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Conformal field theory correlators from classical scalar field theory on anti–de Sitter space

W. Mück* and K. S. Viswanathan†
Department of Physics, Simon Fraser University, Burnaby, British Columbia, V5A 1S6 Canada
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We use the correspondence between scalar field theory on AdS_{d+1} and a conformal field theory on $\mathbb{R}^d$ to calculate the first-order contributions to the 3- and 4-point functions of the latter. [S0556-2821(98)50116-6]

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I. INTRODUCTION

Since the suggestion of Maldacena about the equivalence of the large N limit of certain conformal field theories in d dimensions on one hand and supergravity on $(2+1)$ anti–de Sitter space (AdS_{d+1}) on the other hand [1], theories on anti–de Sitter spaces seem to have undergone a renaissance. After detailed investigations in the past (see for example [2–4]), there has been a multitude of papers related to this subject in various aspects in the past months alone (see [5] for a recent list of references). In particular, the suggested correspondence was made more precise in [6–8]. According to these references, one identifies the partition function of the AdS theory (with suitably prescribed boundary conditions for the fields) with the generating functional of the boundary conformal field theory. Thus, one has, schematically,

$$Z_{\mathrm{AdS}}[\phi_0]=\int_{\phi_0}\mathcal{D}\phi\ \exp(-I[\phi])$$
$$\equiv Z_{\mathrm{CFT}}[\phi_0]=\left\langle \exp\left(\int_{\partial\Omega}d^dx\,\mathcal{O}\phi_0\right)\right\rangle. \qquad (1)$$

The path integral on the left-hand side (LHS) is calculated under the restriction that the field ϕ asymptotically approaches ϕ_0 on the boundary. On the other hand, the function ϕ_0 is considered as a current, which couples to the scalar density operator $\mathcal{O}$ in the boundary conformal field theory. Calculating the LHS of Eq. (1) thus allows one to obtain explicitly correlation functions of the boundary conformal field theory. Of course, since the 2- and 3-point functions are fixed (up to a constant) by conformal invariance [9], one is especially interested in calculating the cases $n>3$.

It is not only of pedagogical interest to consider the classical approximation to the AdS partition function, which is obtained by inserting the solutions of the classical field equations into $I[\phi]$. In fact, the suggested AdS–conformal-field-theory (CFT) correspondence [1] involves classical supergravity on the AdS side. Moreover, it is instructive to study toy examples in order to better understand this correspondence. A number of examples, including free massive scalar and $U(1)$ gauge fields were studied in [8] and free fermions were considered in [5,10]. Since a free field theory will inevitably lead to a trivial (i.e., free) boundary CFT, we feel it necessary to consider interactions. A short note on interacting scalar fields is contained in [11]. In this paper we will consider in detail a classical interacting scalar field on AdS_{d+1} to first order in the interactions.

We recall here for convenience the formulas necessary for solving the classical scalar field theory with Dirichlet boundary conditions. Let us start with stating the action for a real scalar field in $d+1$ dimensions (Riemannian signature) with polynomial interactions,

$$I[\phi]=\int_\Omega d^{d+1}x\sqrt{g}\left(\frac{1}{2}((\nabla\phi)^2+m^2\phi^2)+\sum_{n\geq 3}\frac{\lambda_n}{n!}\phi^n\right). \qquad (2)$$

The action (2) yields the equation of motion

$$(\nabla^2-m^2)\phi=\sum_{n\geq 3}\frac{\lambda_n}{(n-1)!}\phi^{n-1}. \qquad (3)$$

