Chiral fermions on the lattice through gauge fixing: Perturbation theory

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We study the gauge-fixing approach to the construction of lattice chiral gauge theories in one-loop weak-coupling perturbation theory. We show how the infrared properties of the gauge degrees of freedom determine the nature of the continuous phase transition at which we take the continuum limit. The fermion self-energy and the vacuum polarization are calculated, and confirm that, in the Abelian case, this approach can be used to put chiral gauge theories on the lattice in four dimensions. We comment on the generalization to the non-Abelian case. [S0556-2821(98)00415-9]

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I. INTRODUCTION

In a recent paper [1] we have shown that one can construct models with chiral fermions on the lattice by using a lattice action which contains a discretization of a covariant continuum gauge-fixing term. The model we investigated is a concrete implementation [2] of the so-called ''Rome approach'' [3,4].

In lattice chiral gauge theories, the gauge symmetry is explicitly broken for nonzero values of the lattice spacing, even in anomaly free models. The basic reason for this is that each fermion species has to contribute its part to the chiral anomaly, and in order to do so, chiral symmetry has to be explicitly broken in the regulated theory [5] (see also Ref. [6] and references therein). On the lattice, the gauge-symmetry breaking induced by quantum effects is not restricted to the anomaly, but includes infinitely many higher-dimensional operators which are suppressed by powers of the lattice spacing (are ''irrelevant'') for smooth external gauge fields. However, for arbitrarily ''rough'' lattice gauge fields, these operators potentially lead to unsuppressed interactions between the fermions and the gauge degrees of freedom (the longitudinal modes of the gauge field). Typically, this phenomenon alters the fermion spectrum of the theory nonperturbatively, leading to a vectorlike rather than a chiral fermion content in the continuum limit (for reviews, see Refs. [7, 6]).

In order to remedy this problem, it is natural to consider gauge-fixed lattice gauge theories [3,4]. It was argued in Ref. [4] that a smooth gauge may lead to a suppression of rough lattice gauge fields such that a location in the phase diagram of the theory exists, where the fermion spectrum remains chiral. In this case, both the transversal and longitudinal

modes are controlled by the bare lattice gauge coupling, so that the lattice theory can be systematically studied in weak-coupling perturbation theory.

In order for the lattice theory to admit a perturbative expansion, the gauge-fixing action should have a global minimum at the perturbative vacuum, $A_\mu = 0$. A discretization of the standard Lorentz gauge-fixing term with this property was proposed in Ref. [2]. A simplified version of this model was then studied nonperturbatively for the Abelian case. In this ''reduced'' model, only the longitudinal modes of the gauge field (or, equivalently, the gauge degrees of freedom) are taken into account. Since these are precisely the degrees of freedom that, without gauge fixing, destroy the chiral nature of the fermions, it is important to study such reduced models first, in order to demonstrate that the fermions remain chiral despite their interactions with the gauge degrees of freedom.

In Refs. [4,2] it was argued that, for small gauge coupling, the gauge-fixed lattice action leads to a continuous phase transition between a Higgs phase, and a novel ''directional'' phase, in which the gauge field condenses. At the phase transition (which belongs to a universality class different from the usual Higgs transition), the gauge field is massless, and a continuum limit can be defined. The existence of this phase transition was confirmed in the reduced Abelian model by high-statistics numerical computations and in the mean-field approximation [8]. In the reduced model, which is always invariant under constant gauge transformations, the Higgs phase corresponds to a phase with broken symmetry, which, however, gets restored at the phase transition between the Higgs and ''directional'' phases. This symmetry restoration is of crucial importance, since it allows us to unambiguously determine the fermionic quantum numbers under the (global remnant of the) gauge group. Using Wilson fermions, the existence of undoubled fermions in the desired chiral representation of the gauge group was confirmed numerically in Ref. [1].

In this paper, we study the reduced model in detail in

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weak-coupling perturbation theory. In Sec. II, we define the fully gauged and reduced models, and explain how perturbation theory may be set up systematically. In Sec. III, where we limit ourselves to the Abelian case, we show how the dynamics of the gauge degrees of freedom leads to the continuous phase transition mentioned above, and how the symmetry gets restored at the phase transition. In Sec. IV, we discuss the one-loop fermion self-energy, and demonstrate that indeed free chiral fermions with the correct quantum numbers emerge at this phase transition in the reduced model. We then go on to discuss the vacuum polarization in Sec. V. We calculate the shift in the location of the phase transition induced by the fermions at one loop. We show that, at the phase transition, the gauge degrees of freedom decouple from the fermions (a result that also follows from, and is consistent with, the fermion self-energy calculated in Sec. IV), and that the expected fermionic contribution to the β function is obtained for the gauge coupling. All these results confirm that, at least in the Abelian case, our lattice theory leads to the desired chiral gauge theory, when the continuum limit is taken at the continuous phase transition at weak gauge coupling. Some of the results of this paper have already been used in a comparison with the numerical results of Refs. [1,8]. In Sec. VI, we discuss the issue of fermion number nonconservation at the level of perturbation theory. Following Ref. [9], we show that a gauge invariant fermion-number current can be constructed with the correct anomaly in the continuum limit. In the last section, we summarize our results, and outline some of the most important open problems. We refer to Refs. [10,11] for a less technical account of our work.

II. THE MODEL

Let us start with the action for the fully gauged lattice chiral fermion theory. We will assume that all physical fermions are left handed, and that they transform in some (not necessarily irreducible) representation of a gauge group G. This representation will have to be anomaly free if a unitary continuum limit is to exist. The complete action can be written as a sum of terms, each of which we will introduce below:

$$S_V = S_{\text{plaq}} + S_{\text{gf}} + S_{\text{ghost}} + S_{\text{fermion}} + S_{\text{ct}}. \quad (1)$$

For S_{plaq} we will assume the usual plaquette term with the link variables $U_{x,\mu} = \exp(iA_{x,\mu})$ in the fundamental representation. For S_{gf} we will take the lattice version of the square of the Lorentz gauge condition that we proposed in Ref. [2]:

$$S_{\text{gf}} = \frac{1}{2\xi g^2} \text{tr}\left(\sum_{xyz} \Box_{xy}(U)\Box_{yz}(U) - \sum_x B_x^2(V(U)) \right), \quad (2)$$

where

$$\Box_{xy}(U) = \sum_\mu (\delta_{x+\mu,y}U_{x,\mu} + \delta_{x-\mu,y}U_{y,\mu}^\dagger) - 8\delta_{x,y} \quad (3)$$

is the covariant lattice Laplacian, and

$$B_x(V) = \sum_\mu \left(\frac{V_{x-\mu,\mu} + V_{x,\mu}}{2} \right)^2, \quad (4)$$

with

$$V_{x,\mu} = \frac{1}{2i}(U_{x,\mu} - U_{x,\mu}^\dagger) = A_{x,\mu} + O(A^3). \quad (5)$$

g is the bare gauge coupling and ξ is the bare gauge-fixing parameter. It is straightforward to show that, in the classical continuum limit,

$$S_{\text{gf}} = \frac{1}{2\xi g^2} \text{tr}(\partial_\mu A_{x,\mu})^2 + \text{irrelevant operators}. \quad (6)$$

Of course there are many possible choices for S_{gf} with the same classical continuum limit. Our choice here is motivated by two important properties obeyed by Eq. (2) [2]: S_{gf} has a unique absolute minimum at $U_{x,\mu} = I$, validating weak-coupling perturbation theory in g; our choice of S_{gf} leads to a critical behavior suitable for taking a continuum limit in the limit $g \to 0$.

Both properties will be used and discussed in this paper. The fact that this gauge-fixing action has a unique minimum is closely related to the fact that, on the lattice, it is not the square of a local gauge-fixing condition. As a result, the action S_V (even without the fermions) is not Becchi-Rouet-Stora-Tyutin (BRST) invariant. This situation allows us to avoid a theorem stating that expectation values of gauge-invariant operators would vanish in a lattice model with exact BRST invariance, due to the existence of lattice Gribov copies in such lattice models [12].

