Chiral quark model in a Tamm-Dancoff inspired approximation

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A procedure inspired by the Tamm-Dancoff method is applied to the chiral quark model. It is illustrated here by the chiral σ model embedded in the chiral bag environment. This simple model is used as an example and as the test for the Tamm-Dancoff inspired approximation (TDIA). Here the TDIA is employed in two ways: (i) a set of field operator equations of motion is solved between quark states and (ii) the Hamilton field operator is averaged between suitable hadron states, and the equations of motion are derived for these mean fields. The second approach is analogous to the usual one which employs hedgehog quarks, which is also reproduced here. It turns out that the energy minimum (i.e., hadron masses) can be found with the TDIA also. Model predictions for the axial-vector coupling constant and for the nucleon magnetic moment obtained in the TDIA are comparable to those obtained using the usual hedgehog-based approximation. The TDIA can be used in more complex models too. [S0556-2821(98)04913-3]

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I. INTRODUCTION

The Tamm-Dancoff method (TDM) [1,2] has been applied to strong interaction problems and was intensively investigated during the 1950's [3]. Recently it has been revived [4] in the form of the light-front Tamm-Dancoff field theory (LFTD) where problems which occur in equal-time field theory [3] are either averted or redefined.

For our purposes it is important to recall that the TDM is a much better approximation than perturbation theory. In the case of the electromagnetic interaction it leads to the Coulomb potential used in the Schrödinger equation [3]. One would not get a good approximation of a bound state problem at all by using perturbation theory on free particle states [5].

Chiral bag models [6,7] are a simple effective theory of quark bound states, hadrons. Thus it is not unreasonable to hope that the TDM might be useful in that case. We intend to use the chiral bag model in the study of the electroweak transitions. For that one needs either currents or products of currents which contain strong corrections. Thus it might be more convenient to work in the Heisenberg picture.

In usual applications of the TDM [1–4,8–11] the state vector of the system under consideration is expanded in terms of the eigenfunctions of the number operators of the free field. The field operators retain the free-field form containing only one creation (annihilation) operator for the particle (antiparticle) [9]. In the Tamm-Dancoff inspired approximation (TDIA) which is used here, one does just the opposite. Working in the Heisenberg picture [12] we expand field operators in the free field creation or annihilation operators. The state vectors of the system are given in terms of the free particle operators. The TDIA looks similar to a reversed picture (or photographic negative) of TDM. A brief comparison, using a nonrelativistic Yukawa model [3] is sketched in the Appendix. Here the TDIA will be developed and tested for a simple chiral σ model [6,7] leading to results which are quite close to the ones obtained by using the hedgehog *Ansatz* [6,13–16].

The chiral σ model has been used as a transparent easily treatable example. The TDIA is equally suitable for models containing nonlinear interaction with meson fields [17] including dielectric binding of the quarks [18].

A rudimentary form of the TDIA has already appeared in our earlier papers [19]. In the next section this TDIA will be presented systematically so that it can be easily adopted to related models [17].

Comparison with the well-established hedgehog *Ansatz* [6,13–16], Sec. IV, is very encouraging. The energy minimum (i.e., hadron masses) can be found in the TDIA. The TDIA predictions for the axial-vector coupling constant and for the nucleon magnetic moment are comparable with the hedgehog-based results. Moreover, in the TDIA isospin and spin are separately conserved. The analogous situation holds for the theories with SU(3) flavors.

II. TAMM-DANCOFF INSPIRED APPROXIMATION

Working in the Heisenberg picture [12], we expand field operators in the operators of the free fields. Probability amplitudes which weigh those operators [see Eq. (2.3) below, and the Appendix] should satisfy c-number equations which follow from the Euler-Lagrange equations of motions (2.4). In that way one ends with an infinite set of coupled differential equations instead of integral ones, which appear in the TDM [3,4]. These differential equations are closely related to the familiar chiral quark model equations.

All this will become quite transparent when illustrated in a particular case of the linear σ model and the bag formal-

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ism. The Lagrangian containing the linear σ model embedded in the bag environment has the usual form [6,16,20]

$$\mathcal{L}=\mathcal{L}_\psi\Theta+\mathcal{L}_{\rm int}\delta_S+[\mathcal{L}_{\sigma\pi}-U(\sigma,\vec{\pi})]\overline{\Theta}, \tag{2.1}$$

where

$$\begin{aligned}
\mathcal{L}_\psi&=\frac{1}{2}[\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x)-\partial_\mu\overline{\psi}(x)\gamma^\mu\psi(x)],\\
\mathcal{L}_{\rm int}&=\frac{g}{2}\,\overline{\psi}(x)[\sigma(x)+i\vec{\tau}\vec{\pi}(x)\gamma_5]\psi(x),\\
\mathcal{L}_{\sigma\pi}&=\frac{1}{2}\,\partial^\mu\sigma(x)\partial_\mu\sigma(x)+\frac{1}{2}\,\partial^\mu\vec{\pi}(x)\partial_\mu\vec{\pi}(x),\\
U(\sigma,\vec{\pi})&=\frac{\lambda^2}{4}[\sigma^2(x)+\vec{\pi}^2(x)-\nu^2]^2-f_\pi m_\pi^2\sigma(x),
\end{aligned} \tag{2.2}$$

and $f_\pi=0.093$ GeV. The Θ function signals that $\mathcal{L}_\psi$ is different from zero inside the bag ($r<R_{\rm bag}$). The surface δ function δ_S gives the surface quark-π (or σ) interaction, and $\overline{\Theta}$ ensures that the potential U and the $(\sigma,\vec{\pi})$ kinetic-energy terms exist (only) outside the bag. In the *spherical* bag Θ and $\overline{\Theta}$ become $\theta(R_{\rm bag}-r)$ and $\theta(r-R_{\rm bag})$, respectively. The self-interaction potential U contains the symmetry-breaking (SB) term $c\sigma(x)\equiv-f_\pi m_\pi^2\sigma(x)$. The values of other constants are fixed by the creation of mass terms for the $\vec{\pi}$ and σ fields, by PCAC (partial conservation of axial-vector current) and by the requirement $U^{(\rm min)}=0$. In the framework of this particular model, m_σ and m_π are not necessarily equal to the physical sigma and pion mass, but play the role of model parameters. The Lagrangian (2.1) defines an effective empirical quantum field theory which describes quark dynamics, as an approximant for the underlying, fundamental and exact QCD.

The equations of motion can be obtained from the Lagrangian (2.1) using standard variational methods. The field operators ψ, $\vec{\pi}$, and σ are then expanded in terms of the free field number operators. For the quark field one introduces

$$\psi_f^c(x)=\phi_m^f(x)b_{m,f}^c+\tilde{\phi}_m^f(x)d_{m,f}^{c\dagger}+\chi_{m_1m_2m_3}^{fgh}(x)b_{m_1,f}^c d_{m_2,g}^{e\dagger}b_{m_3,h}^e+\cdots. \tag{2.3}$$

Here c is a quark color and f is a quark flavor, whereas m is the spin projection. $b_{m,f}^c$ and $d_{m,f}^c$ are quark and antiquark annihilation operators, respectively. This infinite expansion is truncated leading to a physically motivated finite basis, which defines the Tamm-Dancoff inspired approximation.

