\prime}_{\mathrm{S(t)}}+\frac{2}{\tau}\phi^{(2)\prime}_{\mathrm{S(t)}}-\frac{6}{\tau^2}\phi^{(2)}_{\mathrm{S(t)}}=\frac{\tau^2}{9}\varphi^{,ij}\nabla^2\chi^{(1)\top}_{ij}-\frac{1}{6}\chi^{(1)\top ij\prime}\chi^{(1)\top\prime}_{ij}+\frac{2}{3\tau}\chi^{(1)\top ij\prime}\chi^{(1)\top}_{ij}$$

$$-\frac{1}{3}\chi^{(1)\top ij}\nabla^2\chi^{(1)\top}_{ij}+\frac{1}{\tau^2}(\chi^{(1)\top ij}\chi^{(1)\top}_{ij}-\chi^{(1)\top ij}_{0}\chi^{(1)\top}_{0ij})\equiv\mathcal{Q}(\mathbf{x},\tau). \tag{C1}$$

Energy constraint:

$$\frac{2}{\tau}\phi^{(2)\prime}_{\mathrm{S(t)}}-\frac{1}{3}\nabla^2\phi^{(2)}_{\mathrm{S(t)}}+\frac{6}{\tau^2}\phi^{(2)}_{\mathrm{S(t)}}-\frac{1}{12}\chi^{(2)ij}_{\mathrm{S(t)},ij}=\frac{5\tau}{18}\chi^{(1)\top ij\prime}\varphi_{,ij}+\frac{5}{9}\chi^{(1)\top ij}\varphi_{,ij}-\frac{\tau^2}{18}\nabla^2\chi^{(1)\top ij}\varphi_{,ij}-\frac{\tau^2}{36}\chi^{(1)\top ij,k}\varphi_{,ijk}$$

$$-\frac{1}{24}\chi^{(1)\top ij\prime}\chi^{(1)\top\prime}_{ij}-\frac{2}{3\tau}\chi^{(1)\top ij\prime}\chi^{(1)\top}_{ij}+\frac{1}{6}\chi^{(1)\top ij}\nabla^2\chi^{(1)\top}_{ij}+\frac{1}{8}\chi^{(1)\top ij,k}\chi^{(1)\top}_{ij,k}$$

$$-\frac{1}{12}\chi^{(1)\top ij,k}\chi^{(1)\top}_{kj,i}-\frac{1}{\tau^2}(\chi^{(1)\top ij}\chi^{(1)\top}_{ij}-\chi^{(1)\top ij}_{0}\chi^{(1)\top}_{0ij}). \tag{C2}$$

Momentum constraint:

$$2\phi^{(2)\prime}_{\mathrm{S(t)},j}+\frac{1}{2}\chi^{(2)i\prime}_{\mathrm{S(t)}j,i}=\frac{\tau^2}{3}\left[(\chi^{(1)\top ik\prime}_{,j}-\chi^{(1)\top k,i\prime}_{j})\varphi_{,ik}+\frac{1}{2}\chi^{(1)\top ik\prime}\varphi_{,ikj}-\frac{1}{2}\chi^{(1)\top k\prime}_{j}\nabla^2\varphi_{,k}\right]$$

$$+\frac{\tau}{3}\chi^{(1)\top ik}{}_{,j}\varphi_{,ik}+\frac{5}{3}\chi^{(1)\top i\prime}_{j}\varphi_{,i}+\chi^{(1)\top ik}(\chi^{(1)\top\prime}_{kj,i}-\chi^{(1)\top\prime}_{ki,j})-\frac{1}{2}\chi^{(1)\top ik}{}_{,j}\chi^{(1)\top\prime}_{ik}. \tag{C3}$$