In the BRST approach, the gauge-fixing part of the action is not complete without a Faddeev-Popov term S_{ghost}. However, we will not specify this term here, as we will be mostly concerned with the Abelian case $G = U(1)$, in which case no ghosts need to be introduced, or with one-loop calculations not involving ghost loops. For more discussion of the non-Abelian case, see Ref. [4]. (A recent claim [13] that the theorem of Ref. [12] would also apply here rests on a misunderstanding about the choice of the ghost action. This is explained in detail in Ref. [14].)

For the fermion action, we will choose to use Wilson fermions. For each left-handed fermion ψ_L, we introduce a right-handed "spectator" fermion ψ_R. This allows us to construct a Wilson term that will serve to remove the fermion doublers, of course at the expense of gauge invariance [15]. The fermion action is

$$S_{\text{fermion}} = \frac{1}{2} \sum_{x,\mu} (\bar{\psi}_x \gamma_\mu (U_{x,\mu} P_L + P_R) \psi_{x+\mu}$$
$$- \bar{\psi}_{x+\mu} \gamma_\mu (U_{x,\mu}^\dagger P_L P_R) \psi_x$$
$$- r(\bar{\psi}_x \psi_{x+\mu} + \bar{\psi}_{x+\mu} \psi_x - 2\bar{\psi}_x \psi_x)). \quad (7)$$

$P_{L(R)}$ are the left(right)-handed projectors $\frac{1}{2}(1 \mp \gamma_5)$, and r is the Wilson parameter. Since the Wilson term breaks the

left-handed G invariance anyway, we choose not to put any gauge fields in, and S_{fermion} is therefore invariant under the shift symmetry [16]

$$\psi_R \rightarrow \psi_R + \epsilon_R, \quad \bar{\psi}_R \rightarrow \bar{\psi}_R + \bar{\epsilon}_R. \qquad (8)$$

Since gauge invariance (or more precisely, BRST invariance) is broken by the fermion action and by the gauge-fixing action, we will need to add counterterms, S_{ct}. In principle, all relevant and marginal counterterms which are allowed by the exact symmetries of the lattice theory will be needed [3]. The most important one for our purposes in this paper is the gauge-boson mass counterterm, which is the only dimension-two counterterm. All other counterterms are of dimension four, since a fermion-mass counterterm is forbidden by shift symmetry (lower dimension counterterms involving ghost fields are excluded by lattice symmetries as well [3]). So we will choose

$$S_{\text{ct}} = -\kappa \operatorname{tr} \sum_{x,\mu} (U_{x,\mu} + U^\dagger_{x,\mu}) + \text{marginal terms}, \qquad (9)$$

where we do not need to specify the marginal terms for this paper. They could be constructed from their continuum form, by replacing $A_{x,\mu} \rightarrow V_{x,\mu}$ [cf. Eq. (5)] and partial derivatives by difference operators.

Since the action S_V is not gauge invariant, we may introduce a Stückelberg field $\phi_x \in G$, and write the action as

$$S_H = S_{\text{plaq}} + S^\phi_{\text{gf}} + S^\phi_{\text{ghost}} + S^\phi_{\text{fermion}} + S^\phi_{\text{ct}}, \qquad (10)$$

with

$$S^\phi_{\text{gf}} = \frac{1}{2\xi g^2} \operatorname{tr} \sum_x (\phi^\dagger_x(\Box^2(U)\phi)x - B^2_x(V^\phi(U))),$$

$$\begin{aligned}
S^\phi_{\text{fermion}} = \frac{1}{2} \sum_{x,\mu} (&\bar{\psi}_x \gamma_\mu(U_{x,\mu} P_L + P_R) \psi_{x+\mu} \\
&- \bar{\psi}_{x+\mu} \gamma_\mu(U^\dagger_{x,\mu} P_L + P_R) \psi_x \\
&- r[(\bar{\psi}_x(\phi^\dagger_{x+\mu} P_L + \phi_x P_R) \psi_{x+\mu} + \text{H.c.}) \\
&- 2\bar{\psi}_x(\phi^\dagger_x P_L + \phi_x P_R) \psi_x]),
\end{aligned}$$

$$S^\phi_{\text{ct}} = -\kappa \operatorname{tr} \sum_x \phi^\dagger_x(\Box(U)\phi)_x + \text{marginal terms}, \qquad (11)$$

and in which $V_{x,\mu}$ is replaced by $V^\phi_{x,\mu}$ with

$$V^\phi_{x,\mu} = \frac{1}{2i}(\phi^\dagger_x U_{x,\mu} \phi_{x+\mu} - \phi^\dagger_{x+\mu} U^\dagger_{x,\mu} \phi_x). \qquad (12)$$

Note that S_{plaq} and the $r=0$ part of S_{fermion} do not change because they are gauge invariant.

S_H is gauge invariant under the transformation

$$U_{x,\mu} \rightarrow h_{Lx} U_{x,\mu} h^\dagger_{Lx+\mu},$$

$$\phi_x \rightarrow h_{Lx} \phi_x,$$

$$\psi_x \rightarrow (h_{Lx} P_L + P_R) \psi_x, \qquad (13)$$

where $h_{Lx} \in G$. Because of this, ϕ_x may be completely eliminated from S_H by a gauge transformation, and doing so we recover, as expected,

$$S_V(U,\psi) = S_H(\phi,U,\psi)|_{\phi=I}. \qquad (14)$$

We will refer to $S_{V(H)}$ as the action in the "vector" ("Higgs") picture. The two formulations are entirely equivalent: observables in the vector picture are mapped into (gauge invariant) observables in the Higgs picture, and *vice versa* [17].

Next, we introduce the "reduced" model, which is obtained from S_H by setting the *gauge field* $U_{x,\mu}$ equal to one. The reason that this reduced model is of interest is that, if the full model is to yield a theory of fermions chirally coupled to gluons in the continuum limit, the reduced model should lead to a theory of free chiral fermions (in the correct representation of the gauge group G) in the corresponding continuum limit. Ignoring the marginal counterterms, we obtain the reduced model action

$$S_{\text{reduced}} = \tilde{\kappa} \operatorname{tr} \sum_x (\phi^\dagger_x(\Box^2\phi)_x - B^2_x(V^r(\phi))) - \kappa \operatorname{tr} \sum_x \phi^\dagger_x(\Box\phi)_x$$

$$+ \frac{1}{2} \sum_{x,\mu} (\bar{\psi}_x \gamma_\mu \psi_{x+\mu} - \bar{\psi}_{x+\mu} \gamma_\mu \psi_x - r[(\bar{\psi}_x(\phi^\dagger_{x+\mu} P_L + \phi_x P_R) \psi_{x+\mu} + \text{H.c.}) - 2\bar{\psi}_x(\phi^\dagger_x P_L + \phi_x P_R) \psi_x]), \qquad (15)$$

where now $\Box$ is the standard lattice Laplacian [cf. Eq. (3) with $U_{x,\mu} = I$],

$$V^r_{x,\mu} = \frac{1}{2i}(\phi^\dagger_x \phi_{x+\mu} - \phi^\dagger_{x+\mu} \phi_x), \qquad (16)$$

and we abbreviated

$$\tilde{\kappa} \equiv \frac{1}{2\xi g^2}. \qquad (17)$$

S_{reduced} is invariant under the transformation Eq. (13) for constant $h_{Lx}=h_L$, as well as under the transformation

$$\phi \to \phi h_R^\dagger,$$

$$\psi \to (P_L + h_R P_R)\psi, \tag{18}$$

with $h_R \in G$, i.e., S_{reduced} has a global $G_L \times G_R$ symmetry. Weak-coupling perturbation theory in g corresponds to perturbation theory in $1/\tilde{\kappa}$. Note that in the original action in the vector picture, the gauge-fixing term corresponds to a kinetic term for the longitudinal part of the gauge field $U_{x,\mu}$. Therefore S_V is manifestly renormalizable, and can be treated systematically in perturbation theory in g, even though it is not gauge invariant [3,4]. In the reduced model, we expand

