$USp(2k)$ matrix model: Nonperturbative approach to orientifolds

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We discuss the theoretical implications of the large k $USp(2k)$ matrix model in zero dimensions. The model appears as the matrix model of type IIB superstrings on a large $T^6/\tilde{\mathcal{Z}}^2$ orientifold via the matrix twist operation. In the small volume limit, the model behaves four dimensionally and its T dual is a six-dimensional worldvolume theory of type I superstrings in ten spacetime dimensions. Several theoretical considerations including the analysis on planar diagrams, the commutativity of the projectors with supersymmetries, and the cancellation of gauge anomalies are given, providing us with the rationale for the choice of the Lie algebra and the field content. A few classical solutions are constructed which correspond to Dirichlet p-branes and some fluctuations are evaluated. The particular scaling limit with matrix T duality transformation is discussed which derives the F theory compactification on an elliptic fibered $K3$. [S0556-2821(98)00214-8]

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I. INTRODUCTION

This paper discusses in depth the theoretical implications of the $USp(2k)$ matrix model in zero dimensions introduced in Ref. [1]. A particular emphasis will be given to the aspects of the model as a nonperturbative framework to deal with orientifold compactification.

Gauge fields and strings have governed our thoughts on a unified theory of all forces including gravity and constituents for more than two decades. One of our current theoretical endeavors is, it seems, to take gauge fields as dynamical variables of noncommuting matrix coordinates [2] to construct string theory from matrices. This approach strives to overcome some of the difficulties of the first quantized superstring theory, which have led to an inevitable impasse: one may list, among other things, the existence of infinitely degenerate perturbative vacua, the problem of supermoduli, etc. The one-dimensional matrix model [3] of M theory [4] has obtained success on the agreement of the spectrum and other properties with the low energy eleven-dimensional supergravity theory while the zero-dimensional model [5] of type IIB superstrings lays its basis on the correspondence [6–8,5] with the first quantized action of the Schild type gauge [9] and appears to be numerically accessible. We will often refer to the latter case as the reduced model. See Refs. [10] for some of the references on the subsequent developments.

We would like to show that the reduced model presented in this paper descends from the first quantized nonorientable type I superstring theory [11], which is believed to be related to heterotic string theory [12] by S duality [4,13]. In this sense, it is expected that the model is exposed to phenomenological questions of particle physics by the presence of gauge bosons, matter fermions, and other properties. As pointed out in Ref. [1], the model, at the same time, captures one of the exact results in string theory, namely the F theory compactification on an elliptic fibered $K3$, which is originally deduced geometrically [14] from the $SL(2,Z)$ duality [15]: it is nonetheless exact quantum mechanically.

In the next section, the definition of the $USp(2k)$ matrix model introduced in Ref. [1] is recalled. The relationship of the parts not involving the fields in the fundamental representation with the type IIB matrix model is given precisely, by introducing projectors onto the USp adjoint as well as the antisymmetric representation. This is found to be useful in developing the analysis in the remaining sections. The definition of the model appears to be rather *ad hoc* at first. In the subsequent three sections, we will show that our model passes in fact several stringent criteria which the large k reduced model of orientifold must satisfy. We will be able to provide the rationales for our choice of the Lie algebra usp, for the choice of the number of the noncommuting coordinates belonging to the adjoint representation and that to the antisymmetric representation, and finally for the number of multiplets needed, denoted by n_f, belonging to the fundamental representation.

The most basic notion of the large k reduced models is that the dense set of Feynman diagrams in the large k limit forms the string worldsheet [16]. This is not limited to a combinatorial equivalence. The reduced $U(2k)$ Yang-Mills model goes to the string action in the Schild gauge. The Lie algebra $u(2k)$ becomes isomorphic to the area preserving diffeomorphisms on a sphere. In Sec. III, we begin with showing how this fact is extended to nonorientable strings. We examine the role played by the matrix F in large k USp Feynman diagrams, ignoring the diagrams coming from the fields belonging to the fundamental representation. This is combined with the analysis relating F to the worldsheet involution in the large k limit, telling us that the surfaces created by the dense set of Feynman diagrams are nonorientable. The correspondence with the first quantized operator approach confirms that F is a matrix analog of the twist operation. This is strengthened by showing that Eq. (3.9) changes sign under the matrix T duality transformation [17]. In Sec. IV, we examine the commutativity of the projectors with dynamical as well as kinematical supersymmetry. The cases which pass this criterion with eight dynamical and eight kinematical supercharges are found to be very scarce. The field content of our model stands as the most natural choice. In Sec. V, we discuss the role played by the fields in the fundamental representation and the cancellation of gauge anomalies. Obviously, these fields create boundaries of the surfaces.