Evolution equation:

$$
\begin{aligned}
&-\left(\phi^{(2)''}_{\mathrm{S(t)}}+\frac{4}{\tau}\phi^{(2)'}_{\mathrm{S(t)}}\right)\delta^i{}_j+\frac{1}{2}\left(\chi^{(2)i''}_{\mathrm{S(t)}j}+\frac{4}{\tau}\chi^{(2)i'}_{\mathrm{S(t)}j}\right)+\phi^{(2),i}_{\mathrm{S(t)},j}-\frac{1}{4}\chi^{(2)kl}_{\mathrm{S(t)},kl}\delta^i{}_j+\frac{1}{2}\chi^{(2)ki}_{\mathrm{S(t)},kj}+\frac{1}{2}\chi^{(2)k}_{\mathrm{S(t)}j,k}{}^{,i}-\frac{1}{2}\nabla^2\chi^{(2)i}_{\mathrm{S(t)}j}\\
&\quad=-\frac{2\tau}{3}\chi^{(1)\top}_{kj}{}^{\prime}\varphi^{,ik}-\frac{2\tau}{3}\chi^{(1)\top ik\prime}\varphi_{,kj}+\frac{\tau}{3}\chi^{(1)\top i}_{j}\nabla^2\varphi+\frac{\tau}{6}\chi^{(1)\top kl\prime}\varphi_{,kl}\delta^i{}_j\\
&\quad+\frac{10}{3}\chi^{(1)\top i}_{j}\nabla^2\varphi+\frac{25}{3}\chi^{(1)\top kl}\varphi_{,kl}\delta^i{}_j-\frac{10}{3}\chi^{(1)\top ik}\varphi_{,jk}-\frac{10}{3}\chi^{(1)\top}_{jk}\varphi^{,ik}+\frac{10}{3}\nabla^2\chi^{(1)\top i}_{j}\varphi\\
&\quad+\frac{\tau^2}{3}(\chi^{(1)\top i,kl}_{j}+\chi^{(1)\top kl,i}_{,j}-\chi^{(1)\top ki,l}_{,j}-\chi^{(1)\top k,il}_{j})\varphi_{,kl}\\
&\quad+(\chi^{(1)\top i,k}_{j}-\chi^{(1)\top ik}_{,j}-\chi^{(1)\top k,i}_{j})\left(\frac{5}{3}\varphi_{,k}+\frac{\tau^2}{6}\nabla^2\varphi_{,k}\right)\\
&\quad+\frac{\tau^2}{6}(\chi^{(1)\top kl,i}\varphi_{,klj}+\chi^{(1)\top kl}_{,j}\varphi_{,kli}-\nabla^2\chi^{(1)\top kl}\varphi_{,kl}\delta^i{}_j)\\
&\quad-\frac{\tau^2}{12}\chi^{(1)\top kl,m}\varphi_{,klm}\delta^i{}_j+\chi^{(1)\top ik\prime}\chi^{(1)\top}_{kj}{}^{\prime}-\frac{1}{8}\chi^{(1)\top kl\prime}\chi^{(1)\top}_{kl}{}^{\prime}\delta^i{}_j\\
&\quad+\chi^{(1)\top kl}(\chi^{(1)\top i}_{k\ ,jl}+\chi^{(1)\top}_{kj\ ,l}{}^{,i}-\chi^{(1)\top i}_{j\ ,kl}-\chi^{(1)\top}_{kl\ ,j}{}^{,i})+\chi^{(1)\top ki}_{\ \ ,l}(\chi^{(1)\top l}_{j\ ,k}-\chi^{(1)\top}_{kj}{}^{,l})\\
&\quad-\frac{1}{2}\chi^{(1)\top kl}_{\ \ ,j}\chi^{(1)\top}_{kl}{}^{,i}+\frac{1}{2}\chi^{(1)\top kl}\nabla^2\chi^{(1)\top}_{kl}\delta^i{}_j+\frac{3}{8}\chi^{(1)\top km}_{\ \ ,l}\chi^{(1)\top}_{km}{}^{,l}\delta^i{}_j-\frac{1}{4}\chi^{(1)\top km}_{\ \ ,l}\chi^{(1)\top l}_{m\ ,k}\delta^i{}_j. \qquad \text{(C4)}
\end{aligned}
$$

The Raychaudhuri equation can be easily solved for $\phi^{(2)}_{\mathrm{S(t)}}$ by means of the Green method. The resulting expression has been given in Eq. (4.32) of the main text. By replacing it in the remaining equations one can in principle obtain the traceless tensor $\chi^{(2)}_{\mathrm{S(t)}ij}$ by integration.