$$\phi_x = \exp(i\theta_x/\sqrt{2\tilde{\kappa}}) = \exp(ig\sqrt{\xi}\theta_x), \tag{19}$$

in order to develop perturbation theory. This leads to tree-level scalar and fermion propagators $\langle \theta(p)\theta(q) \rangle = \delta(p+q)G(p)$ and $\langle \psi(p)\bar{\psi}(q) \rangle = \delta(p+q)S(p)$ with

$$G(p) = \frac{1}{\hat{p}^2(\hat{p}^2+m^2)}, \quad m^2 \equiv \frac{\kappa}{\tilde{\kappa}},$$

$$S(p) = (i\slashed{s}(p) + rM(p))^{-1}$$

$$= (-i\slashed{s}(p) + rM(p))/D(p),$$

$$D(p) = s^2(p) + r^2 M^2(p), \tag{20}$$

where $\hat{p}_\mu = 2\sin(p_\mu/2)$, $\slashed{s}(p) = \Sigma_\mu \gamma_\mu \sin p_\mu$, $s^2(p) = \Sigma_\mu \sin^2 p_\mu$ and $M(p) = \frac{1}{2}\hat{p}^2$. The vertices can also be read off from S_{reduced} after expanding ϕ in terms of θ. A vertex with n θ lines has a coupling constant of order $\tilde{\kappa}^{-(n-2)/2}$, while a vertex involving the fermions and n θ lines has a coupling of order $\tilde{\kappa}^{-n/2}$.

III. THE FM-FMD TRANSITION AND THE CONTINUUM LIMIT

In this section, we will discuss in detail the properties of the phase transition that occurs for a critical value κ_c of the parameter κ. We will assume that $\tilde{\kappa}$ is large and positive (for details on the complete phase diagram, see Refs. [8,4,2]). We will limit ourselves to the case without fermions, and postpone their inclusion to a later section. We will also simplify the discussion by restricting ourselves to the Abelian case, $G = U(1)$.

An indication that a continuous phase transition occurs can be obtained from the θ propagator [Eq. (20)]: if $\kappa < 0$, m^2 becomes negative, signaling an instability at $\kappa_c = 0$ against the condensation of plane waves with nonzero momentum, which breaks lattice space-time symmetries. (This value for κ_c is just its tree-level value; its true value will be shifted by quantum corrections.) We first observe that, for large $\tilde{\kappa}$, ϕ acquires an expectation value, and in fact $|\langle \phi_x \rangle| \to 1$ for $\tilde{\kappa} \to \infty$ (as long as we stay away from the phase transition line, see below). This breaks the global $G_L \times G_R$ symmetry down to the diagonal symmetry $h_L = h_R$ [cf. Eqs.

(13),(18)]. In order to analyze the situation for small $|\kappa|$, we substitute $\phi_x = \exp(iqx)$ into the bosonic part of S_{reduced}, which gives us a potential density $V(q)$:

$$V(q) = \tilde{\kappa}\left[4\left(\sum_\mu (1-\cos q_\mu)\right)^2 - \left(\sum_\mu \sin^2 q_\mu\right)^2 \right.$$

$$\left. + 2m^2\sum_\mu (1-\cos q_\mu)\right]. \tag{21}$$

It is easy to see that for $m^2 > 0$, $V(q) \geq 0$ and that $V(q) = 0 \Leftrightarrow q = 0$. But for m^2 negative, the absolute minimum of $V(q)$ occurs at a nonzero value of q; for m^2 small and negative it occurs at [2]

$$q_\mu = \pm\left(\frac{|m^2|}{6}\right)^{1/4}, \quad \text{all } \mu. \tag{22}$$

Hence, for large values of $\tilde{\kappa}$, a continuous phase transition takes place from a phase with broken symmetry and $q = 0$, which we will call the FM (ferromagnetic) phase, to a phase with broken symmetry and $q \neq 0$, which we will call the FMD (directional ferromagnetic) phase. In the full model, this condensation of q corresponds to the condensation of the vector field A_μ, and m^2 corresponds to the gauge field mass [2]. The critical point $\kappa = \kappa_c$ (=0 at tree level), $\tilde{\kappa} \to \infty$ or $g \to 0$ should therefore correspond to the desired continuum limit, with the desired chiral fermions and massless gluons, in perturbation theory [4].

The discussion of the order parameter $\langle q_\mu \rangle$, however, does not complete our discussion of the phase transition at $\kappa = \kappa_c$. Let us consider the expectation value $v = \langle \phi_x \rangle$ for $\kappa > 0$, where the tree-level scalar propagator is given by the expression in Eq. (20). To leading order in $1/\tilde{\kappa}$, we obtain

$$\langle \phi_x \rangle = 1 - \frac{1}{4\tilde{\kappa}}\langle \theta_x^2 \rangle + \cdots$$

$$= 1 - \frac{1}{4\tilde{\kappa}}\int_p \frac{1}{\hat{p}^2(\hat{p}^2+m^2)} + \cdots, \tag{23}$$

where $\int_p = \int d^4p/(2\pi)^4$ is the integral over the Brillouin zone. For $m^2 \to 0$ this is infrared divergent, and we need to resum the series in order to obtain a finite answer:

$$\langle \phi_x \rangle = \exp\left(-\frac{1}{4\tilde{\kappa}}\int_p G(p)\right)\left(1 + O\left(\frac{1}{\tilde{\kappa}^2}\right)\right)$$

$$\sim (m^2)^\eta\left(1 + O\left(\frac{1}{\tilde{\kappa}^2}\right)\right), \tag{24}$$

with

$$\eta \equiv \frac{1}{64\pi^2\tilde{\kappa}} + O\left(\frac{1}{\tilde{\kappa}^2}\right). \tag{25}$$

The $O(1/\tilde{\kappa}^2)$ corrections come from θ self-interactions, which we will discuss below. We see that for $\kappa \searrow \kappa_c$, v goes

to zero with a $\tilde{\kappa}$-dependent critical exponent η. This situation is very reminiscent of that with massless scalars in two dimensions, cf. the Coleman/Mermin-Wagner theorem [18]. It is simply a consequence of the fact that the scalar propagator goes like $1/(p^2)^2$ for $m^2 = 0$.

Equation (24) has a very important consequence: for $m^2 \to 0$ (i.e., $\kappa \to \kappa_c$), $\langle \phi_x \rangle$ goes to zero, and the full $U(1)_L \times U(1)_R$ symmetry [cf. Eqs. (13),(18)] is restored at $\kappa = \kappa_c$. This implies that the $U(1)_L$ [and $U(1)_R$] charges of massless fermions are well defined at the critical point.

Interactions can be taken into account systematically in perturbation theory. To order $1/\tilde{\kappa}^2$, Eq. (24) is replaced by

$$\langle \phi_x \rangle = \exp\left(-\frac{1}{4\tilde{\kappa}} \int_p G^{1-loop}(p) \right) \sim (\kappa - \kappa_c^{1-loop})^{\eta}, \tag{26}$$

where G^{1-loop} differs from G by finite wave function and mass renormalizations. Also the critical value of κ is shifted from its (vanishing) tree-level value to [8]

$$\kappa_c^{1-loop} = 0.02993(1). \tag{27}$$

The fact that the renormalizations are finite originates in the fact that the interactions are irrelevant (in the Abelian case), and therefore do not change the long-distance behavior of correlation functions. See Ref. [8] for a much more detailed analysis of the order parameter $\langle \phi_x \rangle$ in both the FM and FMD phases, where it is shown that perturbation theory agrees very well with numerical results.