Combining these analyses in Secs. III, IV, and V, we conclude that the model in its original form is the large k reduced model of type IIB superstrings on a large $T^6/\mathcal{Z}^2$ orientifold. In the other limit, namely the small volume limit in which the model behaves as in four-dimensional flat spacetime, the T duality transformation takes this model into the six-dimensional worldvolume theory representing type I superstrings in ten spacetime dimensions. The anomaly cancellation of this worldvolume gauge theory in Sec. V selects $n_f=16$, telling us that this is the matrix counterpart of the original Green-Schwarz cancellation leading to $SO(32)$ type I nonorientable superstrings.

In Sec. VI, we turn to constructing classical solutions which correspond to a D-string and two (anti-)parallel D-strings. A formula for the one-loop effective action on a general background is obtained. This is used to evaluate the potential between two antiparallel D-strings. Evidently, two additional dimensions are not generated in this naive large k limit. These solutions are straightforwardly generalized to solutions representing a Dp-brane and parallel Dp-branes, which we illustrate in the case of $p=3$ in Sec. VII. In Sec. VIII, applying some of the results obtained in Secs. III, IV, and V, we supplement the discussion of Ref. [1] on the connection with the F theory compactification on an elliptic fibered $K3$.

II. DEFINITION OF THE $USp(2k)$ MATRIX MODEL

We adopt a notation that the inner product of the two $2k$ dimensional vectors u_i and v_i invariant under $USp(2k)$ are

$$\langle u,v\rangle = u_i F^{ij} v_j, \tag{2.1}$$

$$F^{ij}=\begin{pmatrix} 0 & I_k \\ -I_k & 0\end{pmatrix}. \tag{2.2}$$

I_k is the unit matrix. The raising and lowering of the indices are done by $F=F^{ij}$ and $F^{-1}=F_{ij}$. The element X of the $usp(2k)$ Lie algebra satisfying $X^tF+FX=0$ and $X^\dagger=X$ can be represented as

$$X=\begin{pmatrix} M & N \\ N^* & -M^t\end{pmatrix} \tag{2.3}$$

with $M^\dagger=M$ and $N^t=N$. It is sometimes convenient to adopt the tensor product notation:

$$X=\left(\frac{1+\sigma^3}{2}\right)\otimes M+\left(\frac{1-\sigma^3}{2}\right)\otimes(-M^t)+\sigma^+\otimes N+\sigma^-\otimes N^*, \tag{2.4}$$

where σ^1, σ^2, and σ^3 are Pauli matrices, and $\sigma^\pm\equiv(\sigma^1\pm i\sigma^2)/2$. On the other hand, the element Y of the antisymmetric representation of the $USp(2k)$ is

$$Y=\begin{pmatrix} A & B \\ C & A^t\end{pmatrix} \tag{2.5}$$

with $B^t=-B$ and $C^t=-C$. The hermiticity condition can be imposed. In the tensor product notation, Eq. (2.5) becomes then

$$\left(\frac{1+\sigma^3}{2}\right)\otimes A+\left(\frac{1-\sigma^3}{2}\right)\otimes A^t+\sigma^+\otimes B+\sigma^-\otimes(-B^\infty) \tag{2.6}$$

with $A^\dagger=A$ and $B^t=-B$.

Let us recall the definition of the $USp(2k)$ matrix model in zero dimensions introduced in Ref. [1]. Our zero-dimensional model can be written, by borrowing $\mathcal{N}=1$, $d=4$ superfield notation in the Wess-Zumino gauge. One simply drops all spacetime dependence of the fields but keeps all Grassmann coordinates as they are

$$S\equiv S_{\rm vec}+S_{\rm asym}+S_{\rm fund},$$

$$S_{\rm vec}=\frac{1}{4g^2}\,{\rm Tr}\left(\int d^2\theta W^\alpha W_\alpha+{\rm H.c.}+4\int d^2\theta d^2\overline{\theta}\Phi^\dagger e^{2V}\Phi e^{-2V}\right),$$

$$S_{\rm asym}=\frac{1}{g^2}\int d\theta^2 d\overline{\theta}^2[T^{*ij}(e^{2V({\rm asym})})_{ij}{}^{kl}T_{kl}+\widetilde{T}^{ij}(e^{-2V({\rm asym})})_{ij}{}^{kl}\widetilde{T}^*_{kl}]+\frac{\sqrt{2}}{g^2}\left\{\int d\theta^2\widetilde{T}^{ij}(\Phi_{({\rm asym})})_{ij}{}^{kl}T_{kl}+{\rm H.c.}\right\},$$

$$S_{\rm fund}=\frac{1}{g^2}\sum_{f=1}^{n_f}\left[\int d^2\theta d^2\overline{\theta}[Q^{*i}_{(f)}(e^{2V})_i{}^jQ_{(f)j}+\widetilde{Q}^i_{(f)}(e^{-2V})_i{}^j\widetilde{Q}^*_{(f)j}]+\left\{\int d^2\theta(m_{(f)}\widetilde{Q}_{(f)}{}^iQ_{(f)i}+\sqrt{2}\widetilde{Q}_{(f)}{}^i(\Phi)_i{}^jQ_{(f)j})+{\rm H.c.}\right\}\right]. \tag{2.7}$$