APPENDIX D: TENSOR CONTRIBUTION TO THE SECOND-ORDER GAUGE TRANSFORMATION

In this appendix we show how to compute the contribution from linear tensor modes to the perturbed metric in the Poisson gauge by performing a gauge transformation from the synchronous gauge perturbed metric obtained from Appendix C. The equations for the gauge transformation parameters involved are obtained straightforwardly from Eqs. (5.12)–(5.14):

$$
\begin{aligned}
&\nabla^2\nabla^2\beta^{(2)}_{(\mathrm{t})}=-\frac{3}{4}\chi^{(2)ij}_{\mathrm{S(t)},ij}-\frac{\tau}{2}\left(\chi^{(1)\top}_{ij}{}^{\prime}+\frac{4}{\tau}\chi^{(1)\top}_{ij}\right)\varphi^{,ij}-\frac{5\tau^2}{12}\chi^{(1)\top}_{ij,k}\varphi^{,ijk}-\frac{\tau^2}{3}\chi^{(1)\top}_{ij}\nabla^2\varphi^{,ij}+\frac{\tau^2}{6}\nabla^2\chi^{(1)\top}_{ij}\varphi^{,ij},\\
&\nabla^2\alpha^{(2)}_{(\mathrm{t})}=\nabla^2\beta^{(2)}_{(\mathrm{t})}{}^{\prime}+\frac{2\tau}{3}\chi^{(1)\top}_{ij}\varphi^{,ij},\\
&\nabla^2 d^{(2)}_{(\mathrm{t})i}=-\frac{4}{3}\nabla^2\beta^{(2)}_{(\mathrm{t}),i}-\chi^{(2)j}_{\mathrm{S(t)}i,j}-\frac{2\tau}{3}\left(\chi^{(1)\top}_{ij}{}^{\prime}+\frac{4}{\tau}\chi^{(1)\top}_{ij}\right)\varphi^{,j}-\frac{2\tau^2}{3}\chi^{(1)\top}_{ij,k}\varphi^{,jk}-\frac{\tau^2}{3}\chi^{(1)\top}_{ij}\nabla^2\varphi^{,j}-\frac{\tau^2}{9}\chi^{(1)\top}_{jk}\varphi^{,jk}_{\ \ ,i}+\frac{2\tau^2}{9}\chi^{(1)\top}_{jk,i}\varphi^{,jk}.
\end{aligned}
\qquad \text{(D1)}
$$

The energy constraint (C2) can be used to replace $\chi^{(2),ij}_{\mathrm{S(t)}\ ij}$ in terms of $\phi^{(2)}_{\mathrm{S(t)}}$ and products of first-order quantities.

On the other hand, we have from Eqs. (5.8)–(5.11) that the contributions to the perturbed metric that we are interested in are

$$\psi^{(2)}_{\mathrm{P(t)}}=\alpha^{(2)}_{(\mathrm{t})}{}^{\prime}+\frac{2}{\tau}\alpha^{(2)}_{(\mathrm{t})}, \qquad \text{(D2)}$$

$$\omega^{(2)}_{\mathrm{P(t)}\ i}=\frac{2\tau}{3}\chi^{(1)\top}_{ij}\varphi^{,j}-\alpha^{(2)}_{(\mathrm{t}),i}+\beta^{(2)}_{(\mathrm{t}),i}{}^{\prime}+d^{(2)}_{(\mathrm{t})i}{}^{\prime}, \qquad \text{(D3)}$$

$$\phi^{(2)}_{\mathrm{P(t)}}=\phi^{(2)}_{\mathrm{S(t)}}-\frac{\tau^2}{9}\chi^{(1)\top}_{ij}\varphi^{,ij}-\frac{2}{\tau}\alpha^{(2)}_{(\mathrm{t})}-\frac{1}{3}\nabla^2\beta^{(2)}_{(\mathrm{t})}, \qquad \text{(D4)}$$