IV. FERMION SPECTRUM IN THE REDUCED MODEL

In this section we will present one of the key results of this paper: the fermion self-energy to one loop in the reduced model. But let us first discuss what we would expect, if the reduced model is to pass the test outlined in Sec. II. The fermion action in Eq. (15) is formulated in terms of a charged left-handed field $\psi_L^c = P_L \psi$ [i.e., it transforms under the symmetry Eq. (13)], and a neutral right-handed field $\psi_R^n = P_R \psi$ [which does not transform under Eq. (13)]. In the continuum limit, the neutral right-handed fermion is free because of the shift symmetry Eq. (8) [16]. Moreover, at least naively, the charged left-handed fermion is also free in the continuum limit, because the interaction terms in Eq. (15) with the field θ are irrelevant [in the usual technical sense, i.e., dimension greater than four; θ has mass dimension zero, cf. Eq. (20)], as can be seen by inserting and expanding Eq. (19). However, this argument does not take into account the nonstandard infrared behavior of the scalar field θ, and might therefore be misleading. We will therefore study the fermion propagator at one loop in perturbation theory, and see that, to this order, the argument just given is nevertheless correct.
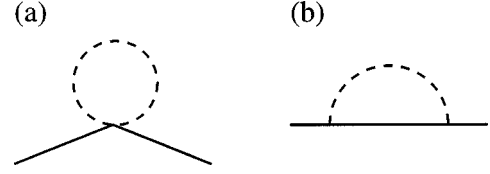

FIG. 1. One-loop fermion self-energy.

For a quicker, but more heuristic argument leading to the same result, see Ref. [10].

In order to perform actual perturbation theory calculations, it is advantageous to reformulate the reduced action, Eq. (15) by a field redefinition of the fermion variables. By redefining $\psi_R^n = \phi^\dagger \psi_R^c$ or $\psi_L^c = \phi \psi_L^n$, we can write the action in terms of respectively charged or neutral fermion fields only. This has the advantage of improving the infrared behavior of loop corrections. Here we will choose the charged option. To order $1/\tilde{\kappa}$, for $G = U(1)$, the reduced action becomes

$$\begin{aligned}
S_{reduced}^{fermion} = \frac{1}{2} \sum_{x,\mu} & \left[\bar{\psi}_x^c \gamma_\mu \psi_{x+\mu}^c - \bar{\psi}_{x+\mu}^c \gamma_\mu \psi_x^c - r \bar{\psi}_x^c (\Box \psi^c)_x \right. \\
& + \frac{i}{\sqrt{2\tilde{\kappa}}} (\partial_\mu^+ \theta)_x (\bar{\psi}_x^c \gamma_\mu P_R \psi_{x+\mu}^c + \bar{\psi}_{x+\mu}^c \gamma_\mu P_R \psi_x^c) \\
& - \frac{1}{4\tilde{\kappa}} (\partial_\mu^+ \theta)_x^2 (\bar{\psi}_x^c \gamma_\mu P_R \psi_{x+\mu}^c - \bar{\psi}_{x+\mu}^c \gamma_\mu P_R \psi_x^c) \\
& - r \left(\frac{i}{\sqrt{2\tilde{\kappa}}} (\partial_\mu^+ \theta)_x (\bar{\psi}_x^c \psi_{x+\mu}^c - \bar{\psi}_{x+\mu}^c \psi_x^c) \right. \\
& \left. \left. - \frac{1}{4\tilde{\kappa}} (\partial_\mu^+ \theta)_x^2 (\bar{\psi}_x^c \psi_{x+\mu}^c + \bar{\psi}_{x+\mu}^c \psi_x^c) \right) \right], \tag{28}
\end{aligned}$$

where ∂_μ^+ is the forward derivative: $(\partial_\mu^+ f)_x = f_{x+\mu} - f_x$. If we would have chosen to use the neutral formulation, the action would have been similar, but for a parity transformation $P_L \leftrightarrow P_R$, $\theta \to -\theta$, and the omission of scalar-fermion couplings proportional to r. Note that, in both formulations, the θ field always appears with derivatives, improving the infrared behavior of perturbation theory in the limit $m^2 \to 0$. (In the non-Abelian case, there would have been extra scalar-fermion couplings involving the commutator $[\theta_x, \theta_{x+\mu}]$. We believe that in this case the infrared finiteness in the limit $m^2 \to 0$ of observables invariant under the symmetries of the model can be proven adapting the methods of Ref. [19].) The calculation of the charged fermion one-loop self-energy proceeds in a straightforward manner. There are two contributions, depicted in Fig. 1. The tadpole diagram of Fig. 1(a) gives a contribution

$$\Sigma^{(a)}(p) = \frac{1}{8\tilde{\kappa}} \sum_\mu (-i\gamma_\mu \sin p_\mu P_R + r \cos p_\mu) \int_k \sum_\nu (1 - \cos k_\nu) G(k), \tag{29}$$

while the diagram of Fig. 1(b) leads to a more complicated contribution

$$\Sigma^{(b)}(p) = \frac{1}{8\widetilde{\kappa}} \sum_{\mu\nu} e^{-ip_\mu + ip_\nu} \int_k G(k)(e^{-ik_\mu}-1)(e^{ik_\nu}-1)\{-\gamma_\mu i\slashed{s}(k+p)\gamma_\nu P_R D^{-1}(k+p)(e^{i(k+2p)_\mu}+1)(e^{-i(k+2p)_\nu}+1)$$

$$-rS(k+p)\gamma_\nu P_R(e^{i(k+2p)_\mu}-1)(e^{-i(k+2p)_\nu}+1) - r\gamma_\mu[i\slashed{s}(k+p)P_L - rM(k+p)P_R]D^{-1}(k+p)(e^{i(k+2p)_\mu}+1)$$

$$\times(e^{-i(k+2p)_\nu}-1) - r^2 S(k+p)(e^{i(k+2p)_\mu}-1)(e^{-i(k+2p)_\nu}-1)\}. \tag{30}$$

The total one-loop self-energy is given by $\Sigma(p) = \Sigma^{(a)}(p) + \Sigma^{(b)}(p)$.

First, substituting $p=0$, we find $\Sigma(0)=0$, which tells us that no mass counterterm is needed in order to keep the fermion massless. In the neutral formulation, this is a direct consequence of shift symmetry [16], and what we find here in the charged formulation is consistent with that.

Next, we are interested in the nonanalytic behavior of the self-energy in the continuum limit. To start, let us see what happens to the doublers, i.e., for momenta $p = \pi_A + \widetilde{p}$, where we take $\widetilde{p}$ small and

$$\pi_A \in \{(\pi,0,0,0),\dots,(\pi,\pi,\pi,\pi)\}. \tag{31}$$

The only pole in the fermion propagator in $\Sigma^{(b)}$ occurs for $k = \pi_A + \widetilde{k}$ with $\widetilde{k}$ small, but in that region $G(k)$ is of order one, and therefore these regions do not lead to any nonanalytic terms in $\widetilde{p}$ in the continuum limit. For small k of course $G^{-1}(k) \approx k^2(k^2 + m^2)$, but now $S(k+p)$ is of order one (thanks to the Wilson term), and again there are no nonanalytic terms coming from this region. (Note that the derivative couplings of θ play an important role here.) We conclude that, for these momenta, $\Sigma(p)$ constitutes a small regular correction of order $1/\widetilde{\kappa}$, and that therefore the doublers are still removed by the tree-level Wilson term.

For p small (i.e., $\pi_A = 0$) all nonanalytic behavior comes from the region around $k = 0$. We obtain the nonanalytic terms by cutting out a small region with radius δ around $k = 0$, with $k \ll \delta \ll 1$, so that we can replace the integrand inside this region by its covariant (continuum) expression [5]. (Any explicit δ dependence coming from the region $k < \delta$ must cancel against the explicit δ dependence coming from the region $k > \delta$, leaving the complete result independent of the arbitrary parameter δ.) Power counting tells us that no contribution comes from any of the terms proportional to a power of r, and we find, in the continuum limit,

$$\Sigma_{\text{nonan}}(p) = \frac{-i}{2\widetilde{\kappa}} \int_{|k|<\delta} G(k)\slashed{k}(\slashed{k}+\slashed{p})\slashed{k}P_R(k+p)^{-2}$$

$$= \frac{-i\slashed{p}P_R}{32\widetilde{\kappa}\pi^2}\left\{\log\frac{p^2}{\delta^2} + \frac{1}{2}\left[\left(\frac{p^2}{m^2} + \frac{m^2}{p^2} + 2\right)\log\left(1 + \frac{m^2}{p^2}\right) - \frac{m^2}{p^2}\log\frac{m^2}{p^2} - 1\right]\right\}$$

$$\to \frac{-i\slashed{p}P_R}{32\widetilde{\kappa}\pi^2}\log\frac{p^2}{\delta^2}, \quad m^2 \to 0, \tag{32}$$

for small p^2/δ^2. This result shows that nonanalytic terms occur only in the right-handed kinetic part of the charged fermion propagator. The left-handed kinetic term receives only a finite renormalization coming from contact terms in the fermion self-energy. This tells us that the left-handed charged fermion is a free particle, with a simple pole in its two-point function.