The chiral superfields introduced above are

$$W_\alpha=-\frac{1}{8}\overline{DD}e^{-2V}D_\alpha e^{2V},\quad \Phi=\Phi+\sqrt{2}\theta\psi+\theta\theta F_\Phi, \tag{2.8}$$

$$Q_i = Q_i + \sqrt{2}\,\theta\psi_{Qi} + \theta\theta F_{Qi}, \quad T_{ij} = T_{ij} + \sqrt{2}\,\theta\psi_{Tij} + \theta\theta F_{Tij}, \tag{2.9}$$

while

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha}, \quad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}}, \tag{2.10}$$

$$V = -\theta\sigma^m\bar{\theta}v_m + i\,\theta\theta\bar{\theta}\bar{\lambda} - i\,\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\,\theta\theta\bar{\theta}\bar{\theta}D. \tag{2.11}$$

We represent the antisymmetric tensor superfield T_{ij} as

$$Y \equiv (TF)_i^{\;j} = \begin{pmatrix} A & B \\ C & A^t \end{pmatrix} \tag{2.12}$$

with $B^t = -B,\ C^t = -C$. We define $\tilde{Y}$ similarly.

In terms of components, the action reads, with indices suppressed,

$$S_{\text{vec}} = \frac{1}{g^2}\,\text{Tr}\big(-\tfrac{1}{4}v_{mn}v^{mn} - [\mathcal{D}_m,\Phi]^\dagger[\mathcal{D}^m,\Phi] - i\lambda\sigma^m[\mathcal{D}_m,\bar{\lambda}] - i\bar{\psi}\bar{\sigma}^m[\mathcal{D}_m,\psi] - i\sqrt{2}[\lambda,\psi]\Phi^\dagger - i\sqrt{2}[\bar{\lambda},\bar{\psi}]\Phi\big)$$
$$+\frac{1}{g^2}\,\text{Tr}\left(\frac{1}{2}DD - D[\Phi^\dagger,\Phi] + F_\Phi^\dagger F_\Phi\right), \tag{2.13}$$

$$S_{\text{asym}} = \frac{1}{g^2}\{-(\mathcal{D}_mT)^*(\mathcal{D}^mT) - i\bar{\psi}_T\bar{\sigma}^m\mathcal{D}_m\psi_T + i\sqrt{2}T^*\lambda^{(\text{asym})}\psi_T - i\sqrt{2}\bar{\psi}_T\bar{\lambda}^{(\text{asym})}T$$
$$-(\mathcal{D}_m\tilde{T})(\mathcal{D}^m\tilde{T})^* - i\bar{\psi}_{\tilde{T}}\bar{\sigma}^m\mathcal{D}_m\psi_{\tilde{T}} - i\sqrt{2}\tilde{T}^*\lambda^{(\text{asym})}\psi_{\tilde{T}} + i\sqrt{2}\bar{\psi}_{\tilde{T}}\bar{\lambda}^{(\text{asym})}\tilde{T}$$
$$-2(\Phi^*_{(\text{asym})}T^*)(\Phi_{(\text{asym})}T) - 2(\tilde{T}\Phi_{(\text{asym})})(\tilde{T}^*\Phi^*_{(\text{asym})}) - \sqrt{2}(\psi_{\tilde{T}}\psi^{(\text{asym})}T + \tilde{T}\psi^{(\text{asym})}\psi_{\text{T}} + \psi_{\tilde{T}}\Phi_{(\text{asym})}\psi_T)$$
$$-\sqrt{2}(\bar{\psi}_T\bar{\psi}^{(\text{asym})}\tilde{T}^* + T^*\bar{\psi}^{(\text{asym})}\bar{\psi}_{\tilde{T}} + \psi_T\Phi^*_{(\text{asym})}\bar{\psi}_{\tilde{T}})$$
$$+\sqrt{2}\tilde{T}F_\Phi^{(\text{asym})}T + \sqrt{2}\tilde{T}^*F_\Phi^{*(\text{asym})}T^* + \tilde{T}D^{(\text{asym})}T + \tilde{T}^*D^{(\text{asym})}T^*\}, \tag{2.14}$$