The truncation of the ψ field (2.3) as well as the *Ansätze* for the $\vec{\pi}$ and σ fields are best discussed using the model (2.1) as an example. This model is equivalent to the following set of the Euler-Lagrange equations and boundary conditions:

$$\begin{aligned}
i\gamma_\mu\partial^\mu\psi(r)&=0 \quad (r<R_{\rm bag}),\\
\partial^\mu\partial_\mu\pi^a(r)+\lambda^2\pi^a(r)[\sigma(r)^2+\vec{\pi}(r)^2-\nu^2]&=0 \quad (r>R_{\rm bag}),\\
\partial_\mu\partial_\mu\sigma(r)+\lambda^2\sigma(r)[\sigma(r)^2+\vec{\pi}(r)^2-\nu^2]+f_\pi m_\pi^2&=0 \quad (r>R_{\rm bag}),\\
[\partial^\mu\pi^a(r)]n_\mu\delta_S-\frac{g_\pi}{2}\,\overline{\psi}(r)i\tau^a\gamma_5\psi(r)\delta_S&=0 \quad (r=R_{\rm bag}),\\
[\partial^\mu\sigma(r)]n_\mu\delta_S-\frac{g_\sigma}{2}\,\overline{\psi}(r)\psi(r)\delta_S&=0 \quad (r=R_{\rm bag}).
\end{aligned} \tag{2.4}$$

For the intended application one needs a static, time independent solution. Here the term ''solution'' is to be understood in the TDIA sense. In the field expansion one keeps just the terms which are needed to obtain a nontrivial coupled system of differential equations.

The bag formalism leads to considerable simplifications as the first equation in Eq. (2.4) is coupled to the rest only through boundary conditions. The leading approximation follows if in Eq. (2.4) one keeps just the two first terms in the expansion (2.3).

The result is then sandwiched between the initial quark or antiquark states

$$|q_{f,r}^a\rangle=b_{f,r}^{q\dagger}|0\rangle \quad \langle\overline{q}_{f,r}^a|=\langle 0|d_{f,r}^q, \tag{2.5}$$

and the vacuum, leading to terms such as

$$\langle 0|\gamma_\mu\partial^\mu\psi(x)|q_{f,m}^c\rangle=\gamma_\mu\partial^\mu\phi_m^f(x). \tag{2.6}$$

This Dirac equation for the free quark inside the bag [19] leads to the following approximate TDIA *Ansatz* for the quark field:

$$\psi_f^c(x)=\frac{N}{\sqrt{4\pi}}\begin{pmatrix} j_0 \\ i(\vec{\sigma}\hat{r})j_1\end{pmatrix}\chi_m^f b_{m,f}^c+\frac{N}{\sqrt{4\pi}}\begin{pmatrix}(\vec{\sigma}\hat{r})j_1 \\ ij_0\end{pmatrix}\chi_m^f d_{m,f}^{c\dagger}. \tag{2.7}$$

Here the quantities $j_{0,1}(r\omega/R)$ are spherical Bessel functions of the order (0,1) and X_m^f is the quark isospinor ($\tilde{\chi}^f$)-spinor (χ_m) product. The Lagrangian (2.1) leads to the decoupled equations (2.4) thus serving as a simple and yet nontrivial model which can be used to illustrate and test the TDIA. The decoupling allows us to use the Bessel functions $j_{\not\ell}(r\omega/R)$ in the *Ansatz* (2.7). If one used a more general Lagrangian, continuous in the whole space, the first and second equation in Eq. (2.4) would not decouple. One would have to use an *Ansatz* containing unknown functions $f_{\not\ell}(r)$. In a simple case (2.1) ψ, π, and σ fields are coupled through boundary conditions only.

The inspection of the boundary conditions (2.4) shows that they can be satisfied with the first two terms in the expansion (2.3) if the corresponding $\vec{\pi}$ field expansion contains terms such as

$$b_{m,f}^{c\ \dagger}b_{m',f'}^c. \tag{2.8}$$

Similar conclusion can be reached for the σ field too. The boundary conditions and the equations of motion (2.4) are compatible with the approximation (2.7) if one keeps just a few terms in the $\vec{\pi}$ and σ field expansion.

All terms in Eq. (2.4), either the bispinor ones ($\overline{\psi}\Gamma\psi$) or the meson ones ($\vec{\pi},\sigma$) must contain the same number and the same kind of the creation (annihilation) quark operators. Various coordinate-dependent pieces, which multiply quark operators have to be equal. Thus the σ-field TDIA ansatz is given by the s-wave component, in terms of chiral-quark operators, together with the symmetry-breaking term (f_π):

$$\sigma(r)=\sigma_s(r)(b_{m,f}^{c\dagger}b_{m,f}^c+d_{m,f}^{c\dagger}d_{m,f}^c)-f_\pi. \tag{2.9}$$

The pion field contains the s- and p-wave components

$$\begin{aligned}\pi^a(r)=&\pi_s(r)(b_{m,f}^{c\dagger}d_{m',f'}^{c\dagger}+d_{m,f}^c b_{m',f'}^c)[\chi_{m,f}^\dagger\tau^a\chi_{m',f'}]\\&+\pi_p(r)(b_{m,f}^{c\dagger}b_{m',f'}^c+d_{m,f}^c d_{m',f'}^{c\dagger})\\&\times[\chi_{m,f}^\dagger(\vec{\sigma}\hat{r})\tau^a\chi_{m',f'}].\end{aligned} \tag{2.10}$$

At this level of TDIA expansion only the quark operators are important. The boson operators can be introduced later on or one can assume that theory (2.1) and (2.2) contains the fermions only. Then terms such as σ^2, $\vec{\pi}^2$, etc., describe various nonlinear interactions among fermions (quarks) which have to be coupled in scalar (pseudoscalar) combinations. Such models (theories) [7] would be effective nonrenormalizable field theories. In the following the terms meson, pion, or sigma are used in that generalized sense refering to expressions such as Eqs. (2.9) and (2.10).

Although expansions (2.9) and (2.10) for bosonic quantities might look strange they appeare quite naturally. They have been encountered in the past applications of the Tamm-Dancoff procedure, as for example in the formula (4.6) of Ref. [11]. As the operators b and d have the opposite parity [21], both terms in the expansion (2.10) have the same negative parity. The TDIA conserves parity throughout, as will become apparent in the applications.

The expansions (2.7), (2.9), and (2.10) represent the leading terms in the TDIA. Further corrections would be obtained by introducing higher terms in the expansions, such as indicated in Eq. (2.3). Corresponding additional terms would appear in the expansions for all fields. This would enlarge the system of the coupled equations, as discussed below.

The boundary conditions in Eq. (2.4) are now specified using the *Ansätze* (2.7)–(2.10). The result is then sandwiched between the final and initial states

$$\langle f|=\langle 0|b_{f,t}^c,\qquad |i\rangle=b_{i,u}^c|0\rangle, \tag{2.11}$$

or

$$\langle f|=\langle 0|,\qquad |i\rangle=d_{i',u'}^{c\dagger}b_{i',u'}^{c\dagger}|0\rangle. \tag{2.12}$$

One finds

$$\begin{aligned}\frac{\partial}{\partial r}\sigma_s(r)\Big|_{r=R_{\rm bag}}&=-\frac{N^2}{4\pi}\frac{g_{\sigma/s}}{2}[j_0^2(\omega)-j_1^2(\omega)]\Big|_{r=R_{\rm bag}},\\ \frac{\partial}{\partial r}\pi_s(r)\Big|_{r=R_{\rm bag}}&=-\frac{N^2}{4\pi}\frac{g_{\pi/s}}{2}[j_0^2(\omega)+j_1^2(\omega)]\Big|_{r=R_{\rm bag}},\\ \frac{\partial}{\partial r}\pi_p(r)\Big|_{r=R_{\rm bag}}&=-\frac{N^2}{4\pi}\frac{g_{\pi/p}}{2}[j_0(\omega)j_1(\omega)]\Big|_{r=R_{\rm bag}}.\end{aligned} \tag{2.13}$$

At spatial infinity the σ and π "fields" (i.e., solitons) have to vanish:

$$\sigma_s(r)|_{r\to\infty}=0,\ \ \pi_s(r)|_{r\to\infty}=0,\ \ \pi_p(r)|_{r\to\infty}=0. \tag{2.14}$$

Varying $\mathcal{L}$ (2.1) with respect to the fermion field and its derivative and collecting the corresponding surface terms, one obtains an additional boundary condition

$$\begin{aligned}i(\vec{\gamma}\hat{r})\psi(r)|_{r=R_{\rm bag}}=&ig_\sigma\sigma(r)(\vec{\gamma}\hat{r})\psi(r)|_{r=R_{\rm bag}}-g_\pi\vec{\tau}\vec{\pi}(r)\\&\times(\vec{\gamma}\hat{r})\gamma_5\psi(r)|_{r=R_{\rm bag}}.\end{aligned} \tag{2.15}$$