$$\chi^{(2)}_{\mathrm{P(t)}\,ij}=\chi^{(2)}_{\mathrm{S(t)}\,ij}+\frac{2\tau}{3}\varphi\left(\chi^{(1)\top\prime}_{ij}+\frac{4}{\tau}\chi^{(1)\top}_{ij}\right)+\frac{\tau^2}{3}(\chi^{(1)\top}_{ij,k}\varphi^{,k}+\chi^{(1)\top}_{ik}\varphi^{,k}_{,j}+\chi^{(1)\top}_{jk}\varphi^{,k}_{,i})-\frac{2\tau^2}{9}\delta_{ij}\chi^{(1)\top}_{lk}\varphi^{,lk}+2(d^{(2)}_{(\mathrm{t})(i,j)}+\mathrm{D}_{ij}\beta^{(2)}_{(\mathrm{t})}). \tag{D5}$$

We can obtain expressions in terms of the synchronous gauge perturbed metric as follows.

1. Lapse perturbation

Replacing the expression for $\alpha^{(2)}_{(\mathrm{t})}$ into Eq. (D2) we obtain an expression in terms of $\phi^{(2)}_{\mathrm{S(t)}}$, its derivatives and products of first-order quantities, which with the help of the Raychaudhuri equation (C1) can be written as

$$\begin{aligned}\nabla^2\nabla^2\psi^{(2)}_{\mathrm{P(t)}}=&\frac{6}{\tau^2}\nabla^2\phi^{(2)}_{\mathrm{S(t)}}+\frac{6}{\tau}\chi^{(1)\top\prime}_{ij}\varphi^{,ij}-3\nabla^2\chi^{(1)\top}_{ij}\varphi^{,ij}+\frac{5}{8}\nabla^2(\chi^{(1)\top\prime\,ij}\chi^{(1)\top\prime}_{ij})+\frac{1}{2\tau}\chi^{(1)\top\prime\,ij}\nabla^2\chi^{(1)\top}_{ij}\\&-\frac{1}{4}\nabla^2\chi^{(1)\top}_{ij}\nabla^2\chi^{(1)\top\,ij}-\frac{1}{\tau}\nabla^2\chi^{(1)\top\prime}_{ij}\chi^{(1)\top\,ij}+\frac{1}{2}\chi^{(1)\top\,ij}\nabla^2\nabla^2\chi^{(1)\top}_{ij}-\frac{1}{2\tau}\chi^{(1)\top\prime}_{ij,k}\chi^{(1)\top ij,k}\\&+\frac{5}{4}\chi^{(1)\top\,ij,k}\nabla^2\chi^{(1)\top}_{ij,k}-\frac{3}{2}\chi^{(1)\top\prime}_{ij,k}\chi^{(1)\top\prime\,kj,i}+\frac{3}{\tau}\chi^{(1)\top}_{ij,k}\chi^{(1)\top\prime\,kj,i}-\frac{3}{2}\chi^{(1)\top\,kj,i}\nabla^2\chi^{(1)\top}_{ij,k}\\&+\frac{3}{\tau^2}(2\chi^{(1)\top\,ij}\nabla^2\chi^{(1)\top}_{ij}+2\chi^{(1)\top}_{ij,k}\chi^{(1)\top\,ij,k}-2\nabla^2\chi^{(1)\top}_{0\,ij}\chi^{(1)\top\,ij}_0-2\chi^{(1)\top}_{0\,ij,k}\chi^{(1)\top\,ij,k}_0).\end{aligned} \tag{D6}$$