$$S_{\text{fund}} = +\frac{1}{g^2}\sum_{f=1}^{n_f}[-(\mathcal{D}_mQ_{(f)})^*(\mathcal{D}^mQ_{(f)}) - i\bar{\psi}_{Q(f)}\bar{\sigma}^m\mathcal{D}_m\psi_{Q(f)} + i\sqrt{2}Q^*_{(f)}\lambda\psi_{Q(f)} - i\sqrt{2}\bar{\psi}_{Q(f)}\bar{\lambda}Q_{(f)}]$$
$$+\frac{1}{g^2}\sum_{f=1}^{n_f}[-(\mathcal{D}_m\tilde{Q}_{(f)})(\mathcal{D}^m\tilde{Q}_{(f)})^* - i\bar{\psi}_{\tilde{Q}(f)}\bar{\sigma}^m\mathcal{D}_m\psi_{\tilde{Q}(f)} - i\sqrt{2}\tilde{Q}_{(f)}\lambda\psi_{\tilde{Q}(f)} + i\sqrt{2}\bar{\psi}_{\tilde{Q}(f)}\bar{\lambda}\tilde{Q}^*_{(f)}]$$
$$+\frac{1}{g^2}\sum_{f=1}^{n_f}(Q^*_{(f)}DQ_{(f)} + \tilde{Q}_{(f)}D\tilde{Q}^*_{(f)}) + \frac{1}{g^2}\sum_{f=1}^{n_f}\{-(m_{(f)})^2(Q^*_{(f)}Q_{(f)} + \tilde{Q}_{(f)}\tilde{Q}^*_{(f)}) - m_{(f)}(\tilde{\psi}_{Q(f)}\psi_{Q(f)} + \bar{\tilde{\psi}}_{Q(f)}\bar{\psi}_{Q(f)})$$
$$-\sqrt{2}m_{(f)}(Q^*_{(f)}\Phi^\dagger Q_{(f)} + \tilde{Q}_{(f)}\Phi^\dagger\tilde{Q}^*_{(f)} + Q^*_{(f)}\Phi Q_{(f)} + \tilde{Q}_{(f)}\Phi\tilde{Q}^*_{(f)}) - 2Q^*_{(f)}\Phi^\dagger\Phi Q_{(f)} - 2\tilde{Q}_{(f)}\Phi^\dagger\Phi\tilde{Q}^*_{(f)}$$
$$-\sqrt{2}(\psi_{\tilde{Q}(f)}\psi Q_{(f)} + \tilde{Q}_{(f)}\psi\psi_{Q(f)} + \psi_{\tilde{Q}(f)}\Phi\psi_{Q(f)}) - \sqrt{2}(\bar{\psi}_{Q(f)}\bar{\psi}\tilde{Q}^*_{(f)} + Q^*_{(f)}\overline{\psi\psi}_{\tilde{Q}(f)} + \bar{\psi}_{Q(f)}\Phi^\dagger\bar{\psi}_{\tilde{Q}(f)})$$
$$+\sqrt{2}\tilde{Q}_{(f)}F_\Phi Q_{(f)} + \sqrt{2}\tilde{Q}^*_{(f)}F_\Phi^\dagger Q^*_{(f)}\}, \tag{2.15}$$

where

$$D_i^{\;j} = [\Phi^\dagger,\Phi]_i^{\;j} + \sum_{f=1}^{n_f}(Q^{*j}_{(f)}Q_{(f)i} + \tilde{Q}^j_{(f)}\tilde{Q}^*_{(f)i}) + 2T^{*jk}T_{ki} + 2\tilde{T}^{jk}\tilde{T}^*_{ki}, \tag{2.16}$$

$$F_{\Phi i}^{\;\;j} = -\sum_{f=1}^{n_f}(\sqrt{2}Q^{*j}_{(f)}\tilde{Q}^*_{(f)i}) - \sqrt{2}T^{*jk}T^*_{ki}, \tag{2.17}$$

Here $\mathcal{D}_m = iv_m$ with v_m in appropriate representations. $\Phi_{(\mathrm{asym})}$, $\psi^{(\mathrm{asym})}$, and $F_\Phi^{(\mathrm{asym})}$ are the fields in antisymmetric representation.

Let us now find a relationship of $S_{\mathrm{vec}}+S_{\mathrm{asym}}$ in Eq. (2.7) with the reduced action of the four-dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory written again in terms of superfields. This latter action in turn is related in the component form to the reduced action of the ten-dimensional $\mathcal{N}=1$ Yang-Mills theory, which is nothing but the type IIB matrix model [5].

First note that $S_{\mathrm{vec}}+S_{\mathrm{asym}}$ in Eq. (2.7) is written as

$$S_{\mathrm{vec}}+S_{\mathrm{asym}} \equiv S_{\mathrm{adj+asym}} = \frac{1}{4g^2}\,\mathrm{Tr}\left(\int d^2\theta W^\alpha W_\alpha + \mathrm{H.c.} + 4\int d^2\theta d^2\bar{\theta}\Phi^{\dagger i} e^{2V}\Phi_i e^{-2V}\right) + \frac{1}{\sqrt{2}g^2}\,\mathrm{Tr}\left(\int d^2\theta d^2\bar{\theta}\epsilon^{ijk}[\Phi_i,[\Phi_j,\Phi_k]] + \mathrm{H.c.}\right), \tag{2.18}$$

where we have introduced the notation

$$\Phi_1 \equiv \Phi, \quad \Phi_2 \equiv Y, \quad \Phi_3 \equiv \tilde{Y}. \tag{2.19}$$