This boundary condition is "sandwiched" between quark states, as done with the equations of motion. Between σ-ψ and between $\vec{\pi}$-ψ one inserts the complete set of states. Depending on the type of states, one obtains relations between the coupling constants and radial functions evaluated at $r=R_{\rm bag}$. This is a straightforward but somewhat lengthy procedure. The following are some details we use as an example. With the *Ansätze* (2.7)–(2.10), the boundary condition (2.15) takes the following form:

$$\begin{aligned}\begin{pmatrix} j_0 \\ i(\vec\sigma\hat r)j_1\end{pmatrix}\chi^f_m b^c_{m,f}+\begin{pmatrix}(\vec\sigma\hat r)j_1\\ ij_0\end{pmatrix}\chi^f_m d^{c\dagger}_{m,f}=&-g_{\pi/p}\pi_p(R)\begin{pmatrix}(\vec\sigma\hat r)j_0\\ -ij_1\end{pmatrix}\chi^f_m b^{d\dagger}_{m_1,f_1}(\vec\tau\cdot\vec\tau)b^d_{m_2,f_2}[\chi^{f_1\dagger}_{m_1}(\vec\sigma\hat r)\chi^{f_2}_{m_2}]b^c_{m,f}-g_{\pi/p}\pi_p(R)\\
&\times\begin{pmatrix}(\vec\sigma\hat r)j_0\\ -ij_1\end{pmatrix}\chi^f_m d^d_{m_1,f_1}(\vec\tau\cdot\vec\tau)d^{d\dagger}_{m_2,f_2}[\chi^{f_1\dagger}_{m_1}(\vec\sigma\hat r)\chi^{f_2}_{m_2}]b^c_{m,f}-g_{\pi/p}\pi_p(R)\\
&\times\begin{pmatrix}j_1\\ -i(\vec\sigma\hat r)j_0\end{pmatrix}\chi^f_m b^{d\dagger}_{m_1,f_1}(\vec\tau\cdot\vec\tau)b^d_{m_2,f_2}[\chi^{f_1\dagger}_{m_1}(\vec\sigma\hat r)\chi^{f_2}_{m_2}]d^{c\dagger}_{m,f}-g_{\pi/p}\pi_p(R)\\
&\times\begin{pmatrix}j_1\\ -i(\vec\sigma\hat r)j_0\end{pmatrix}\chi^f_m d^d_{m_1,f_1}(\vec\tau\cdot\vec\tau)d^{d\dagger}_{m_2,f_2}[\chi^{f_1\dagger}_{m_1}(\vec\sigma\hat r)\chi^{f_2}_{m_2}]d^{c\dagger}_{m,f}+ig_\sigma\sigma_s(R)\\
&\times\begin{pmatrix}ij_1\\ -(\vec\sigma\hat r)j_0\end{pmatrix}\chi^f_m b^{d\dagger}_{m_1,f_1}b^d_{m_2,f_2}b^c_{m,f}+ig_\sigma\sigma_s(R)\begin{pmatrix}ij_1\\ -(\vec\sigma\hat r)j_0\end{pmatrix}\chi^f_m d^{d\dagger}_{m_1,f_1}d^d_{m_2,f_2}b^c_{m,f}-ig_\sigma f_\pi\\
&\times\begin{pmatrix}ij_1\\ -(\vec\sigma\hat r)j_0\end{pmatrix}\chi^f_m b^c_{m,f}+ig_\sigma\sigma_s(R)\begin{pmatrix}i(\vec\sigma\hat r)j_0\\ -j_1\end{pmatrix}\chi^f_m b^{d\dagger}_{m_1,f_1}b^d_{m_2,f_2}d^{c\dagger}_{m,f}+ig_\sigma\sigma_s(R)\\
&\times\begin{pmatrix}i(\vec\sigma\hat r)j_0\\ -j_1\end{pmatrix}\chi^f_m d^{d\dagger}_{m_1,f_1}d^d_{m_2,f_2}d^{c\dagger}_{m,f}-ig_\sigma f_\pi\begin{pmatrix}i(\vec\sigma\hat r)j_0\\ -j_1\end{pmatrix}\chi^f_m d^{c\dagger}_{m,f}-g_{\pi/s}\pi_s(R)\\
&\times\begin{pmatrix}(\vec\sigma\hat r)j_0\\ -ij_1\end{pmatrix}\chi^f_m b^{d\dagger}_{m_1,f_1}(\vec\tau\cdot\vec\tau)d^{d\dagger}_{m_2,f_2}b^\dagger_{m,f}-g_{\pi/s}\pi_s(R)\\
&\times\begin{pmatrix}(\vec\sigma\hat r)j_0\\ -ij_1\end{pmatrix}\chi^f_m d^d_{m_1,f_1}(\vec\tau\cdot\vec\tau)b^d_{m_2,f_2}b^c_{m,f}-g_{\pi/s}\pi_s(R)\\
&\times\begin{pmatrix}j_1\\ -i(\vec\sigma\hat r)j_0\end{pmatrix}\chi^f_m b^{d\dagger}_{m_1,f_1}(\vec\tau\cdot\vec\tau)d^{d\dagger}_{m_2,f_2}d^{c\dagger}_{m,f}-g_{\pi/s}\pi_s(R)\\
&\times\begin{pmatrix}j_1\\ -i(\vec\sigma\hat r)j_0\end{pmatrix}\chi^f_m d^d_{m_1,f_1}(\vec\tau\cdot\vec\tau)b^d_{m_2,f_2}d^{c\dagger}_{m,f}.\end{aligned}\tag{2.16}$$

This boundary condition can be sandwiched between the final antiquark state $\langle f|=\langle\overline{q}^a_{p,r}|$ and the initial vacuum state $|i\rangle=|0\rangle$. It is easy to see by inspection that many terms drop out, so that one ends up with very simple relations. On the left-hand side (LHS) one has

$$\mathrm{LHS}=\begin{pmatrix}(\vec\sigma\hat r)j_1\\ ij_0\end{pmatrix}\chi^f_m\langle 0|d^n_{p,q}d^{c\dagger}_{m,f}|0\rangle.\tag{2.17a}$$

On the right-hand side (RHS) one has to insert the complete set of intermediate states $|s\rangle\langle s|$:

$$\begin{aligned}\mathrm{RHS}=&ig_\sigma\sigma_s(R)\begin{pmatrix}i(\vec\sigma\hat r)j_0\\ -j_1\end{pmatrix}\chi^f_m\langle 0|d^a_{p,r}d^d_{m_2,f_2}|s\rangle\langle s|d^{c\dagger}_{m,f}|0\rangle\\
&-g_\sigma f_\pi\begin{pmatrix}i(\vec\sigma\hat r)j_0\\ -j_1\end{pmatrix}\chi^f_m+3g_{\pi/s}\pi_p(R)\\
&\times\begin{pmatrix}j_1\\ -i(\vec\sigma\hat r)j_0\end{pmatrix}\chi^f_m(\vec\sigma\hat r).\end{aligned}\tag{2.17b}$$

Thus one obtains

$$\begin{aligned}\begin{pmatrix}(\vec\sigma\hat r)j_1\\ ij_0\end{pmatrix}=&ig_\sigma\sigma_s(R)\begin{pmatrix}i(\vec\sigma\hat r)j_0\\ -j_1\end{pmatrix}-ig_\sigma f_\pi\begin{pmatrix}i(\vec\sigma\hat r)j_0\\ -j_1\end{pmatrix}\\
&+3g_{\pi/p}\pi_p(R)\begin{pmatrix}(\vec\sigma\hat r)j_1\\ -ij_0\end{pmatrix}.\end{aligned}\tag{2.18}$$

Two equations follow from the above expression (here $R=R_{\rm bag}$):

$$j_0(\omega)g_\sigma[f_\pi-\sigma_s(R)]-j_1(\omega)[1-3g_{\pi/p}\pi_p(R)]=0,$$

$$j_0(\omega)[1+3g_{\pi/p}\pi_p(R)]-j_1(\omega)g_\sigma[f_\pi-\sigma_s(R)]=0.\tag{2.19}$$

These two equations constitute a homogeneous system for the functions $j_{0,1}(\omega)$, so the determinant of the system should vanish.