2. Shift perturbation

From Eq. (D3) and using the momentum constraint (C3) and the Raychaudhuri equation (C1) we obtain

$$\begin{aligned}\nabla^2\nabla^2\omega^{(2)}_{\mathrm{P(t)}\,i}=&\nabla^2(-4\chi^{(1)\top\prime}_{ij}\varphi^{,j}-2\chi^{(1)\top\prime}_{ik,j}\chi^{(1)\top\,jk}+2\chi^{(1)\top\prime}_{jk,i}\chi^{(1)\top\,jk}+\chi^{(1)\top\prime}_{jk}\chi^{(1)\top\,jk}_{,i})\\&+(4\chi^{(1)\top\prime}_{kj}\varphi^{,kj}-\chi^{(1)\top\prime}_{kj}\nabla^2\chi^{(1)\top\,kj}-2\chi^{(1)\top\,kj}\nabla^2\chi^{(1)\top\prime}_{kj}-3\chi^{(1)\top\prime}_{kj,l}\chi^{(1)\top\,kj,l}+2\chi^{(1)\top\prime}_{kj,l}\chi^{(1)\top\,lj,k})_{,i}.\end{aligned} \tag{D7}$$

3. Spatial metric, trace

From Eq. (D4) and using the Raychaudhuri equation (C1) we obtain

$$\begin{aligned}\nabla^2\nabla^2\phi^{(2)}_{\mathrm{P(t)}}=&\frac{18}{\tau^2}\nabla^2\phi^{(2)}_{\mathrm{S(t)}}+\frac{1}{8}\nabla^2(\chi^{(1)\top\prime\,ij}\chi^{(1)\top\prime}_{ij})-\frac{1}{2}\nabla^2(\chi^{(1)\top}_{ij}\nabla^2\chi^{(1)\top\,ij})-\frac{3}{4}\chi^{(1)\top}_{ij,k}\nabla^2\chi^{(1)\top\,ij,k}\\&-\frac{3}{4}\chi^{(1)\top}_{ij,kl}\chi^{(1)\top\,ij,kl}+\frac{1}{2}\chi^{(1)\top}_{ij,k}\nabla^2\chi^{(1)\top\,kj,i}+\frac{1}{2}\chi^{(1)\top}_{ij,kl}\chi^{(1)\top\,kj,il}-\nabla^2(\chi^{(1)\top}_{ij}\varphi^{,ij})-8\varphi^{,ij}\nabla^2\chi^{(1)\top}_{ij}\\&+\frac{1}{2\tau}\chi^{(1)\top\prime\,ij}\nabla^2\chi^{(1)\top}_{ij}+\frac{6}{\tau}\chi^{(1)\top\prime}_{ij}\varphi^{,ij}+\frac{12}{\tau^2}\chi^{(1)\top\prime}_{ij}\chi^{(1)\top\prime\,ij}-\frac{1}{\tau}\chi^{(1)\top\,ij}\nabla^2\chi^{(1)\top\prime}_{ij}\\&-\frac{1}{2\tau}\chi^{(1)\top\prime}_{ij,k}\chi^{(1)\top\,ij,k}+\frac{3}{\tau}\chi^{(1)\top}_{ij,k}\chi^{(1)\top\prime\,kj,i}+\frac{3}{\tau^2}\nabla^2(\chi^{(1)\top}_{ij}\chi^{(1)\top\,ij}+\chi^{(1)\top\,ij}_0\chi^{(1)\top}_{0\,ij})\\&+\frac{24}{\tau^2}\chi^{(1)\top}_{ij}\nabla^2\chi^{(1)\top\,ij}-\frac{48}{\tau^3}\chi^{(1)\top\prime}_{ij}\chi^{(1)\top\,ij}-\frac{72}{\tau^4}(\chi^{(1)\top}_{ij}\chi^{(1)\top\,ij}-\chi^{(1)\top}_{0\,ij}\chi^{(1)\top\,ij}_0).\end{aligned} \tag{D8}$$

4. Spatial metric, traceless part

Replacing from Eqs. (D1) the expressions for $\alpha^{(2)}_{(\mathrm{t})}$ and $d^{(2)}_{(\mathrm{t})i}$ in Eq. (D5) we obtain