Using the covariant Green's function, which satisfies

$$(\nabla^2-m^2)G(x,y)=\frac{\delta(x-y)}{\sqrt{g(x)}} \qquad (4)$$

and the boundary condition $G(x,y)|_{x\in\partial\Omega}=0$, the classical field ϕ satisfying the equation of motion (3) and a Dirichlet boundary condition on $\partial\Omega$ satisfies the integral equation

$$\phi(x)=\int_{\partial\Omega}d^dy\,\sqrt{h}n^\mu\frac{\partial}{\partial y^\mu}G(x,y)\phi(y)$$
$$+\int_\Omega d^{d+1}y\,\sqrt{g}G(x,y)\sum_{n\geq 3}\frac{\lambda_n}{(n-1)!}\phi(y)^{n-1}, \qquad (5)$$

where h is the determinant of the induced metric on $\partial\Omega$ and n^μ the unit vector normal to $\partial\Omega$ and pointing outwards. A perturbative expansion in the couplings λ_n is obtained by using Eq. (5) recursively. We shall denote the surface term in Eq. (5) by $\phi^{(0)}$ and the remainder by $\phi^{(1)}$. Then, substituting the classical solution (5) into Eq. (2), integrating by parts, and using the properties of the Green's function, one obtains

*Email address: wmueck@sfu.ca
†Email address: kviswana@sfu.ca

$$I[\phi]=\frac{1}{2}\int_{\partial\Omega}d^dx\sqrt{h}n^\mu\phi^{(0)}\partial_\mu\phi^{(0)} + \sum_{n\geq 3}\frac{\lambda_n}{n!}\int_\Omega d^{d+1}x\sqrt{g}(\phi^{(0)})^n+\mathcal{O}(\lambda^2). \qquad (6)$$

A short outline of the remainder of this paper is as follows. In Sec. II we consider the free field on AdS_{d+1}. We explicitly calculate the solutions to the wave equation, the Green's function, solve the Dirichlet boundary problem, and find the 2-point function of the boundary conformal field theory. In Sec. III we perform the calculations to first order in the interaction parameters λ_n. An explicit closed formula for the n-point function does not seem attainable for $n>3$. However, we will stay general as far as possible and only then specialize in the cases $n=3$ and $n=4$. Finally, Sec. IV contains conclusions.

II. FREE FIELD THEORY ON AdS_{d+1}

We will use the representation of AdS_{d+1} as the upper half space $(x_0>0)$ with the metric

$$ds^2=\frac{1}{x_0^2}\sum_{i=0}^{d}dx_i^2, \qquad (7)$$

which possesses the constant curvature scalar $R=-d(d+1)$. The boundary $\partial\Omega$ is given by the space $\mathbb{R}$ with $x_0=0$ plus the single point $x_0=\infty$ [8]. In the sequel we shall adopt the notations $x=(x_0,\mathbf{x})$, $x^*=(-x_0,\mathbf{x})$ and $x^2=x_0^2+\mathbf{x}^2$.

Let us first solve the massive wave equation

$$(\nabla^2-m^2)\phi=\left(x_0^2\sum_{i=0}^{d}\partial_i^2-x_0(d-1)\partial_0-m^2\right)\phi=0. \qquad (8)$$

The linearly independent solutions of Eq. (8) are found to be

$$x_0^{d/2}e^{-i\mathbf{k}\cdot\mathbf{x}}\begin{cases}I_\alpha(kx_0)\\K_\alpha(kx_0)\end{cases},\quad \text{where } \alpha=\sqrt{\frac{d^2}{4}+m^2}, \qquad (9)$$

$\mathbf{k}$ is a momentum d-vector and $k=|\mathbf{k}|$. It is easy to check that these modes are not square integrable, if $m^2\geq -d^2/4$.

The modes can now be used to calculate the Green's function in Eq. (4). Making the ansatz

$$G(x,y)=\int\frac{d^dk}{(2\pi)^d}x_0^{d/2}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}f(k,y_0)\begin{cases}I_\alpha(kx_0)K_\alpha(ky_0) & \text{for } x_0<y_0,\\K_\alpha(kx_0)I_\alpha(ky_0) & \text{for } x_0>y_0,\end{cases} \qquad (10)$$

we explicitly satisfy the boundary condition at $x_0=0$ and ∞ and ensure continuity at $x_0=y_0$. Matching the two regions at the discontinuity yields $f=-y_0^{d/2}$. The ansatz (10) can be integrated and gives