A similar analysis of the neutral propagator at one loop can be performed by expressing Eq. (15) in terms of the neutral fermion field $\psi^n = \phi^\dagger \psi^c$. One finds similar nonanalytic terms only in the left-handed kinetic part of the neutral fermion propagator, telling us that in this case, the right-handed neutral fermion is free. The finite one-loop renormalization of the right-handed kinetic term actually vanishes in this case, in accordance with shift symmetry.

If indeed the neutral right-handed fermion and the charged left-handed fermion are the only free fermions that exist at the critical point $m^2 = 0$ in the reduced model, one would expect that the two-point functions of ψ_R^c and ψ_L^n correspond to two-point functions of fermion-scalar composite operators, with a cut starting at $p = 0$ (for $m^2 = 0$). In fact, in the continuum limit, we would expect to find that these correlation functions factorize:

$$\langle \psi_{Rx}^c \bar{\psi}_{Ry}^c \rangle \sim \langle \psi_{Rx}^n \bar{\psi}_{Ry}^n \rangle \langle \phi_x^\dagger \phi_y \rangle, \tag{33}$$

and similar for the neutral left-handed fermion. We will show now that the nonanalytic behavior found for the charged right-handed fermion is exactly what one would obtain from calculating the right-hand side of Eq. (33) in momentum space, expanded to order $1/\widetilde{\kappa}$. An analogous argument can be given for the neutral left-handed fermion.

The bosonic two-point function in Eq. (33) is

$$\langle \phi_x^\dagger \phi_y \rangle = \exp \frac{1}{2\widetilde{\kappa}} [G(x-y)-G(0)] \left[1 + O\left(\frac{1}{\widetilde{\kappa}^2}\right) \right]$$

$$= 1 + \frac{1}{2\widetilde{\kappa}} [G(x-y)-G(0)] + \cdots , \qquad (34)$$

where

$$G(x-y) = \int_p e^{i(x-y)} G(p). \qquad (35)$$

In order to calculate $\langle \psi_{Rx}^n \bar{\psi}_{Ry}^n \rangle$, we need to repeat the self-energy calculation, but now with S_{reduced} in the neutral fermion formulation. This calculation is analogous, but simpler than the one we outlined above, so we will not repeat it here, but just quote the results as we need them. One finds that in this case, the only nonanalytic term occurs for the left-handed fermion, i.e., the nonanalytic neutral self-energy is the parity-transformed version of Eq. (32). We have

$$\sum_{xy} e^{-ipx+iqy} \langle \psi_{Rx}^n \bar{\psi}_{Ry}^n \rangle \langle \phi_x^\dagger \phi_y \rangle = \delta(p-q) \exp[-G(0)/2\widetilde{\kappa}] \left[P_R S^n(p) P_L + \frac{1}{2\widetilde{\kappa}} \int_k P_R S^n(p-k) P_L G(k) + O\left(\frac{1}{\widetilde{\kappa}^2}\right) \right], \qquad (36)$$

where S^n is the neutral fermion propagator, and we wish to calculate the right-hand side of Eq. (36) to order $1/\widetilde{\kappa}$. The first term in square brackets does not contain any nonanalytic terms in the continuum limit because of the chiral projectors. The second term, in which we may replace $S^n(p-k)$ by $S(p-k)$ to the desired accuracy, yields the following nonanalytic terms in the continuum limit:

$$\frac{1}{2\widetilde{\kappa}} \int_k P_R S^n(p-k) P_L G(k) \rightarrow \frac{1}{32\pi^2\widetilde{\kappa}} \left(\frac{-i\not{p}}{p^2}\right) i\not{p} P_R \left\{ \log \frac{p^2}{m^2} + \frac{1}{2}\left[\left(\frac{p^2}{m^2} + \frac{m^2}{p^2} + 2\right) \log\left(1 + \frac{m^2}{p^2}\right) - \frac{m^2}{p^2} \log \frac{m^2}{p^2} - 1 \right] \right\} \left(\frac{-i\not{p}}{p^2}\right). \qquad (37)$$

If we amputate the two massless fermion propagators, this expression is not quite equal to minus the self-energy given in Eq. (32) yet. For this, we need to include the nonanalytic part coming from expanding the factor $\exp(-G(0)/2\widetilde{\kappa})$ with

$$\frac{G(0)}{2\widetilde{\kappa}} = \frac{1}{2\widetilde{\kappa}} \int_p G(p)$$

$$= -\frac{1}{32\pi^2\widetilde{\kappa}} \log \frac{m^2}{\delta^2} + \text{const}, \qquad (38)$$

where we again isolated the nonanalytic term by cutting out a spherical region with radius δ from the integration region. Combining this with the tree-level part of Eq. (36) and with Eq. (37), we recover exactly the expression Eq. (32) for the charged right-handed fermion self-energy.

The dynamics of the scalar field ϕ plays a crucial role in obtaining this state of affairs. A very similar model, the Smit-Swift model [15], has been studied in the past with hopes of enforcing the situation described above. Without gauge fields, the Smit-Swift model corresponds to Eq. (15) with $\widetilde{\kappa}=0$. For no values of κ and the Wilson-Yukawa coupling r does one obtain the desired result: if the global $G_L \times G_R$ symmetry is unbroken, neutral or charged massless fermions always come in left- and right-handed pairs (for a review see Ref. [7]). This is in accordance with a general argument about the applicability of the Nielsen-Ninomiya theorem [20] to interacting theories [21]. Here we see that addition of an extra parameter, $\widetilde{\kappa}$, which has its origin in gauge fixing, makes it possible to construct a continuum limit in which the symmetry is unbroken, and the chiral fer-

mions undoubled. For a discussion as to how this is not in contradiction with the Nielsen-Ninomiya theorem, see Ref. [11].

We will end this section with some remarks. First, the calculation of the neutral and charged fermion propagators could have been done starting directly from Eq. (15), in what we will call the "mixed formulation." For the two-point functions which are invariant under the global symmetry, $\langle \psi_{Rx}^c \bar{\psi}_{Ry}^c \rangle$, $\langle \psi_{Lx}^c \bar{\psi}_{Ly}^c \rangle$, $\langle \psi_{Rx}^n \bar{\psi}_{Ry}^n \rangle$, and $\langle \psi_{Lx}^n \bar{\psi}_{Ly}^n \rangle$, we would have found exactly the same results. (For noninvariant quantities, resummations are necessary in order to remove infrared divergences; the simplest example of this is $\langle \phi_x \rangle$ discussed in the previous section.) This holds only for the connected correlation functions, and not for "auxiliary" quantities such as the self-energy.

Second, we believe that all these arguments can be extended to higher orders in perturbation theory. This is based on the observation that the infrared structure of the reduced model is very similar to that of two-dimensional theories with massless scalars. There is a vast literature on such two-dimensional models, see, e.g., Refs. [22,19], and we expect that some of the arguments and methods can be adapted to our four-dimensional case.
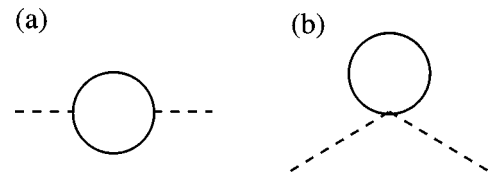

(a)   (b)

FIG. 2. One-loop θ self-energy.