$$\nabla^2\nabla^2\chi^{(2)}_{\mathrm{P(t)}ij}=\nabla^2\nabla^2\chi^{(2)}_{\mathrm{S(t)}ij}-2\nabla^2\chi^{(2),k}_{\mathrm{S(t)}k(i,j)}+\frac{1}{2}\chi^{(2)kl}_{\mathrm{S(t)},klij}+\frac{1}{2}\delta_{ij}\nabla^2\chi^{(2)kl}_{\mathrm{S(t)},kl}-\nabla^2\left\{\frac{4\tau}{3}\left[\varphi^{,k}\left(\chi^{(1)\top\prime}_{k(i}+\frac{4}{\tau}\chi^{(1)\top}_{k(i}\right)\right]_{,j)}\right.$$
$$+\frac{4\tau^2}{3}\varphi^{,kl}\chi^{(1)\top}_{k(i,j)l}+\frac{2\tau^2}{3}\nabla^2\varphi^{,l}\chi^{(1)\top}_{l(i,j)}-\frac{4\tau^2}{9}\left(\chi^{(1)\top}_{lk,ij}\varphi^{,lk}-\frac{1}{2}\chi^{(1)\top}_{lk}\varphi^{,lk}_{,ij}+\frac{1}{2}\chi^{(1)\top}_{lk,(i}\varphi_{,j)}{}^{,lk}\right)-\frac{2\tau^2}{3}\nabla^2\chi^{(1)\top}_{k(i}\varphi_{,j)}{}^{,k}$$
$$-\nabla^2\left[\frac{2\tau}{3}\varphi\left(\chi^{(1)\top\prime}_{ij}+\frac{4}{\tau}\chi^{(1)\top}_{ij}\right)+\frac{\tau^2}{3}\varphi^{,k}\chi^{(1)\top}_{ij,k}\right]-\frac{1}{2}\delta_{ij}\left[\frac{2\tau}{3}\varphi^{,kl}\left(\chi^{(1)\top\prime}_{kl}+\frac{4}{\tau}\chi^{(1)\top}_{kl}\right)-\frac{\tau^2}{3}\varphi^{,klm}\chi^{(1)\top}_{kl,m}\right.$$
$$\left.\left.-\frac{2\tau^2}{3}\varphi^{,kl}\nabla^2\chi^{(1)\top}_{kl}\right]\right\}+\frac{1}{2}\left[\frac{2\tau}{3}\varphi^{,kl}\left(\chi^{(1)\top\prime}_{kl}+\frac{4}{\tau}\chi^{(1)\top}_{kl}\right)+\frac{5\tau^2}{9}\varphi^{,klm}\chi^{(1)\top}_{kl,m}\right.$$
$$\left.+\frac{4}{9}\tau^2\chi^{(1)\top}_{kl}\nabla^2\varphi^{,kl}-\frac{2\tau^2}{9}\varphi^{,kl}\nabla^2\chi^{(1)\top}_{kl}\right]_{,ij}. \tag{D9}$$

[1] P. J. E. Peebles, *The Large Scale Structure of the Universe* (Princeton University Press, Princeton, 1980).

[2] E. Bertschinger, in *Cosmology and Large Scale Structure*, Proceedings of the Les Houches School, edited by R. Shaeffer, J. Silk, M. Spiro, and V. Zinn-Justin (Elsevier, Netherland, 1996), Sec. LX.

[3] S. Matarrese, O. Pantano, and D. Sáez, Phys. Rev. Lett. **72**, 320 (1994); Mon. Not. R. Astron. Soc. **271**, 513 (1994).

[4] S. Matarrese and D. Terranova, Mon. Not. R. Astron. Soc. **283**, 400 (1996).

[5] P. J. E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, Princeton, 1993).

[6] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational Lenses* (Springer-Verlag, Berlin, 1992).

[7] L. Kofman and D. Pogosyan, Astrophys. J. **442**, 30 (1995).

[8] Ya. B. Zel'dovich, Astron. Astrophys. **5**, 84 (1970).

[9] S. Matarrese and C. Daffara (in preparation).

[10] M. Bruni, S. Matarrese, and O. Pantano, in *Dynamical Systems in Cosmology*, edited by G. F. R. Ellis and J. Wainwright (Cambridge University Press, Cambridge, England, 1997).

[11] M. Rees and D. W. Sciama, Nature (London) **217**, 511 (1968).

[12] T. Pyne and S. M. Carroll, Phys. Rev. D **53**, 2920 (1996).