The other projection between the vacuum and the one-quark state $|i\rangle=|q^a_{p,q}\rangle$ gives a system similar to that above:

$$j_0(\omega)-j_1(\omega)[g_\sigma f_\pi+3g_{\pi/s}\pi_s(R)]=0,\tag{2.20}$$

$$j_0(\omega)[g_\sigma f_\pi - 3g_{\pi/s}(R)] - j_1(\omega) = 0.$$

The quark eigenenergy ω will be determined from the compatibility of the boundary conditions (2.13) and (2.15). One takes the smallest possible ω, which corresponds to the ground state in the standard bag model [6,7]. In that case, instead of a common meson coupling constant g [Eq. (2.2)] flavor- and angular-momentum-dependent couplings $g_{\sigma/s}$, $g_{\pi/s}$ and $g_{\pi/p}$ appear. This reflects chiral symmetry breaking. As shown in Eq. (2.21) below, this appears naturally when the nonlinear system (2.2) is solved using the *Ansätze* (2.7)–(2.10). One can solve the system of equations (2.19) and (2.20). One solution for $g_{\pi/p} = \pi_p(R)/3$ gives a trivial solution for g_σ, i.e., $g_\sigma = 0$. The other gives

$$g_\sigma = \frac{J^2+1}{2 f_\pi J},$$

$$g_{\pi/s} = \frac{1-J^2}{6J\pi_s(R)},$$

$$g_{\pi/p} = \frac{J^2-1}{3(J^2+1)\pi_p(R)},$$

$$\sigma_s(R) = f_\pi \frac{J^4 - 3J^2+1}{(1+J^2)^2},$$

$$J = j_1(\omega)/j_0(\omega). \qquad (2.21)$$

To extract the equations for the s- and p- wave components from the operator equations of motion, Eqs. (2.4) are sandwiched between the final state $\langle f| = \langle q^c_{f,t}| = \langle 0| b^c_{f,t}$ and the initial state $|i\rangle = |q^c_{i,u}\rangle = b^{c\dagger}_{i,u}|0\rangle$. This choice yields the equation for $\sigma_s(r)$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right] \sigma_s(r) = \lambda^2[\sigma_s(r) - f_\pi]\{[\sigma_s(r)-f_\pi]^2 + 3\pi_p^2(r) - \nu^2\} + f_\pi m_\pi^2, \qquad (2.22)$$

and for $\pi_p(r)$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2}{r^2}\right]\pi_p(r) = \lambda^2 \pi_p(r)\{[\sigma(r)-f_\pi]^2 + 3\pi_p^2(r) - \nu^2\}. \qquad (2.23)$$

The other choice, i.e., $\langle f| = \langle 0|$ and $|i\rangle = |q^c_{i,u}\overline{q}^c_{i',u'}\rangle = d^{c\dagger}_{i',u'} b^{c\dagger}_{i,u}|0\rangle$, gives the pion s-wave component

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right]\pi_s(r) = \lambda^2\pi_s(r)[f_\pi^2 + 36\pi_s^2(r) - \nu^2]. \qquad (2.24)$$

The problem is to find a set of solutions of the differential equations (2.4), (2.23), and (2.24), $\{\sigma(r), \pi_s(r), \pi_p(r)\}$, which satisfy the *mathematical* boundary conditions (2.13) and (2.14). These solutions must be compatible with Eq. (2.21) which is independent of r. Of course, J contains information on the system of differential equations, so one has a strongly correlated algebraic system (2.19) and (2.20) and the system of differential equations.

The parameters (λ, ν) which enter $\mathcal{L}$ (2.2) are restricted by the symmetry-breaking behavior of the theory. Usually [6,20], the σ particle is considered to be a 1.2 GeV resonance, whereas the pion ''mass'' is a parameter which, for simplicity (and lack of knowledge), is assigned the value of the physical pion mass (0.137 GeV). In the present application, these values have also been used, although m_σ and m_π can, in principle, be considered as additional parameters.

The usage of the bag model has to some extent decoupled the equation for the quark expansion functions ϕ^f_m, $\tilde{\phi}^f_m$, etc. (2.3) from the rest. It communicates with the $\pi_n(r)$ ($n = s,p$) and $\sigma(r)$ functions only through algebraic relations (2.19) and (2.20). In some more sophisticated model a nonlinear differential equation for ϕ^f_m would be a part of the system containing also π_s, π_p and σ_s (2.22)–(2.24).

Higher order terms in the expansion, such as the third term in Eq. (2.3) for example, would enlarge the system of the coupled equations. As in the TDM the whole system would be coupled sector by sector. That would be governed by the number of creation (annihilation) operators and by some additional $|i_a\rangle$ ($\langle f_a|$) states besides those in Eqs. (2.11) and (2.12). The end results would be completely analogous to the relations among different sectors in the Fock space in the TDM, as one should expect from its reversed picture.

The results obtained in the leading order of the TDIA for this simple model, can be used to calculate the nucleon magnetic moment, the axial-vector coupling constant, and the meson mass, by expressing them in terms of functions $j_l(\omega r/R)$, $\pi_p(r)$, and $\sigma_s(r)$. The magnetic moment operator is [22]

$$\vec{\mu}(\vec{r}) = \frac{1}{2}[\vec{r} \times \vec{j}_{\rm EM}(\vec{r})]. \qquad (2.25)$$

Here

$$j^\mu_{\rm EM}(r) = \overline{\psi}(r)\gamma^\mu Q\psi(r) + \epsilon_{3ij}\pi_i(r)\partial^\mu\pi_j(r) \qquad (2.26a)$$

and

$$Q = \frac{2}{3}\frac{1+\tau_3}{2} - \frac{1}{3}\frac{1-\tau_3}{2}. \qquad (2.26b)$$

The quark contribution to μ_N is

$$\mu^{(Q)} = \frac{2}{3}\frac{R}{\omega^4}\frac{(\omega/2) - (3/8)\sin 2\omega + (\omega/4)\cos 2\omega}{j_0^2(\omega) + j_i^2(\omega) - 2j_0(\omega)j_1(\omega)/\omega}. \qquad (2.27)$$

The meson contribution is

$$\mu_P^{(M)} = \frac{16\pi}{3}\frac{11}{3}\int_{R_{\rm bag}}^\infty r^2 dr[\pi_p(r)]^2\mu_p. \qquad (2.28)$$

The proton magnetic moment is given by

$$\mu_p = \mu^{(Q)} + \mu_p^{(M)}. \tag{2.29}$$

Here and in the following, the expressions like quark contributions, meson contributions, meson phase, etc., are used as verbal shortcuts. One actually deals with complex and/or complicated stationary quark fields. The contributions (2.27) and (2.28) were obtained by calculating the matrix element

$$\left\langle p \middle| \int d^3 r \vec{\mu}(r) \middle| p \right\rangle. \tag{2.30}$$

Here $|p\rangle$ is the standard three quark proton state

$$|p,\uparrow\rangle = \frac{\varepsilon^{abc}}{\sqrt{18}} ([u^\dagger_{(\uparrow,a)} d^\dagger_{(\downarrow,b)} - u^\dagger_{(\downarrow,a)} d^\dagger_{(\uparrow,b)}] u^\dagger_{(\uparrow,c)})|0\rangle. \tag{2.31}$$