$$G(x,y)=-\frac{c}{2\alpha}\xi^{-\Delta}F\left(\frac{d}{2},\Delta;\alpha+1;\frac{1}{\xi^2}\right), \qquad (11)$$

where F denotes the hypergeometric function [12],

$$\xi=\frac{1}{2x_0y_0}\left[\frac{1}{2}((x-y)^2+(x-y^*)^2)+\sqrt{(x-y)^2(x-y^*)^2}\right] \qquad (12)$$

and the new constants are defined by $\Delta=d/2+\alpha$ and $c=\Gamma(\Delta)/(\pi^{d/2}\Gamma(\alpha))$. The Green's function (11) coincides with the one found by Burgess and Lütken [4] after using a transformation formula for the hypergeometric function [12, formula 9.134 2.]. Our form has the advantage that for even d the result can, using either special value formulas or the definition as a series, be expressed in terms of rational functions. For example, for $d=2$ we can use

$$F(1,1+\alpha;1+\alpha;z)=\frac{1}{1-z}.$$

We shall in this paper make use only of the boundary behavior of the Green's function. Since the induced metric diverges on the boundary of AdS_{d+1} $(x_0=0)$, one has to consider the standard formalism described in Sec. I on a near-boundary surface $x_0=\epsilon>0$ and then take the limit $\epsilon\to 0$. It has been pointed out recently by Freedman, Mathur, Matusis, and Rastelli [13] that this limit has to be taken carefully, in particular at the very end of those calculations, which involve only the boundary behavior of the classical solution. It is therefore necessary to find the Green's function, which vanishes not at $x_0=0$, but at $x_0=\epsilon$. One can easily change Eq. (10) to accommodate this. Denoting the new Green's function by G_ϵ, we find

$$G_\epsilon(x,y)=G_0(x,y)+\int\frac{d^dk}{(2\pi)^d}(x_0y_0)^{d/2}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}K_\alpha(kx_0)K_\alpha(ky_0)\frac{I_\alpha(k\epsilon)}{K_\alpha(k\epsilon)}, \qquad (13)$$

where G_0 is given by Eqs. (10) and (11). It does not seem possible to perform the momentum integral, but this is not necessary in order to obtain the desired boundary behavior. In particular, we find the normal derivative on the boundary as

$$\frac{\partial}{\partial y_0}G_\epsilon(x,y)\bigg|_{y_0=\epsilon}=-x_0^{d/2}\epsilon^{(d/2)-1}\times\int\frac{d^dk}{(2\pi)^d}\,e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\frac{K_\alpha(kx_0)}{K_\alpha(k\epsilon)},\tag{14}$$

giving $-\epsilon^{d-1}\delta(\mathbf{x}-\mathbf{y})$ for $x_0=\epsilon$.

The bulk behavior of the free field can be obtained from Eq. (5) using the asymptotic behavior of the Bessel function in the denominator of Eq. (14) for $\epsilon\to 0$. We note that for AdS_{d+1}, one has $\sqrt{h(y)}=\epsilon^{-d}$ and $n^\mu=(-\epsilon,\mathbf{0})$. The minus sign comes from n^μ pointing outward. One finds

$$\phi^{(0)bulk}(x)=c\,\epsilon^{\Delta-d}\int d^dy\,\phi_\epsilon(\mathbf{y})\left(\frac{x_0}{x_0^2+|\mathbf{x}-\mathbf{y}|^2}\right)^\Delta,\tag{15}$$

where ϕ_ϵ denotes the Dirichlet boundary value at $x_0=\epsilon$. We define

$$\phi_0(\mathbf{x})=\epsilon^{\Delta-d}\phi_\epsilon(\mathbf{x})\tag{16}$$

in order to make contact with the conformal field theory on the boundary of AdS_{d+1}.