The form of Eq. (2.18) is nothing but the reduced action of $d=4$, $\mathcal{N}=4$ super Yang-Mills theory, which we denote by $S_{\mathcal{N}=4}^{d=4}$:

$$S_{\mathrm{adj+asym}} = S_{\mathcal{N}=4}^{d=4}. \tag{2.20}$$

It is expedient to introduce the projector acting on $U(2k)$ matrices:

$$\hat{\rho}_{\mp}\bullet \equiv \frac{1}{2}\,(\bullet \mp F^{-1}\bullet^t F). \tag{2.21}$$

The action of $\hat{\rho}_-$ and that of $\hat{\rho}_+$ take any $U(2k)$ matrix into the matrix lying in the adjoint representation of $USp(2k)$ and that in the antisymmetric representation, respectively. We can therefore write

$$V = \hat{\rho}_- \underline{V}, \quad \Phi_1 = \hat{\rho}_- \underline{\Phi}_1, \quad \Phi_i = \hat{\rho}_+ \underline{\Phi}_i, \quad i=2,3, \tag{2.22}$$

where the symbols with underlines lie in the adjoint representation of $U(2k)$.

We now invoke the well-known fact that the action of $d=4$, $\mathcal{N}=4$ super Yang-Mills theory can be obtained from the dimensional reduction of $d=10$, $\mathcal{N}=1$ super Yang-Mills theory down to four dimensions [18]. This is stated as

$$S_{\mathcal{N}=4}^{d=4}(\underline{v}_m,\underline{\Phi}_i,\underline{\lambda},\underline{\psi}_i,\underline{\bar{\Phi}}_i,\underline{\bar{\lambda}},\underline{\bar{\psi}}_i) = S_{\mathcal{N}=1}^{d=10}(\underline{v}_M,\underline{\Psi}),$$

$$S_{\mathcal{N}=1}^{d=10}(\underline{v}_M,\underline{\Psi}) = \frac{1}{g^2}\,\mathrm{Tr}\left(\frac{1}{4}\,[\underline{v}_M,\underline{v}_N][\underline{v}^M,\underline{v}^N] - \frac{1}{2}\,\underline{\bar{\Psi}}\Gamma^M[\underline{v}_M,\underline{\Psi}]\right). \tag{2.23}$$

Here

$$\Phi_i = \frac{1}{\sqrt{2}}\,(v_{3+i}+iv_{6+i}),$$

and

$$\Psi = (\lambda,0,\psi_1,0,\psi_2,0,\psi_3,0,0,\bar{\lambda},0,\bar{\psi}_1,0,\bar{\psi}_2,0,\bar{\psi}_3)^t, \tag{2.24}$$

which is a 32-component Majorana-Weyl spinor satisfying

$$C\bar{\Psi}^t = \Psi, \quad \Gamma_{11}\Psi = \Psi. \tag{2.25}$$

With regard to Eqs. (2.24) and (2.25), the same is true for objects with underlines. The ten-dimensional gamma matrices have been denoted by Γ^M. We will not spell out their explicit form which is determined from Eqs. (2.23) and (2.24).

What we have established through the argument above are summarized as the following formulas useful in later sections:

$$S_{\mathrm{adj+asym}} = S_{\mathcal{N}=1}^{d=10}(\hat{\rho}_{b\mp}\underline{v}_M,\hat{\rho}_{f\mp}\underline{\Psi}), \tag{2.26}$$

where $\hat{\rho}_{b\mp}$ is a matrix with Lorentz indices and $\hat{\rho}_{f\mp}$ is a matrix with spinor indices:

$$\hat{\rho}_{b\mp} = \mathrm{diag}(\hat{\rho}_-,\hat{\rho}_-,\hat{\rho}_-,\hat{\rho}_-,\hat{\rho}_-,\hat{\rho}_+,\hat{\rho}_+,\hat{\rho}_-,\hat{\rho}_+,\hat{\rho}_+),$$

$$\hat{\rho}_{f\mp} = \hat{\rho}_- 1_{(4)} \otimes \begin{pmatrix} 1_{(2)} & & & \\ & 0 & & \\ & & 1_{(2)} & \\ & & & 0 \end{pmatrix} + \hat{\rho}_+ 1_{(4)} \otimes \begin{pmatrix} 0 & & & \\ & 1_{(2)} & & \\ & & 0 & \\ & & & 1_{(2)} \end{pmatrix}. \tag{2.27}$$

The notable properties of the model discussed in [1] are, among other things, (1) it possesses eight dynamical and eight kinematical supersymmetries, and (2) translations in six out of ten directions are broken. We will discuss implications of these in subsequent sections.

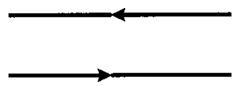

FIG. 1. Propagator.