Last, we note that all arguments in this section generalize to the non-Abelian case.

V. VACUUM POLARIZATION

Let us first consider the effects of the fermions on the dynamics of the scalar field, θ. Since in the continuum limit, the gauge degrees of freedom, which are represented by the field θ, are supposed to decouple (after suitable adjustment of local counterterms), we expect the lattice dynamics to conform with this expectation. In particular, we expect that no nonanalytic terms survive in the continuum limit of the θ two-point function which come from fermion loops. We will verify this explicitly at the one-loop level.

It is convenient to perform this calculation using the neutral-fermion language. Of course, one would obtain the same result using the charged-fermion form of S_{reduced} [Eq. (28)]. Expanded to order $1/\tilde{\kappa}$, the reduced action with neutral fermions is

$$S_{\text{reduced}}^{\text{fermion}} = \frac{1}{2} \sum_{x,\mu} \left(\bar{\psi}_x^n \gamma_\mu \psi_{x+\mu}^n - \bar{\psi}_{x+\mu}^n \gamma_\mu \psi_x^n - r \bar{\psi}_x^n (\Box \psi^n)_x \right.$$

$$- \frac{i}{\sqrt{2\tilde{\kappa}}} (\partial_\mu^+ \theta)_x (\bar{\psi}_x^n \gamma_\mu P_L \psi_{x+\mu}^n + \bar{\psi}_{x+\mu}^n \gamma_\mu P_R \psi_x^n)$$

$$\left. - \frac{1}{4\tilde{\kappa}} (\partial_\mu^+ \theta)_x^2 (\bar{\psi}_x^n \gamma_\mu P_L \psi_{x+\mu}^n - \bar{\psi}_{x+\mu}^n \gamma_\mu P_L \psi_x^n) \right).$$

$$(39)$$

We define the θ self-energy Σ_θ from the full θ two-point function G_{full} by

$$G_{\text{full}}^{-1}(p) = \hat{p}^2(\hat{p}^2 + m^2) + \Sigma_\theta(p). \qquad (40)$$

To one loop, the fermionic contribution to $\Sigma_\theta(p)$ is $\Sigma_\theta^{\text{fermion}}(p) = \Sigma_\theta^{(a)}(p) + \Sigma_\theta^{(b)}(p)$ (cf. Fig. 2), with

$$\Sigma_\theta^{(a)}(p) = \frac{1}{2\tilde{\kappa}} \int_k \left[\sum_{\mu\nu} 8 \sin \frac{1}{2} p_\mu \sin \frac{1}{2} p_\nu \cos\left(k_\mu - \frac{1}{2} p_\mu\right) \cos\left(k_\nu - \frac{1}{2} p_\nu\right) \right.$$

$$\times \left(\sin k_\mu \sin(k_\nu - p_\nu) + \sin k_\nu \sin(k_\mu - p_\mu) - \delta_{\mu\nu} \sum_\lambda \sin k_\lambda \sin(k_\lambda - p_\lambda) \right) D^{-1}(k) D^{-1}(k-p) \right],$$

$$\Sigma_\theta^{(b)}(p) = \frac{1}{2\tilde{\kappa}} \int_k 8 \sin^2 \frac{1}{2} p_\mu \sin^2 k_\mu D^{-1}(k), \qquad (41)$$

where D is given in Eq. (20). In order to find the continuum limit of this expression, we need to expand it to order p^4 [cf. Eq. (40)]. First, the order p^2 term is

$$\frac{1}{2\tilde{\kappa}} p^2 \int_k \left[\left(\sum_\mu \sin^2 k_\mu \cos^2 k_\mu - \frac{1}{2} \sum_\mu \sin^2 k_\mu \sum_\nu \cos^2 k_\nu \right) D^{-2}(k) + \frac{1}{2} \sum_\mu \sin^2 k_\mu D^{-1}(k) \right]$$

$$\approx 0.05464 \times \frac{1}{2\tilde{\kappa}} p^2 \quad (\text{for } r = 1), \qquad (42)$$

leading to a one-loop contribution to κ_c (cf. Sec. III)

$$\kappa_c^{1-loop} = 0.02993(1) - 0.02732(1) n_f \quad (\text{for } r = 1), \qquad (43)$$

where n_f is the number of left-handed fermions in the Abelian case.

Next, we are interested in the $O(p^4)$ term. We will not calculate the complete coefficient of this term, but restrict ourselves to inspection of the nonanalytic part. Like before, we can do this by restricting the loop-momentum integration to the region $|k| < \delta$, and replacing the integrand by its covariant form:

$$\Sigma_\theta(p) \sim \frac{1}{2\tilde{\kappa}} \sum_{\mu\nu} p_\mu p_\nu I_{\mu\nu}(p), \qquad (44)$$

where

$$I_{\mu\nu}(p) = 2 \int_{|k|<\delta} \frac{2k_\mu k_\nu - k_\mu p_\nu - k_\nu p_\mu - \delta_{\mu\nu}(k^2 - k \cdot p)}{k^2(k-p)^2}$$

$$= \frac{1}{24\pi^2} (p_\mu p_\nu - \delta_{\mu\nu} p^2) \log (p^2/\delta^2) + \text{regular terms}$$

$$(45)$$

for $p^2 \ll \delta^2$. We see that, because the nonanalytic part of $I_{\mu\nu}$ is transversal, there is indeed no nonanalytic contribution to the θ two-point function from the fermions. This again demonstrates that the reduced model leads to a theory of free chiral fermions decoupled from the gauge degrees of freedom in the continuum limit, after a suitable tuning of local counterterms (in this case the κ term).
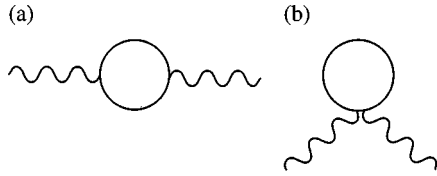
FIG. 3. One-loop contributions to the vacuum polarization.

FIG. 4. Two-loop contributions to the vacuum polarization.

It is straightforward to verify that, in the Abelian case, the same conclusion holds for the one-loop contribution from the θ self-interactions, in accordance with the fact that these self-interactions correspond to irrelevant operators.

Next, we would like to discuss the effective action for the gauge field $A_{x,\mu}$, obtained by integrating out all other degrees of freedom. An important test of our approach consists of the following. Take the external gauge field to be smooth. The effective action can now be defined in two ways: (1) We may set $\phi_x = I$ [cf. Eqs. (1),(14)], and integrate over the fermions; (2) we may integrate over both ϕ_x and the fermions [cf. Eq. (10)].

Both methods (which, in the terminology of Sec. II, correspond to respectively the ''vector'' and ''Higgs'' picture) should yield the same gauge-invariant effective action in the continuum limit, modulo local counterterms (if the fermion representation is anomaly free). The second method verifies that the integration over the (lattice) gauge orbit of the external gauge field does not change the (long-distance part of the) effective action.

We will now examine this using the example of the fermionic contribution to the Abelian vacuum polarization $\Pi_{\mu\nu}(p)$. Starting from Eq. (1) (i.e., following method 1), the vacuum polarization is just the sum of the two one-loop diagrams of Fig. 3. We find that $\Pi_{\mu\nu}(p)$ is given by the expression Eq. (44) for the θ self-energy with a factor $p_\mu p_\nu /(2\widetilde{\kappa})$ omitted. This leads to a one-loop gauge-field mass counterterm in Eq. (9) with κ given by Eq. (43). For the nonanalytic part, we find $\Pi_{\mu\nu}(p) = I_{\mu\nu}(p)$ [Eqs. (44),(45)], leading to the one-loop β function

$$\beta(g) \equiv \frac{\partial g}{\partial \log a} = -n_f \frac{g^3}{24\pi^2} \qquad (46)$$

(in the non-Abelian case this has to be multiplied by the appropriate quadratic Casimir). This is exactly the result we expect for ''chiral QED'' with n_f left-handed fermions.