Equation (15) is the solution to the Dirichlet problem with the boundary at $x_0=0$ [8]. However, for the two-point function, we need to calculate the surface integral in Eq. (6), i.e., we need the near-boundary behavior for a boundary at $x_0=\epsilon$. Using the exact expression (14) we find

$$\partial_0\phi|_{x_0=\epsilon}=\frac{1}{\epsilon}\int d^dy\,\phi_\epsilon(\mathbf{y})\int\frac{d^dk}{(2\pi)^d}\,e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\times\left(\frac{d}{2}-\alpha+k\frac{\partial}{\partial k}\ln((k\epsilon)^\alpha K_\alpha(k\epsilon))\right).$$

The first two terms in the squared bracket yield δ function contact terms in the two-point function, which are of no interest to us. In the third term, the divergence of the Bessel function for $\epsilon\to 0$ is exactly canceled by the power of ϵ in front of it. Using the series expansion,

$$z^\alpha K_\alpha(z)=2^{\alpha-1}\Gamma(\alpha)\left[1-\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\left(\frac{z}{2}\right)^{2\alpha}+\cdots\right],$$

where the dots denote terms of order z^n and $z^{2\alpha+n}$, one can approximate the logarithm and then evaluate the integral to obtain

$$\partial_0\phi|_{x_0=\epsilon}=2\alpha c\,\epsilon^{2\alpha-1}\int d^dy\,\frac{\phi_\epsilon(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{2\Delta}}+\cdots.\tag{17}$$

Inserting Eq. (17) into Eq. (6), we find the value of the free field action as

$$I^{(0)}=-\frac{1}{2}\int d^dx\,d^dy\,2\alpha c\,\epsilon^{2(\Delta-d)}\frac{\phi_\epsilon(\mathbf{x})\phi_\epsilon(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{2\Delta}}+\cdots.\tag{18}$$

Taking the limit $\epsilon\to 0$ with the definition (16) we hence obtain, in agreement with [13], the two-point function for the boundary conformal operators:

$$\langle\mathcal{O}(\mathbf{x})\mathcal{O}(\mathbf{y})\rangle=\frac{2\alpha c}{|\mathbf{x}-\mathbf{y}|^{2\Delta}}.\tag{19}$$

III. FIRST-ORDER CALCULATIONS

For interactions, one can take the limit $\epsilon\to 0$ beforehand, which makes the considerations somewhat easier. The reason is that these calculations involve bulk integrals over AdS_{d+1}, as in the second term of Eq. (6). Hence only the bulk behavior of the free field will be needed, which was obtained in Sec. II. Inserting Eq. (15) (with $\epsilon\to 0$) into the interaction term of the action, one obtains

$$I^{(1)}[\phi_0]=\sum_{n\geqslant 3}\frac{c^n\lambda_n}{n!}\int d^dx_1\ldots d^dx_n\,\phi_0(\mathbf{x}_1)\ldots\phi_0(\mathbf{x}_n)I_n(\mathbf{x}_1,\ldots,\mathbf{x}_n),\tag{20}$$

with

$$I_n(\mathbf{x}_1,\ldots,\mathbf{x}_n)=\int d^{d+1}y\,\frac{y_0^{-(d+1)+n\Delta}}{[(y_0^2+|\mathbf{y}-\mathbf{x}_1|^2)\ldots(y_0^2+|\mathbf{y}-\mathbf{x}_n|^2)]^\Delta}.\tag{21}$$

We can read off the first-order contribution to the connected part of the n-point functions ($n \geqslant 3$) for the operator $\mathcal{O}$ from Eq. (20),

$$\langle \mathcal{O}(\mathbf{x}_1)\ldots\mathcal{O}(\mathbf{x}_n)\rangle_{\text{conn.}} = -\lambda_n c^n I_n(\mathbf{x}_1,\ldots,\mathbf{x}_n). \tag{22}$$