III. $USp(2k)$ PLANAR DIAGRAMS, MATRIX TWIST AND MATRIX T DUAL

We now discuss $USp(2k)$ planar diagrams to see how they create nonorientable surfaces approximated by the dense set of Feynman diagrams. We set aside the fields lying in the fundamental representation in this section. We ignore fermions as well. It is well known that the large k expansion of ordinary $U(2k)$ pure Yang-Mills theory in arbitrary dimensions is a topological (genus) expansion of the two-dimensional (discretized) surfaces created by the Feynman diagrams [16]. It is simple to see how this is modified by $USp(2k)$ Feynman diagrams where some of them are in the adjoint while the others are in the nonadjoint (antisymmetric).

Recall that the propagator in the $U(2k)$ gauge theory is

$$\langle \underline{v}_{m\ r}{}^{s} \underline{v}_{n\ i}{}^{j} \rangle = \delta_i{}^s \delta_r{}^j \delta_{mn} D = \text{Fig. 1}. \tag{3.1}$$

From now on we ignore the D function as its dependence on the arguments is irrelevant to the present discussion. The three and four point vertices are depicted in Fig. 2.

Let $\underline{\mathcal{G}}$ be a $U(2k)$ Feynman diagram. Its dependence on g^2 and on k denoted by $r(\underline{\mathcal{G}})$ is known to be

$$r(\underline{\mathcal{G}}) = (g^2 k)^{\mathcal{E}-\mathcal{V}} k^{\chi},$$

$$\chi = \mathcal{F} - \mathcal{E} + \mathcal{V} = 2 - 2\mathcal{H}, \tag{3.2}$$

where $\mathcal{E}$ is the number of external lines in $\underline{\mathcal{G}}$, which is also the number of edges of the surface, while $\mathcal{V}$ is the number of three and four point vertices in $\underline{\mathcal{G}}$ and is on the surface. The number of faces or index loops and the number of holes of the surface are denoted by $\mathcal{F}$ and by $\mathcal{H}$, respectively.

In USp Feynman diagrams, Eq. (3.1) is modified to

$$\langle v_{M\ r}{}^{s} v_{N\ i}{}^{j} \rangle = \sum_{a=1}^{2k^2 \pm k} (t^n)_r{}^s (t^a)_i{}^j \delta_{MN}$$

$$= (\hat{\rho}_{\mp})_r{}^s{}_i{}^j \delta_{MN}$$

$$= \text{Fig. 3}. \tag{3.3}$$

Here we have treated the adjoint and nonadjoint cases collectively. Similarly, let $\mathcal{G}$ be a USp Feynman diagram. As the propagator contains the second term which reduces the number of index loops by one, $r(\mathcal{G})$ depends upon how many times double lines representing propagators cross. Clearly

$$r(\mathcal{G}) = r(\mathcal{G};c) = (g^2 k)^{\mathcal{E}-\mathcal{V}} k^{\chi - c}, \tag{3.4}$$

where c denotes the number of crossings.

We still need to show that c denotes the number of cross caps and not the number of boundaries. Let us recall that, according to the present point of view, the two-dimensional

FIG. 2. Three point vertex and four point vertex.

surface swept by a string is formed by the dense set of Feynman diagrams. To render this more tangible and more than a combinatorial argument, we note that, via the Schild gauge correspondence, the algebra acting on the functions on the string world sheet must be isomorphic to the large k limit of the appropriate Lie algebra acting on matrices. For this, it is enough to adopt the argument of Pope and Romans [19] on area-preserving diffeomorphisms on RP^2 and the large k limit of the $usp(2k)$ Lie algebra in the present context. Consider first the sphere parametrized by three coordinates x^i, $i=1,2,3$ such that $x^i x^i = 1$. The complete set of functions on the sphere is the spherical harmonics represented by

$$Y^{(p)}(x^i) = a_{i_1,\cdots i_p} x^{i_1} \cdots x^{i_p}, \tag{3.5}$$

where $a_{i_1,\cdots i_p}$ are totally symmetric and traceless constants. The algebra of area preserving diffeomorphisms is defined by a bracket of two functions $A(x^i)$ and $B(x^i)$:

$$\{A,B\} \equiv \epsilon_{ijk} x^i \partial_j A \partial_k B. \tag{3.6}$$

When $A = Y^{(m)}$, $B = Y^{(n)}$, a finite sum of irreducible polynomials $Y^{(p)}$, $|m-n| \leq p \leq m+n-1$ is generated. This algebraic structure is obtained by the large k limit of the $su(2k)$ Lie algebra in the form of maximal $su(2)$ embeddings:

$$\Lambda^{(p)} = a_{i_1 \cdots i_p} \Sigma^{i_1} \cdots \Sigma^{i_p},$$

$$p = 1 \sim k-1. \tag{3.7}$$

Here, Σ^i are the $su(2)$ generators in the $2k$-dimensional representation. On the other hand, RP^2 geometry is obtained from the sphere by the antipodal identification $x^i \to -x^i$, under which the harmonics splits into even and odd ones. Only the odd ones are responsible for forming the algebra of area-preserving diffeomorphism on RP^2: this is clear from Eq. (3.6). We see that the diffeomorphisms of RP^2 are generated by the large k limit of the generators

$$\Lambda^{(2p-1)} = a_{i_1 \cdots i_{2p-1}} \Sigma^{i_1} \cdots \Sigma^{i_{2p-1}},$$

$$p = 1,2, \ldots ,k. \tag{3.8}$$

As shown by Pope and Romans, the algebra formed by Eq. (3.8) is the Lie algebra usp. This concludes that the dia-

FIG. 3. Propagator in $USp(2k)$.

grams generated from the propagator [Eq. (3.3)] and the vertices contain RP^2. The theory we are constructing via matrices is the reduced model of nonorientable strings.