The axial-vector coupling constant g_A is the matrix element of the component $A_3^3(\vec{r})$ of the isovector axial-vector current sandwiched between nucleon states and integrated over all space [6,22]. The quark contribution is

$$g_A^{(Q)} = \left\langle p\uparrow \middle| \int d^3\vec{r}\, \overline{\psi}(\vec{r}) \gamma^3 \gamma^5 \frac{\tau^3}{2} \psi(\vec{r}) \middle| n\uparrow \right\rangle = \frac{5}{3}\frac{1}{3} \frac{j_0^2(\omega) + j_1^2(\omega)}{j_0^2(\omega) + j_1^2(\omega) - 2 j_0(\omega) j_1(\omega)/\omega}. \tag{2.32}$$

The meson contribution is

$$g_A^{(M)} = \frac{5}{3}\frac{4\pi}{3}\int_{R_{\text{bag}}}^{\infty} dr\, r^2 \left\{ [\sigma_s(r) - f_\pi]\left[\pi_p'(r) + \frac{2\pi_p(r)}{r}\right] - \pi_p(r)\sigma_s'(r) \right\}. \tag{2.33}$$

Finally,

$$g_A^{(p)} = g_A^{(Q)} + g_A^{(M)}. \tag{2.34}$$

The physical meson mass is

$$m_\pi^{\text{phys}} = m_Q + m_M. \tag{2.35a}$$

Here m_Q is the well known [7,16] quark phase contribution while m_M is given by

$$m_M = 4\pi \int_{R_{\text{bag}}}^{\infty} r^2 dr \left(\left[12(\pi_s')^2 + (\pi_p') + \frac{2\pi_p^2}{r^2} \right] + \frac{\lambda^2}{4}[\nu^4 - 4\nu^2(\sigma_s - f_\pi)^2 - 4\nu^2(\pi_p^2 + 12\pi_s^2)] - 2 f_\pi m_\pi^2 (\sigma_s - f_\pi) \right). \tag{2.35b}$$

III. ALTERNATIVE APPROXIMATION

In order to test the TDIA we will compare it with the well known and successfull hedgehog *Ansätze* [6,16,17] which is described below in Sec. IV. In those *Ansätze* only p-wave meson fields appear. Thus we will develop an alternative version of the TDIA.

We will retain the p-wave component for the pion field only. The corresponding equations of motion are derived from an effective classical Hamiltonian, which is obtained by averaging the quantized Hamiltonian over a baryon (proton, δ). The baryon wave functions in terms of chiral quarks belong to the conventional **56** representations of SU(6) (2.31). The fields appearing in the quantized Hamiltonian are given in the TDIA. For the ψ field one uses Eq. (2.7) while the σ and pion fields are given by

$$\vec{\pi}(r) = \pi_p(r)[\chi_{s'}^{a'} \vec{\tau}(\vec{\sigma}\hat{r})\chi_s^a] b_{s'}^{\dagger a'} b_s^a, \qquad \sigma(r) = \frac{\sigma_s(r)}{3} b_s^{\dagger a}(m) b_s^a(n) - f_\pi. \tag{3.1}$$

The Hamiltonian is

$$\mathcal{H} = \int d^3x \left\{ \psi^\dagger(-i\vec{\alpha}\vec{\partial})\psi\Theta + [\psi^\dagger g\gamma^0(\sigma + i\gamma_5\vec{\tau}\vec{\pi})]\psi\delta_S + \left[(\vec{\partial}\sigma)^2 + \frac{1}{2}(\vec{\partial}\vec{\pi}_a)^2 + U(\sigma,\vec{\pi}) \right] \overline{\Theta} \right\}. \tag{3.2}$$

For the proton (2.31), the expectation value of $\mathcal{H}$ has the following form:

$$\langle p|\mathcal{H}|p\rangle = \mathcal{H}_p = 3\frac{\omega}{R} + 4\pi\int_R^\infty dr r^2 \left\{ \frac{(\sigma_s')^2}{2} + \frac{1}{2}\frac{\Sigma_p}{3} \times \left(\pi_p^2 + \frac{2\pi_p^2}{r^2} \right) + f_\pi m_\pi^2 (\sigma_s - f_\pi) + \frac{\lambda^2}{4}\left[(\sigma_s - f_\pi)^2 + \pi_p^2 \frac{\Sigma_p}{3} - \nu^2 \right]^2 \right\}. \tag{3.3}$$

For Δ, one finds

$$\langle \Delta|\mathcal{H}|\Delta\rangle = \mathcal{H}_\Delta = 3\frac{\omega}{R} + 4\pi\int_R^\infty dr r^2 \left\{ \frac{(\sigma_s')^2}{2} + \frac{1}{2}\frac{\Sigma_p}{3} \times \left((\pi_p')^2 + \frac{2\pi_p^2}{r^2} \right) + f_\pi m_\pi^2 (\sigma - f_\pi) + \frac{\lambda^2}{4} \times \left[(\sigma_s - f_\pi)^2 + \pi_p^2 \frac{\Sigma_\Delta}{3} - \nu^2 \right]^2 + \frac{\lambda^2}{4}\pi_p^4 16 \right\}. \tag{3.4}$$

Here $\Sigma_{p,\Delta}$ are the matrix elements of the spin-isospin operators averaged over the spinor-isospinor part of the p/Δ wave function [7], for example,

$$\Sigma_p = \langle p|(\sigma_i\tau_j)(\sigma_i\tau_j)|p\rangle. \tag{3.5}$$

The variational procedure leads to the equations of motion (corresponding to the proton)

$$\sigma_s''+\frac{2}{r}\,\sigma_s'=\lambda^2(\sigma_s-f_\pi)\left[(\sigma_s-f_\pi)^2+\frac{\Sigma_p}{3}\,\pi_p^2-\nu^2\right],$$
$$\pi_p''+\frac{2}{r}\,\pi_p'-\frac{2}{r^2}\,\pi_p=\lambda^2\pi_p\left[(\sigma_s-f_\pi)^2+\frac{\Sigma_p}{3}\,\pi_p^2-\nu^2\right]+\lambda^2\pi_p^3\,\frac{48}{\Sigma_\Delta}. \tag{3.6}$$

The boundary conditions for the meson profile functions are

$$\frac{\partial}{\partial r}\,\sigma_s(r)\bigg|_{r=R_{\rm bag}}=-\frac{3N^2}{4\pi}\,\frac{g}{2}\,[j_0^2(\omega)-j_1^2(\omega)],$$
$$\frac{\partial}{\partial r}\,\pi_p(r)\bigg|_{r=R_{\rm bag}}=-\frac{3N^2}{4\pi}\,g[j_0(\omega)j_1(\omega)], \tag{3.7}$$

and

$$\sigma_s(r)|_{r\to\infty}=0,\quad \pi_p(r)|_{r\to\infty}=0. \tag{3.8}$$

Using the same method as in the preceding section one obtains the consistency condition for the quark eigenenergies from

$$j_0(\omega)g[f_\pi-\sigma_s(R)]+j_1(\omega)\left(1-\frac{11}{3}\,g\pi_p(R)\right)=0,$$
$$j_0(\omega)\left(1+\frac{11}{3}\,g\pi_p(R)\right)-j_1(\omega)g[f_\pi-\sigma_s(R)]=0. \tag{3.9}$$

Thus the expression for the coupling constant g is [see Eq. (2.21)]

$$g=\frac{1}{\sqrt{[\sigma_s(R)-f_\pi]^2+(11/9)\pi_p(R)}}. \tag{3.10}$$

Here the number 11 arises from the matrix element Σ (3.5). The other equation analogous to Eq. (2.21) is

$$\frac{1}{J}=\frac{1-g(11/9)\pi_p(R)}{g[f_\pi-\sigma_s(R)]}=\frac{1-(\Sigma_\Delta/9)\pi_p(R)/3}{g[f_\pi-\sigma(R)]}. \tag{3.11}$$

The electromagnetic properties are calculated taking into account the electromagnetic current (2.26a) and (2.26b). The quark contribution to the magnetic moment retains the form (2.27) but with the ω determined from (3.9). For the proton, one finds that $\mu_p^{(M)}$ again has the form (2.28).