[13] S. Mollerach and S. Matarrese, Phys. Rev. D **56**, 4494 (1997).

[14] C.-P. Ma and E. Bertschinger, Astrophys. J. **455**, 7 (1995).

[15] M. Bruni, S. Matarrese, S. Mollerach, and S. Sonego, Class. Quantum Grav. **14**, 2585 (1997).

[16] S. Sonego and M. Bruni, Commun. Math. Phys. **193**, 209 (1998).

[17] K. Tomita, Prog. Theor. Phys. **37**, 831 (1967).

[18] H. Russ, M. Morita, M. Kasai, and G. Börner, Phys. Rev. D **53**, 6881 (1996).

[19] D. S. Salopek, J. M. Stewart, and K. M. Croudace, Mon. Not. R. Astron. Soc. **271**, 1005 (1994).

[20] K. Tomita, Prog. Theor. Phys. **45**, 1747 (1971); **47**, 416 (1972).

[21] E. M. Lifshitz, J. Phys. (Moscow) **10**, 116 (1946); E. M. Lifshitz and I. M. Khalatnikov, Adv. Phys. **12**, 185 (1963); L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975).

[22] R. K. Sachs, in *Relativity, Groups, and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).

[23] J. M. Stewart and M. Walker, Proc. R. Soc. London **A341**, 49 (1974).

[24] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[25] G. F. R. Ellis and M. Bruni, Phys. Rev. D **40**, 1804 (1989).

[26] J. M. Stewart, Class. Quantum Grav. **7**, 1169 (1990).

[27] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).

[28] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).

[29] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[30] It is worth pointing out here that the kth order perturbation $\delta^k T$ of a tensor T_λ on $\mathcal{M}_\lambda$ is *defined* as the kth derivative with respect to λ of $T(\lambda)$ (the representation, in a given gauge, of T_λ on the background $\mathcal{M}_0$); see Refs. [15,29]. This is easily checked from Eqs. (2.6), (2.7), and is the reason for the numerical factor $1/k!$ in front of the $\delta^k T$, $\delta^k\tilde{T}$ terms.

[31] R. J. Gleiser, C. O. Nicasio, R. H. Price, and J. Pullin, Class. Quantum Grav. **13**, L117 (1996); Phys. Rev. Lett. **77**, 4483 (1996).

[32] V. F. Mukhanov, L. R. W. Abramo, and R. H. Brandenberger, Phys. Rev. Lett. **78**, 1624 (1997); L. R. W. Abramo, R. H. Brandenberger, and V. F. Mukhanov, Phys. Rev. D **56**, 3248 (1997).

[33] S. Matarrese and S. Mollerach, in *Proceedings of the Spanish Relativity Meeting (ERE-96)*, Valencia, Spain, 1996, edited by J. A. Miralles, J. A. Morales, and D. Sáez (Editions Frontières, Paris, in press), p. 87, astro-ph/9705168.

[34] For nonlinear perturbations the parametrization is not unique (see Ref. [15] for a brief discussion of this point); here we

simply assume that a parametrization has been given, and we work within it.

[35] R. D'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, Oxford, 1992).

[36] H. Stephani, *General Relativity*, 2nd ed. (Cambridge University Press, Cambridge, England, 1990).

[37] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1973).

[38] B. F. Schutz, *Geometrical Methods of Mathematical Physics* (Cambridge University Press, Cambridge, England, 1980).

[39] Note the difference between Eq. (A9), where the functions $\tilde{x}^\mu$ are given by the action of the operator $\exp[\lambda \pounds_\xi]$ on the functions x^μ, and Eq. (A11), where the components $\tilde{Z}^\mu$ of the pullback $\tilde{Z}$ are the μ components of the vector field given by the action of $\exp[\lambda \pounds_\xi]$ on Z.

[40] R. Geroch and L. Lindblom, J. Math. Phys. **26**, 2581 (1985); E. Flanagan and R. Wald, Phys. Rev. D **54**, 6233 (1996); B. F. Schutz, in *Relativistic Astrophysics and Cosmology*, edited by X. Fustero and E. Verdaguer (World Scientific, Singapore, 1984).