To extend the above argument to higher genera with crosscaps, let us note that the role of the matrix F can be seen by the correspondence with the twist operation in the operator formalism of the first quantized string. Ten of the noncommuting coordinates v_M, which are dynamical variables, satisfy

$$v_i^t = -F v_i F^{-1}, \quad i \in \{\{0,1,2,3,4,7\}\} \equiv \mathcal{M}_-,$$
$$v_I^t = F v_I F^{-1}, \quad I \in \{\{5,6,8,9\}\} \equiv \mathcal{M}_+. \tag{3.9}$$

The v_M are noncommuting counterparts of the ten string coordinates X_M. That this is more than just an analogy is clear as the limit exists from our action to the string action of the Schild type gauge. Taking the transpose is interpreted as flipping the direction of an arrow drawn on a string. The operation F is the matrix analog of the twist operation[1] Ω. The classical counterpart of Eq. (3.9) is therefore

$$X_i(\overline{z},z) = -\Omega X_i(z,\overline{z})\Omega^{-1}, \quad i \in \mathcal{M}_-,$$
$$X_I(\overline{z},z) = \Omega X_I(z,\overline{z})\Omega^{-1}, \quad I \in \mathcal{M}_+. \tag{3.10}$$

The presence of four-dimensional fixed surfaces (orientifold surfaces, $O3s$) becomes clear from this equation (3.10). We conclude that our model is a matrix model on a large volume T^6/Z^2 orientifold. This is consistent with the fact that the translations in six out of ten directions are broken.

The T duality transformation plays an interesting role in matrix models as it relates worldvolume theories of various dimensions via Fourier transforms. We will now find how the matrix T dual behaves under F. First, let us impose periodicities with period $2\pi R$ for L out of the ten coordinates. Recall that

$$Y_l \equiv \hat{T}[X_l] \equiv X_{lR} - X_{lL}, \tag{3.11}$$

$$\hat{T}[X_l](\overline{z},z) = \begin{cases} +\Omega \hat{T}[X_l](z,\overline{z})\Omega^{-1} & \text{if } l \in \mathcal{M}_-, \\ -\Omega \hat{T}[X_l](z,\overline{z})\Omega^{-1} & \text{if } l \in M_+. \end{cases} \tag{3.12}$$

To impose periodicities on infinite size matrices v_l, we decompose v_l into blocks of $n \times n$ matrices. Specify each individual block by an L-dimensional row vector $\vec{a}$ and an L-dimensional column vector $\vec{b}$: $(v_l)_{\vec{a},\vec{b}} \equiv \sqrt{\alpha'}\langle \vec{a}|\hat{v}_l|\vec{b}\rangle$. Let the shift vector be

$$(U(i))_{\vec{a},\vec{b}} = \left(\prod_{j(\neq i)} \delta_{a_j,b_j}\right)\delta_{a_i,b_i+1}. \tag{3.13}$$

The condition to be imposed is

$$U(i)v_l U(i)^{-1} = v_l - \delta_{l,i}R/\sqrt{\alpha'}. \tag{3.14}$$

[1]In the context of Ref. [3], see Ref. [20].

The solution in the Fourier transformed space is

$$\langle \vec{x}|\hat{v}_l|\vec{x}'\rangle = -i\left(\frac{\partial}{\partial x^l} + i\tilde{v}_l(\vec{x})\right)\delta^{(L)}(\vec{x}-\vec{x}'), \tag{3.15}$$

$$\tilde{v}_l(\vec{x}) = \sum_{\vec{k}\in \mathcal{Z}^L} \tilde{\tilde{v}}_l(\vec{k})\exp\left(\frac{-i\vec{k}\cdot\vec{x}}{\tilde{R}}\right),$$
$$\tilde{R} \equiv \alpha'/R. \tag{3.16}$$

The matrix T dual is nothing but the Fourier transform: it interchanges the radius parameter R setting the period of the original matrix index with the dual radius $\tilde{R}$, which is the period of the space Fourier conjugate to the matrix index. Let us write

$$\hat{T}[(v_l)_{\vec{a},\vec{b}}] \equiv \langle \vec{x}|\hat{v}_l|\vec{x}'\rangle. \tag{3.17}$$