The axial-vector coupling constant g_A has the quark contribution (2.32) and the meson contribution (2.33). As already mentioned, the ω value and all parameter values corresponds to the model defined by Eqs. (3.3)–(3.11).

The physical pion mass $m_\pi^{\rm phys}$ is given in Eq. (2.35a) but here m_M is given by

$$m_M=4\pi\int_{R_{\rm bag}}^\infty r^2dr\left(\frac{\Sigma}{6}\left[(\pi')^2+\frac{2\pi^2}{r^2}\right]+\frac{\lambda^2}{4}\,[(\sigma-f_\pi)^2+\Sigma\pi^2-\nu^2]^2-f_\pi m_\pi^2(\sigma-f_\pi)\right). \tag{3.12}$$

IV. THE HEDGEHOG *ANSÄTZE*

This section is intended to provide a detailed comparison between the TDIA used in the preceding section and the hedgehog *Ansätze*. At the classical level there is not much difference between the results obtained in this section and the results presented in Sec. III. The equations of motion are similar and their (classical) solutions are almost identical (see Sec. V). There is a slight difference in the quantization procedure. Usually [6,16], one quantizes (hedgehog) quarks and (hedgehog) mesons as elementary fermion and boson fields. Coherent states are used [6,15,16] to provide a quantum representation of the boson fields.

In the example provided here the bosonic phase is quantized in the same way as used in the *Ansätze* (3.1). The end result is the same as that obtained using coherent states.

The baryons are given in hedgehog form

$$|h\rangle=b_1^\dagger b_2^\dagger b_3^\dagger|0\rangle;\quad \langle h|h\rangle=1. \tag{4.1}$$

The pion state is a p wave and it assumes a hedgehog form as well,

$$\pi_a(\vec r)=\hat r_a\pi(r)b_i^\dagger b_i \tag{4.2}$$

and σ is given by the scalar component and the symmetry-breaking term

$$\sigma(\vec r)=\sigma(r)b_i^\dagger b_i-f_\pi. \tag{4.3}$$

The hedgehog baryon is neither a nucleon nor a Δ, and it has to be projected into a spin-isospin eigenstate [6,20].

The hedgehog form (4.2) is closely related to the *Ansätze* (3.1). If the isospinor-spinor combination χ_m^f in Eq. (3.1) is replaced by the hedgehog combination, i.e.,

$$\chi_m^f=\tilde\chi^f\chi_m\to\frac{1}{\sqrt2}\,(\tilde\chi^{f=u}\chi_{m=-1/2}-\tilde\chi^{f=d}\chi_{m=1/2})=\chi_h\,, \tag{4.4a}$$

then one finds

$$\chi_h^\dagger(\vec\sigma\hat r)\tau^a\chi_h=\hat r^a. \tag{4.4b}$$

Thus the mapping (4.4) transforms the *Ansätze* (3.1) into the corresponding Eqs. (4.2) and (4.3). It is not surprising that the (classical) equations of motion barely change. The change comes from the fact that with the hedgehog *Ansätze* there is only one universal baryon $|h\rangle$ (4.1). However, the s-wave components (2.10) do vanish when the replacement (4.4a) is effected. One obtains

$$\chi_h^\dagger \tau^a \chi_h = 0. \tag{4.4c}$$

Thus the solution presented in this section differs in an essentied way from the one discussed in Sec. II.

The expectation value of the normal-ordered Hamiltonian (3.2) is

$$\begin{aligned}\langle h|\mathcal{H}|h\rangle &= \mathcal{H}_h \\ &= 3N^2 4\pi \int_0^R dr r^2 \bigg[j_0(\omega r/R) \\ &\quad \times \left(\frac{2j_1(\omega r/R)}{r} + j_1'(\omega r/R) \right) \\ &\quad - j_1(\omega r/R) j_0'(\omega r/R) \bigg] + 4\pi \int_R^\infty dr r^2 \\ &\quad \times \left\{ \frac{1}{2} \left[\left(\frac{\partial \sigma}{\partial r} \right)^2 + \left(\frac{\partial \pi}{\partial r} \right)^2 + \frac{2}{r^2}\, \pi^2 \right] \right. \\ &\quad \left. + \frac{\lambda^2}{4} (\sigma^2 + \pi^2 - \nu^2)^2 + f_\pi m_\pi^2 \sigma \right\}. \end{aligned} \tag{4.5}$$

The Euler-Lagrange equations are given in terms of mean fields approximated by the static expectation values.

Instead of Eq. (3.6) one finds

$$\begin{aligned}\sigma'' + \frac{2}{r}\,\sigma' &= \lambda^2(\sigma(r) - f_\pi)\{[\sigma(r) - f_\pi]^2 \\ &\quad + [\pi(r)]^2 - \nu^2\} + f_\pi m_\pi^2 \end{aligned} \tag{4.6}$$

and

$$\pi'' + \frac{2}{r}\,\pi' - \frac{2}{r^2}\,\pi = \lambda^2 \pi(r)\{[\sigma(r) - f_\pi]^2 + [\pi(r)]^2 - \nu^2\}. \tag{4.7}$$

The boundary condition for $\sigma(r)$ is

$$\left.\frac{d\sigma}{dr}\right|_{r=R_{\rm bag}} = -\frac{3g}{8\pi}\, N^2[j_0^2(\omega) - j_1^2(\omega)], \tag{4.8}$$

where g is calculated from the fields at the boundary

$$g = \frac{1}{\sqrt{[\sigma(R_{\rm bag})]^2 + [\pi(R_{\rm bag})]^2}}. \tag{4.9}$$

For the pion phase one gets

$$\left.\frac{d\pi}{dr}\right|_{r=R_{\rm bag}} = -\frac{3g}{4\pi}\, N^2[j_0(\omega) j_1(\omega)]. \tag{4.10}$$

The electromagnetic properties are calculated using Eqs. (2.25), (2.26a), and (2.26b).

The quark contribution to the proton magnetic moment retains the form (2.27). Also, $\mu_n^{(Q)} = -\frac{2}{3}\mu_p^{(Q)}$. With the hedgehog *Ansätze* for meson fields one finds [6]

$$\mu^{(M)} = \frac{4\pi}{3} \int_{R_{\rm bag}}^\infty r^2 dr [\pi(r)]^2. \tag{4.11}$$

The quark contribution to the nucleon axial-vector coupling constant g_A retains the form (2.32).

The meson part of the axial-vector constant is [6]

$$\begin{aligned} g_A^{(M)} = \frac{8\pi}{3} \int_{R_{\rm bag}}^\infty r^2 dr \bigg[[\sigma(r) - f_\pi]\pi'(r) - \pi(r)\sigma'(r) \\ + \frac{2[\sigma(r) - f_\pi]\pi(r)}{r} \bigg]. \end{aligned} \tag{4.12}$$

The difference in constant factors between Eqs. (2.28), (2.29) and Eqs. (4.11), (4.12), respectively, can be traced to averaging over Eq. (4.1) rather than over the proton wave function, as done in Sec. III.

The quantum properties (4.2) and (4.3) of boson solitons follow from the hedgehog version of the boundary condition (2.15). Thus our baryon (4.1) differs from the usual form [6] which uses the coherent states. However, with the hedgehog *Ansätze*, both methods lead to an identical expression for the energy $\mathcal{H}_p$ (4.5).

Using the trial wave function of Ref. [6],

$$|h_{\rm coh}\rangle = \exp(A_\sigma^+)\exp(A_\pi^+)|h\rangle, \tag{4.13}$$

one easily finds

$$\mathcal{H}_h = \frac{\langle h_{\rm coh}|\mathcal{H}|h_{\rm coh}\rangle}{\langle h_{\rm coh}|h_{\rm coh}\rangle}. \tag{4.14}$$

Here A_σ^+ contains the elementary σ-field operator $a_0^+(k)$, i.e.,

$$\begin{aligned} A_\sigma^+ &= \int d^3k\, \frac{\omega_\sigma k}{2}\, \tilde{F}(k) a_0^+(k), \\ \sigma(r) &= \frac{2}{(2\pi)^3} \int d^3k e^{i\vec{k}\vec{r}} \tilde{F}(k), \end{aligned} \tag{4.15}$$

and analogously for A_π^+. Variation with respect to $\pi(r)$ and $\sigma(r)$ leads to the above equations of motion.