We shall now elaborate on a detailed calculation of the 3- and 4-point functions. After a Feynman parametrization, the y integral in Eq. (21) can be done yielding

$$I_n = \frac{\pi^{d/2}\Gamma[(n/2)\Delta - d/2]\Gamma[(n/2)\Delta]}{2\Gamma(\Delta)^n} \times \int_0^\infty d\alpha_1\ldots d\alpha_n \delta\left(\sum \alpha_i - 1\right) \times \frac{\Pi \alpha_i^{\Delta-1}}{(\sum_{i<j} \alpha_i\alpha_j x_{ij}^2)^{(n/2)\Delta}},$$

where $x_{ij} = |\mathbf{x}_i - \mathbf{x}_j|$. Now we can introduce new integration variables β_i by $\alpha_1 = \beta_1$ and $\alpha_i = \beta_1\beta_i$ ($i \geqslant 2$). The integration over β_1 is then trivial and leads to

$$I_n = \frac{\pi^{d/2}\Gamma[(n/2)\Delta - d/2]\Gamma[(n/2)\Delta]}{2\Gamma(\Delta)^n} \times \int_0^\infty d\beta_2\ldots d\beta_n \frac{\prod_{i=2}^n \beta_i^{\Delta-1}}{[\sum_{i=2}^n \beta_i(x_{1i}^2 + \sum_{j>i}\beta_j x_{ij}^2)]^{(n/2)\Delta}}. \tag{23}$$

We shall not try to perform the remaining integration in the general formula, but consider the cases $n=3$ and $n=4$. For $n=3$ the integrations can be carried out straightforwardly. Inserting the result into Eq. (22) gives

$$\langle \mathcal{O}(\mathbf{x}_1)\mathcal{O}(\mathbf{x}_2)\mathcal{O}(\mathbf{x}_3)\rangle = -\frac{\lambda_3\Gamma[(1/2)\Delta + \alpha]}{2\pi^d}\left[\frac{\Gamma[(1/2)\Delta]}{\Gamma(\alpha)}\right]^3 \frac{1}{(x_{12}x_{13}x_{23})^\Delta}. \tag{24}$$

For $n=3$ there is no disconnected contribution, hence Eq. (24) describes the full first-order 3-point function.

For $n=4$, we obtain, after integration over β_4 and β_3,

$$I_4 = \frac{\Gamma(2\Delta - d/2)}{2\Gamma(2\Delta)} \frac{\pi^{d/2}}{(x_{12}x_{34})^{2\Delta}} \times \int_0^\infty \frac{d\beta_2}{\beta_2} F\left(\Delta,\Delta;2\Delta;1 - \frac{(x_{13}^2 + \beta_2 x_{23}^2)(x_{14}^2 + \beta_2 x_{24}^2)}{\beta_2 x_{12}^2 x_{34}^2}\right).$$

A change of integration variables and the introduction of the conformal invariants (harmonic ratios) [9]

$$\beta_2 = \frac{x_{13}x_{14}}{x_{23}x_{24}} e^{2z}, \quad \eta = \frac{x_{12}x_{34}}{x_{14}x_{23}}, \quad \zeta = \frac{x_{12}x_{34}}{x_{13}x_{24}},$$

then yields

$$I_4 = \frac{\Gamma(2\Delta - d/2)}{\Gamma(2\Delta)} \frac{2\pi^{3/2}}{(\eta\zeta \prod_{i<j} x_{ij})^{(2/3)\Delta}} \times \int_0^\infty dz F\left(\Delta,\Delta;2\Delta;1 - \frac{(\eta+\zeta)^2}{(\eta\zeta)^2} - \frac{4}{\eta\zeta}\sinh^2 z\right). \tag{25}$$

Obviously, Eq. (25) is of exactly the form dictated for a four point function by conformal invariance [9].

IV. CONCLUSIONS

We have considered an example of the correspondence between field theories on an AdS space and CFTs on its boundary. The classical interacting scalar field has been treated to first order in the interactions and a nontrivial conformal field theory of boundary operators has been obtained. We calculated a nontrivial coefficient of the 3-point function and, for the first time with this method, found an expression for the function $f(\eta,\zeta)$ contained in the first-order contribution to the 4-point function [9]. The analysis could in principle be extended to higher orders, since all necessary tools are at our disposal. The simplest second-order term is a tree diagram contribution to the 4-point function involving two 3-point vertices, which are connected by a bulk Green's function. However, even in this case the integrals involved seem intractable. We believe that the obtained results will also be helpful for studying more complicated field theories containing fermions and gauge fields.

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