Next, we wish to verify that the orbit integration does not change the nonlocal part of the vacuum polarization (cf. method 2). To one loop, this is trivial, since, as before, the vacuum polarization is just the sum of the two diagrams of Fig. 3, which do not contain any θ lines. Diagrams with internal θ lines only show up at two loops and higher, and we did not perform an explicit calculation of these diagrams. But, one can easily understand on general grounds that these higher-loop diagrams with internal θ lines do not contribute to the nonanalytic part of the vacuum polarization, and that therefore their effects can be removed by counterterms. Two- and higher-loop contributions can be conveniently calculated by rewriting the action Eq. (10) in terms of charged fermion fields only, by making the substitution $\psi = (P_L + P_R \phi^\dagger) \psi^c$ in
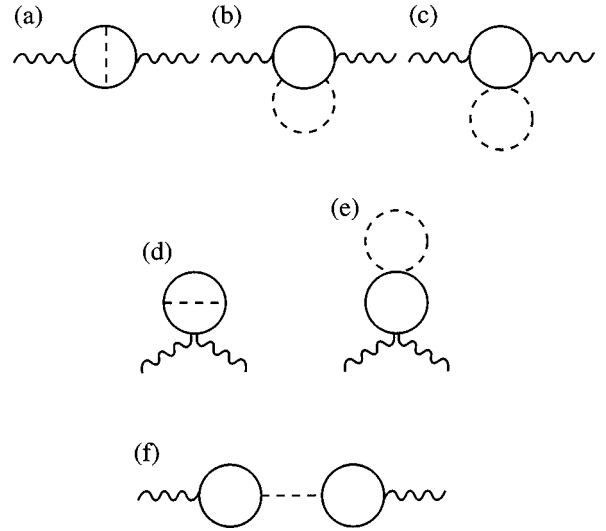
Eq. (10). As in the reduced model, this improves the infrared behavior of perturbation theory, validating standard power-counting arguments in particular. We observe that, in the charged-fermion language, gauge field-fermion vertices only occur in the left-handed kinetic term, while θ-fermion vertices occur only in the right-handed kinetic and Wilson terms. (The reduced model in the charged-fermion formulation is then obtained again by setting $U_{x,\mu} = 1$.)

The topology of the contributing two-loop diagrams is shown in Fig. 4 (where we omitted any diagrams with θ tadpoles, since this tadpole vanishes). As we just explained, fermion-θ vertices either arise from the Wilson term, or contain a factor $\gamma_\mu P_R$. If we start following the fermion loop from one of the A_μ vertices, we either encounter a vertex from the Wilson term, which corresponds to an irrelevant operator, or we encounter a vertex $\gamma_\mu P_R$. In this case, because of the left-handed projector P_L at the A_μ vertex, only the Wilson term part of the fermion propagator [cf. Eq. (20)] contributes, which again corresponds to an irrelevant operator. In both cases, we therefore do expect these diagrams to yield only contact terms in the continuum limit.

This analysis demonstrates explicitly how the ''rough gauge field problem'' is resolved within the gauge-fixing approach [4] also for nontrivial orbits. The resolution is a direct consequence of the fact that the full theory, including the gauge degrees of freedom, can be systematically investigated in perturbation theory.

VI. THE FERMION-NUMBER CURRENT

The fermion action, Eq. (7), is invariant under simple $U(1)$ phase rotations of the fermion field

$$\psi \to e^{i\alpha}\psi, \quad \bar{\psi} \to \bar{\psi}e^{-i\alpha}. \qquad (47)$$

This exact symmetry appears to be problematic, since it seems to imply that we can define a continuum limit containing only left-handed fermions (the right-handed fermions decouple in the continuum limit) with a conserved $U(1)$ quan-

tum number [23]. This would be in contradiction with the fact that this $U(1)$ quantum number should be anomalous, leading to fermion number violating processes through instantons [24]. Here, we analyze this question in perturbation theory, leaving a discussion of nonperturbative issues to future work. In this section, we will work in the vector picture, cf. Eq. (1).

The conserved current corresponding to the symmetry Eq. (47) is

$$J_{x,\mu}=J^L_{x,\mu}+J^R_{x,\mu}+J^W_{x,\mu},$$

$$J^L_{x,\mu}=\tfrac{1}{2}(\bar\psi_x\gamma_\mu P_L U_{x,\mu}\psi_{x+\mu}$$

$$+\bar\psi_{x+\mu}\gamma_\mu P_L U^\dagger_{x,\mu}\psi_x),$$

$$J^R_{x,\mu}=\tfrac{1}{2}(\bar\psi_x\gamma_\mu P_R\psi_{x+\mu}+\bar\psi_{x+\mu}\gamma_\mu P_R\psi_x),$$

$$J^W_{x,\mu}=-\frac{r}{2}(\bar\psi_x\psi_{x+\mu}-\bar\psi_{x+\mu}\psi_x).$$

$$(48)$$

On the lattice, we have

$$\sum_\mu (J_{x,\mu}-J_{x-\mu,\mu})=0. \qquad (49)$$

However, $J_{x,\mu}$ is not gauge invariant, and therefore will not correspond to the appropriate physical current in the continuum limit. Let us consider this in some detail by calculating the expectation value of the current, $\langle J_{x,\mu}\rangle_A$ to quadratic
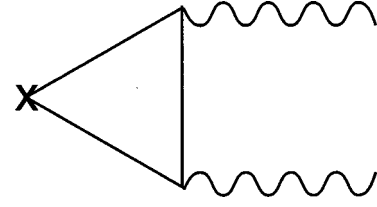

FIG. 5. Triangle diagram [the cross denotes the conserved current of Eq. (48)].

order in the gauge fields $A_{x,\mu}$, in the continuum limit. ($\langle\cdots\rangle_A$ denotes the functional average over ψ and $\bar\psi$ only.) We choose the fermions to be in the fundamental representation of the gauge group G, and we write

$$A_{x,\mu}=A^a_{x,\mu}T_a, \quad A^a_{x,\mu}=\int_p e^{ipx}A^a_\mu(p), \qquad (50)$$

with T_a the Hermitian generators of the group G, normalized by

$$\mathrm{tr}\, T_a T_b=\tfrac{1}{2}\delta_{ab}. \qquad (51)$$

The only diagram that contributes to order A^2 is shown in Fig. 5. (All other ''lattice artifact'' diagrams vanish, as already observed in Ref. [5].) The parity-even part vanishes, and we find the following result, to leading order in the gauge-field momenta k and l:

$$\langle J^L_{x,\mu}\rangle_A=i\int_{kl} e^{i(k+l)x}[I_{\mu\rho\sigma}(k,l)+I_L\epsilon_{\mu\nu\rho\sigma}(k-l)_\nu]A^a_\rho(k)A^a_\sigma(l),$$

$$\langle J^R_{x,\mu}\rangle_A=i\int_{kl} e^{i(k+l)x}I_R\epsilon_{\mu\nu\rho\sigma}(k-l)_\nu A^a_\rho(k)A^a_\sigma(l),$$

$$\langle J^W_{x,\mu}\rangle_A=i\int_{kl} e^{i(k+l)x}I_W\epsilon_{\mu\nu\rho\sigma}(k-l)_\nu A^a_\rho(k)A^a_\sigma(l), \qquad (52)$$

with summation implied over repeated indices. The function $I_{\mu\rho\sigma}(k,l)$ is given by

$$I_{\mu\rho\sigma}(k,l)=2\epsilon_{\alpha\beta\mu\sigma}k_\alpha l_\beta[k_\rho(I_{20}-I_{10})-l_\rho I_{11}]+2\epsilon_{\alpha\beta\mu\rho}k_\alpha l_\beta[k_\sigma I_{11}-l_\sigma(I_{02}-I_{01})]$$

$$+\tfrac{1}{2}\epsilon_{\alpha\mu\rho\sigma}\{k_\alpha[k^2 I_{20}-2k\cdot l I_{11}+l^2(I_{02}-2I_{01})]-l_\alpha[l^2 I_{02}-2k\cdot l I_{11}+k^2(I_{20}-2I_{10})]\}, \qquad (53)$$

where

$$I_{st}\equiv I_{st}(k,l)=\frac{1}{16\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{x^s y^t}{x(1-x)k^2+y(1-y)l^2+2xyk\cdot l}. \qquad (54)$$