A possible generalization of the coherent state which conserves spin and isospin is considerably more complicated than Eq. (4.13). Even the one pion approximation [23] is quite involved.

V. THE NUMERICAL PROCEDURE

Numerics will be illustrated here for the nonlinear system of coupled ordinary differential equations which were derived in Sec. II. The other two approaches, the TDIA with hadron averaging and hedgehog *Ansätze*, lead to very similar systems which differ only in some superficial details.

This system determines fermion and boson radial functions appearing in the *Ansätze*, for example in Eqs. (2.7)–(2.10). The boson radial functions had to satify Eqs. (2.22)–(2.24). These equations were supplemented by the boundary conditions given by Eqs. (2.13) and (2.14). The conditions (2.14) were dictated by the (physical) requirement that the (massive) field solitons should vanish at infinity.

In Eq. (2.13) the normalization constant N can be expressed in terms of Bessel functions and quark eigenfrequencies ω:

$$N^2=\frac{1}{R^3}\left[j_0^2(\omega)+j_0^2(\omega)-\frac{2j_0(\omega)j_1(\omega)}{\omega}\right]. \tag{5.1}$$

The radial parts of the quark wave functions appearing in Eq. (2.7) are Bessel functions $j_l(\omega r/R)$ for any spherical bag with radius R. At the bag boundary, where $r=R$, these functions have to satisfy the relations (2.19) and (2.20) which combine the quark frequency ω with the coupling constants g_σ, g_π, f_π, etc. The algebraic relations among the coupling constants stem from the requirement that the homogeneous system of linear equations should have the vanishing determinant. Therefore, the coupling constants have to satisfy the consistency conditions given by Eq. (2.21)

The linear σ-model parameters satisfy the following relations derived from the symmetry breaking pattern (see Sec. II) [6,16,22]:

$$\lambda^2=\frac{m_\sigma^2-m_\pi^2}{2f_\pi^2},\quad \nu^2=f_\pi^2-\frac{m_\pi^2}{\lambda^2},\quad d=\frac{1}{2}f_\pi^2 m_\pi^2\frac{2m_\sigma^2-3m_\pi^2}{m_\sigma^2-m_\pi^2}. \tag{5.2}$$

Here the value of d is determined by the requirement that $U(\sigma,\vec{\pi})$ should have zero minima. The σ meson is expected to have a mass of about 1 GeV [20]. Thus the *parameter* masses m_σ and m_π are selected to be 1.2 and 0.139 GeV, respectively. The physical pion mass $m_\pi^{\rm phys}$ is determined either by the formula (2.35) or the formula (3.12).

One has to solve simultaneously the system containing nonlinear differential equations (2.22), (2.23), and (2.24), Eqs. (2.13) and (2.14), and the algebraic relations (2.21) and (5.2). This determines the meson functions $\sigma(r)$, $\pi_s(r)$, and $\pi_p(r)$, the quark frequency ω, and various coupling (g_π, g_σ, etc.).

This complex system has been solved using the code COLSYS, the collocation system solver developed by Asher, Christiansen and Russel [24]. The boundary conditions are set at $[R_{\rm bag},R]$, where R is set to be so large that the fields can be approximated by zero at R. The initial guesses have been supplied. From the asymptotic behavior and some earlier experience the input was rather simple and convergence has been achieved quickly.

The problem turns out to be rather sensitive to the derivative boundary conditions which in all cases involve the coupling constant(s). Although the asymptotic behavior of the solutions can be inferred from the system itself (see, also, Ref. [25]), COLSYS is able to handle rather general initial (guess) solutions.

Upon return the routine gives error estimates for components and its derivatives. The problem parameters can be gradually changed (increased) by using a continuation method in COLSYS which is left to choose the initial mesh points, and in the continuation procedure it refines and redistributes the (former) mesh.

There are additional chiral-bag-model parameters, the same as those used in the MIT bag, i.e., B, Z_0, and α_s [6,7,19,21]. They are connected with the bag properties (B,Z_0) and with the effective gluon exchange (α_s) which removes the nucleon (N)-resonance (Δ) mass degeneracy. Some earlier experience (see Ref. [19]) suggested that these parameters would remain within typical chiral-bag-model values. Here these parameters are used to fix the N and Δ masses within 1% accuracy. The numerical values depend on the particular *Ansätze* used. Thus for example for solution described in Sec. IV (see Table I, below) one finds $R=6.0$, $\omega=1.80$, $Z_0=0.12$, $B^{1/4}=0.14$, and $\alpha_s=0.12$ or $R=5.0$, $\omega=2.10$, $Z_0=0.3$, $B^{1/4}=0.15$, and $\alpha_s=0.25$.

The solutions are compared against the consistency conditions (2.21) and the iterative procedure is continued until the matching is obtained. The iteration consists in performing a self-consistent calculation: the coupling constants for the chiral quarks nonhedgehog method are set to be the same at the beginning (their value is set to be equal to 10.00) and after every iteration new coupling constants are calculated from Eq. (2.21). These new values are replaced in the boundary conditions to calculate new solutions. The procedure converges rather rapidly. When the matching is achieved, the magnetic moment, the axial constant and the *physical* pion mass are calculated from the obtained solutions, i.e., from either $\{\sigma(r),\pi_s(r),\pi_p(r)\}$ for the TDIA or $\{\sigma(r),\pi(r)\}$ for the hedgehog *Ansätze*.

VI. RESULTS, COMMENTS, AND CONCLUSION

The Tamm-Dancoff inspired approximation (TDIA) (Sec. II) leads to results which depend strongly on the quark eigenfrequency ω, as shown in Table I. There are several sets of the coupling constants g_i which satisfy the consistency condition (2.21), thus producing several sets of g_A, μ, and $m_\pi^{\rm phys}$ values.

However, one is more interested here in comparison of methods. As shown in Table II, TDIA based mean-field method gives consistently too large g_A values and somewhat better μ values. The pion masses are always too large. For example with $R=6.00\ {\rm GeV}^{-1}$; $g=10.93$ one obtains $m_\pi^{\rm phys}=0.501$ GeV. With $R=4.97\ {\rm GeV}^{-1}$ and $g=11.28$ one finds $m_\pi^{\rm phys}=0.756$ GeV. All predictions are very similar to those found using the hedgehog mean-field method (Sec. IV).

The hedgehog-based [6] results are displayed in Table III. Here they were obtained by using parameters comparable with those used in Tables I and II, which facilitates the comparison. It is not surprising that the values in Tables II and III are similar. Equations (3.6), (4.6), and (4.7) are not very different. The same goes for the theoretical expressions for g_A and μ. The values of μ in Table II look somewhat closer to the $\mu_{\rm epx}$. However, this could be just an accidental effect of a particular parametrization.

The method described in Sec. II treats the quark and meson fields (or phases) as operator equations, which are approximately solved. The $\pi_p(r)$, $\pi_s(r)$, and $\sigma_s(r)$ are

TABLE I. The results for the chiral-quark model in the TDIA. The bag radius is in GeV^{-1} units.

		magn. moment			ax. const. g_A			
R	ω	μ_Q	μ_M	μ_{tot}	$g_{A/Q}$	$g_{A/M}$	$g_{A/\mathrm{tot}}$	m_π^{phys}
4.00	2.10	1.53	1.41	2.94	1.01	0.12	1.13	0.208
5.00	1.90	1.77	0.44	2.21	1.06	0.23	1.29	0.142
5.00	2.10	1.01	0.91	1.92	1.06	0.06	1.12	0.198
6.00	1.80	2.09	0.28	2.37	1.06	0.05	1.21	0.132
6.00	1.90	2.09	0.34	2.43	0.91	0.25	1.16	0.166
7.00	1.80	3.02	0.25	3.27	1.03	0.18	1.21	0.155
7.00	2.10	2.55	0.06	2.61	1.06	0.21	1.27	0.156

parameters

$\lambda = 9.062$	$m_\sigma = 1.2$ GeV	$\mu_{\mathrm{exp}} = 2.79$	$m_\pi^{\mathrm{exp}} = 0.139$ GeV
$\nu = 0.092$	$f_\pi = 0.093$ GeV	$g_{A/\mathrm{exp}} = 1.26$	$m_\pi = 0.140$ GeV

smoothly decreasing with distance, as required by the boundary conditions. The large μ values in Table I are for the same ω always associated with smaller g_A values, thus both being simultaneously closer to the experimental data. In Table I one can see that such behavior is caused by the meson-phase contributions which here contain both s wave and p wave. They are proportionally much larger in the case of μ as it should be. The same richer structure of the pion phase lead to better predictions for m_π^{phys}.

It is interesting that the TDIA can lead to acceptable solutions of the chiral quark model. The results seem to be comparable with those obtained using the hedgehog *Ansätze*. The TDIA leads to some more complex description of the pion phase, what seems to improve the quality of the calculated results. Although everything strongly depends on the parametrization, these preliminary results seem to encourage further application of the TDIA.

The values displayed in Table I are also comparable with the Skyrme model [14] where, typically, $\mu = 2.48$, $g_A = 0.61$ or with the Nambu-Jona-Lasinio model [23], where $\mu = 2.76$ and $g_A = 1.86$. This test of the TDIA was made in a simple chiral quark model, which leads to the least complex, albeit already extensive, numerics. A first tentative conclusion is that the TDIA works for a field theory version of the model. However, this has to be further tested, in more complex and realistic models [16–18].

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APPENDIX: TDIA IN A SIMPLE MODEL

In order to avoid inessential complexity we consider a system consisting of stationary baryon (nucleon) field ψ interacting with a neutral scalar (meson) field ϕ. The baryon field is described nonrelativistically; spin is being ignored. The interaction is of Yukawa type. This simple model is used here merely to illustrate the relation between two ap-

TABLE II. The TDIA based calculation. The nonhedgehog mean-field method has been used to project the physical states. The bag radius is in GeV^{-1} units. The bag parameters are explained in the main text.

			magn. moment			axial const. g_A		
R	ω	g	μ_Q	μ_M	μ_{tot}	$g_{A/Q}$	$g_{A/M}$	$g_{A/\mathrm{tot}}$
4.97	1.0238	9.299	1.20	0.83	2.02	1.51	0.39	1.90
5.00	0.979	9.311	1.155	1.377	2.531	1.53	0.53	2.06
6.00	1.285	9.799	1.741	1.116	2.857	1.42	0.51	1.93
7.00	1.78	10.799	2.52	0.09	2.61	1.22	0.29	1.50

parameters

$\lambda = 9.062$	$m_\sigma = 1.2$ GeV	$\mu_{\mathrm{exp}} = 2.79$
$\nu = 0.092$	$f_\pi = 0.093$ GeV	$g_{A/\mathrm{exp}} = 1.26$

TABLE III. The chiral-bag-model calculation. The hedgehog mean-field method has been used to project the physical states. The bag radius is in GeV^{-1} units.

			magn. moment			axial const. g_A		
R	ω	g	μ_Q	μ_M	$\mu_{\rm tot}$	$g_{A/Q}$	$g_{A/M}$	$g_{A/\rm tot}$
5.00	1.280	11.250	1.45	0.27	1.72	1.43	0.42	1.85
6.00	1.637	10.878	2.060	0.144	2.204	1.28	0.33	1.61
7.00	1.783	10.799	2.519	0.092	2.610	1.22	0.29	1.504
			parameters					
	$\lambda=9.062$			$m_\sigma=1.2$ GeV			$\mu_{\rm exp}=2.79$	
	$\nu=0.092$			$f_\pi=0.093$ GeV			$g_{A/\rm exp}=1.26$	

proaches, the TDM and TDIA.

The Schrödinger equation is given by

$$(H_0+H_{\rm int})|\psi,\vec{q}\rangle=E|\psi,\vec{q}\rangle,$$

$$H_0=\int d^3x\,\psi^*(\vec{x})\left(-\frac{\hbar^2}{2m}\right)\Delta\psi(\vec{x})$$

$$+\frac{1}{2}\int d^3x\{[\nabla\phi(\vec{x})]^2+\mu^2\phi^2(\vec{x})+\vec{\pi}^2(\vec{x})\}$$

$$H_{\rm int}=G\int d^3x\,\psi^*(\vec{x})\psi(\vec{x})\phi(\vec{x}), \tag{A1}$$

with

$$|\psi,\vec{q}\rangle=A(\vec{q})b_q^\dagger+\int d^3r\,d^3s|,B(\vec{r},\vec{s})b_r^\dagger a_s^\dagger+\cdots \tag{A2}$$

one easily finds the first Tamm-Dancoff equation

$$\frac{\vec{q}^2}{2m}A(\vec{q})+g\int d^3\ell\,B(\vec{\ell},\vec{q}-\vec{\ell})=EA(\vec{q}). \tag{A3}$$

Here b_q and a_q are the annihilation operators for the ψ and ϕ fields, respectively.

The Hamiltonian (A1) corresponds, in the Heisenberg picture, to the equation

$$-\frac{\hbar^2}{2m}\Delta\psi+G\phi\psi=E\psi=\mathcal{O}\psi. \tag{A4}$$

By expanding

$$\phi(\vec{x})=\sum_n[f_n(\vec{x})a_n+f_n^*(\vec{x})a_n^+],$$
$$\psi(\vec{x})=\sum_m g_n(\vec{x})b_m+\sum_{a,b}h_{a,b}(\vec{x})b_a a_b+\cdots \tag{A5}$$

one obtains

$$\langle 0|\mathcal{O}\psi|b_t\rangle=E\langle 0|\psi|b_t\rangle$$

$$-\frac{\hbar^2}{2m}\Delta g_t(\vec{x})+G\sum_m h_{t,n}(\vec{x})f_n^*(\vec{x}) \tag{A6}$$

$$=Eg_t(\vec{x}).$$

The Fourier-transform of expression (A6) has the same overall form as the Tamm-Dancoff equation (A3):

$$g_t(\vec{x})=\int d^3k\,e^{i\vec{k}\vec{x}}g_t(\vec{k}),$$

$$f_n^*(\vec{x})=\int d^3p\,e^{i\vec{p}\vec{x}}f_t^*(\vec{p})\frac{1}{(2\pi)^3}\int d^3x\,e^{-i\vec{q}\vec{x}}$$
$$\times\left[\left(-\frac{\hbar^2}{2m}\Delta g_t+G\sum_n h_{t,n}f_n^*\right)-Eg_t\right]$$
$$=\frac{q^2}{2m}g_t(\vec{q})+G\int d^3k\sum_n f_n^*(\vec{q}-\vec{k})g_t(\vec{k})$$
$$-Eg_t(\vec{q})=0. \tag{A7}$$

One can find such parallels for the whole system of TDM or TDIA equations. In the more sophisticated case, investigated in this paper, such a task would be rather forbidding. It seems that the TDM and TDIA lead to comparable although not exactly equivalent approximations. Functions such as $g(\vec{x})$ or their Fourier transform $g(\vec{g})$ $[A(q)]$ are probability amplitudes for finding that system consists of particles whose position is $\vec{x}$ (momenta are $\vec{q}$, $\vec{k}-\vec{k}$, etc.) [26].

The overall structure of Eqs. (A3), (A6), and (A7), regarding the mixture of $|\psi\rangle$ and $|\psi,\phi\rangle$ states, resembles very closely the integral equations (8) and (9), Ref. [3], p. 201 or Eqs. (7) and (8) of Ref. [8]. However, our system is simplified even in comparison with Ref. [3], not to mention the sophisticated approaches [8,9] based on the light front Yukawa model.

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