The constants I_L, I_R, and I_W are given by

$$I_L=r^2\int_p(\tfrac{1}{4}c_1 c_2 c_3 c_4 s^2(p)M^2(p)-c_1 c_2 c_3 s^2_4 s^2(p)M(p))D^{-4}(p),$$

$$I_R = r^2 \int_p \left(\tfrac{1}{2} c_1 c_2 c_3 c_4 M^2(p) D^{-3}(p) - \tfrac{1}{4}[c_1 c_2 c_3 c_4 s^2(p) + 4r^2 c_1 c_2 c_3 s_4^2 M(p)] M^2(p) D^{-4}(p) \right),$$

$$I_W = -r^2 \int_p c_1 c_2 c_3 s_4^2 M(p) D^{-3}(p), \tag{55}$$

in which

$$s_\mu \equiv \sin p_\mu, \quad c_\mu \equiv \cos p_\mu, \quad s^2(p) \equiv \sum_\mu \sin^2 p_\mu. \tag{56}$$

Lattice loop integrals were calculated again by splitting the integration region into a small region with radius δ around $p=0$ ("inner region"), and the rest ("outer region"), taking the double limit $a \to 0$, followed by $\delta \to 0$. [The split into inner and outer regions depends on the routing of the external momenta through the loop: we chose the momentum of the fermion propagator connecting the two gauge-field vertices to be $p - \tfrac{1}{2}(k-l)$. Of course, the sum of inner and outer region contributions does not depend on this.] For $\langle J_{x,\mu}^R \rangle_A$ and $\langle J_{x,\mu}^W \rangle_A$ the inner-region integrals vanish, while the integral $I_{\mu\rho\sigma}(k,l)$ represents the (nonlocal) inner-region contribution for $\langle J_{x,\mu}^L \rangle_A$. The integrals $I_{L,R,W}$ represent outer-region contributions. In other words, only $\langle J_{x,\mu}^L \rangle_A$ is nonlocal, as one expects, since the right-handed fermions are free in the continuum limit. Using [5]

$$I_L + I_R + I_W = -\frac{1}{64\pi^2} \tag{57}$$

(for any nonzero value of r!), and

$$(k+l)_\mu I_{\mu\rho\sigma}(k,l) = \frac{1}{64\pi^2} \epsilon_{\mu\nu\rho\sigma}(k+l)_\mu(k-l)_\nu,$$

we find that indeed $\partial_\mu \langle J_{x,\mu} \rangle_A$ vanishes.

From Eqs. (52), (53), (57) one can show that

$$k_\rho \frac{\delta \langle J_{x,\mu} \rangle_A}{\delta A_\rho^a(k)} = \frac{i}{16\pi^2} \int_l e^{i(k+l)x} \epsilon_{\mu\nu\rho\sigma} k_\rho l_\nu A_\sigma^a(l). \tag{58}$$

[In deriving this result, we used the relation $k^2(I_{10}(k,l) - 2I_{20}(k,l)) = l^2(I_{01}(k,l) - 2I_{02}(k,l))$.] This proves that, as expected, the current $J_{x,\mu}$ is indeed not gauge invariant, as was pointed out in this context in Ref. [9]. A gauge-invariant vector current can be defined by adding an irrelevant term $J_{x,\mu}^{\text{irr}}$ to $J_{x,\mu}$, with an expectation value that goes to K_μ in the continuum limit, where [9]

$$\langle J_{x,\mu}^{\text{irr}} \rangle_A \to K_{x,\mu} = \frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr}(A_{x,\nu} F_{x,\rho\sigma} - \tfrac{1}{3} A_{x,\nu} A_{x,\rho} A_{x,\sigma}). \tag{59}$$

For example, we may take

$$J_{x,\mu}^{\text{irr}} = \frac{1}{32\pi^2 I_W} J_{x,\mu}^W. \tag{60}$$

The current $J_{x,\mu} + J_{x,\mu}^{\text{irr}}$ yields the correct, gauge-invariant, fermion-number current in the continuum limit to order A^2. Its divergence is

$$\partial_\mu \langle J_{x,\mu} + J_{x,\mu}^{\text{irr}} \rangle_A \to \partial_\mu K_{x,\mu} = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr}(F_{x,\mu\nu} F_{x,\rho\sigma}). \tag{61}$$

(An additional irrelevant operator of order A^3 would likely be needed in order to construct a gauge-invariant current to order A^3 in the non-Abelian case.) Note that the vector current that leads to gauge-invariant correlation functions in the continuum limit, is *not* what one might naively guess: $J_{x,\mu}^L + J_{x,\mu}^R$. The reason is that, although this operator is invariant under gauge transformations, the Feynman rules of the theory are not.

VII. CONCLUSION AND DISCUSSION

In this paper, we studied a proposal for the construction of lattice chiral gauge theories in (one-loop) weak-coupling perturbation theory. We considered mostly the Abelian case, and demonstrated that, in perturbation theory, the model defined in Sec. II has a continuum limit with the desired chiral fermions, in which the gauge degrees of freedom decouple, and with the correct one-loop β function for the gauge coupling. Note that, in the reduced model, the field θ, which represents the gauge degrees of freedom in the full model, decouples from the fermions for any fermion content. This is consistent with the fact that the anomaly vanishes for a purely longitudinal gauge field. Together with the nonperturbative results presented in Refs. [1,8], this makes us confident that the gauge-fixing approach can indeed be used to define Abelian chiral gauge theories on the lattice. Of course, when the full dynamics of the gauge field is taken into account, the fermion representation has to be anomaly free. A next step (in the Abelian case) would be to investigate the potential between two static charges. In principle, the full counterterm action S_{ct} will have to be calculated, and it would be interesting to see to what precision the counterterms have to be adjusted in order to obtain the Coulomb potential.

Our results should also apply to other lattice fermion formulations, such as staggered fermions, domain wall fermions, or Weyl fermions with Majorana mass and Wilson terms. (The latter were discussed in Refs. [4,25]. We verified explicitly that at one loop in the reduced model, the bare

Majorana mass can be tuned such that a free Weyl fermion emerges in the continuum limit. Since in this case there is no shift symmetry, the critical value of the bare fermion mass does not vanish.)

We expect that all perturbative results presented in this paper generalize to the non-Abelian case, with suitable modification. For instance, the long distance behavior of the gauge degrees of freedom (without fermions) should be described by the continuum higher-derivative sigma model of Ref. [26], and we expect that it will. The analysis of the fermion self-energy of Sec. IV carries over without change, and therefore we expect the same conclusions about the fermion content as in the Abelian case. The main reason that we have not considered the non-Abelian case in more detail here is that, in our view, nontrivial nonperturbative issues will have to be addressed first. The approach to lattice chiral gauge theories investigated here is inherently based on gauge fixing. This raises the issue of Gribov copies, which should be resolved before the proposal is ''complete'' for non-Abelian gauge theories. A related observation is that the BRST approach to non-Abelian gauge theories has not been defined outside perturbation theory. Until this issue is better understood, it is relatively less important to study the non-

Abelian case in perturbation theory in much detail. We note here that the fact that our lattice gauge-fixing action S_{gf} has a unique global minimum at $U_{x,\mu}=1$, while suppressing rough ''lattice'' Gribov copies, does not tell us anything about long-distance, continuum Gribov copies.

Finally, we addressed the issue of fermion-number non-conservation, but only in perturbation theory. Work on the nonperturbative aspects of this issue is in progress, and we expect to report on it in a future publication. Here we just quote Ref. [27], in which it was shown that the existence of a gauge-noninvariant conserved charge on the lattice does not imply that fermion number is conserved.

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