Multiply Eq. (3.9) written in the bracket notation

$$\langle \vec{d}|\hat{v}_l|\vec{a}\rangle = \mp \sum_{\vec{b},\vec{c}} \langle \vec{a}|\hat{F}|\vec{b}\rangle\langle \vec{b}|\hat{v}_l|\vec{c}\rangle\langle \vec{c}|\hat{F}^{-1}|\vec{d}\rangle \tag{3.18}$$

by $\langle \vec{x}|\vec{a}\rangle\langle \vec{d}|\vec{x}'\rangle = \langle \vec{a}|\vec{x}\rangle^*\langle \vec{x}'|\vec{d}\rangle^*$. Sum over $\vec{a}$ and $\vec{d}$. From the left-hand side, we obtain

$$-\left[-i\frac{\partial}{\partial x'^l} - \tilde{v}_l(-\vec{x}')\right]\delta^{(L)}(\vec{x}'-\vec{x}). \tag{3.19}$$

We find

$$\hat{T}[v_l]^t = \begin{cases} +\hat{T}[F]\hat{T}[v_l]\hat{T}[F^{-1}] & \text{if } l \in \mathcal{M}_-, \\ -\hat{T}[F]\hat{T}[v_l]\hat{T}[F^{-1}] & \text{if } l \in \mathcal{M}_+, \end{cases} \tag{3.20}$$

provided

$$\tilde{v}_l(-\vec{x}') = -\tilde{v}_l(\vec{x}'). \tag{3.21}$$

It is satisfying to see that the sign change of Eq. (3.20) from Eq. (3.9) under the matrix T dual is in accordance with the sign change of Eq. (3.12) from Eq. (3.10).

One can now imagine imposing periodicities with periods depending on the directions and letting some of the radii zero. The T duality provides worldvolume gauge theories in various dimensions. We will discuss a few cases later.

IV. *USp* PROJECTOR AND SUPERSYMMETRY

We will now derive a set of conditions under which the projectors $\hat{\rho}_{b\mp}$, $\hat{\rho}_{f\mp}$, which act respectively on $\underline{v}_M$ and $\underline{\Psi}$, and dynamical $\delta^{(1)}$ as well as kinematical $\delta^{(2)}$ supersymmetry commute. Our choice for $\hat{\rho}_{b\mp}$ and that for $\hat{\rho}_{f\mp}$ emerge as the case which passes the tight constraint of having eight dynamical and eight kinematical supersymmetries. Let us start with

$$\delta^{(1)}\underline{v}_M = i\overline{\epsilon}\Gamma_M\underline{\Psi}, \tag{4.1}$$

$$\delta^{(1)}\underline{\Psi}=\frac{i}{2}[\underline{v}_M,\underline{v}_N]\Gamma^{MN}\epsilon, \tag{4.2}$$

$$\delta^{(2)}\underline{v}_M=0, \tag{4.3}$$

$$\delta^{(2)}\underline{\Psi}=\xi. \tag{4.4}$$

Let us write generically

$$v_M\equiv\delta_M{}^N\hat{\rho}_{b\mp}^{(N)}\underline{v}_N,$$
$$\Psi_A\equiv\delta_{AB}\overline{\rho}_{f\mp}^{(B)}\underline{\Psi}_B. \tag{4.5}$$

The condition $[\hat{\rho}_{b\mp},\delta^{(1)}]\underline{v}_M=0$ together with Eq. (4.1) gives

$$\sum_{A=1}^{32}(\overline{\epsilon}\Gamma_M)_A(\hat{\rho}_{f\mp}^{(A)}-\hat{\rho}_{b\mp}^{(M)})\underline{\Psi}_A=0, \tag{4.6}$$

with index M not summed. The condition

$$[\hat{\rho}_{f\mp},\delta^{(1)}]\underline{\Psi}|_{v_M\to\hat{\rho}_{b\mp}\underline{v}_M}=0 \tag{4.7}$$

together with Eq. (4.2) provides

$$(1-\hat{\rho}_{f\mp}^{(A)})[\hat{\rho}_{b\mp}^{(M)}\underline{v}_M,\hat{\rho}_{b\mp}^{(N)}\underline{v}_N](\Gamma^{MN}\epsilon)_A=0. \tag{4.8}$$

The restriction at Eq. (4.7) comes from the fact that Eq. (4.2) is true only on shell. Equation (4.3) does not give us anything new while $[\hat{\rho}_{f\mp},\delta^{(2)}]\underline{\Psi}=0$ with Eq. (4.4) gives

$$\xi_A 1=\xi_A\hat{\rho}_{f\mp}^{(A)}1, \tag{4.9}$$

with index A not summed.

In order to proceed further, we rewrite Eq. (4.5) explicitly as

$$\hat{\rho}_{b\mp}^{(M)}\equiv\Theta(M\in\mathcal{M}_-)\hat{\rho}_-+\Theta(M\in\mathcal{M}_+)\hat{\rho}_+,$$
$$\hat{\rho}_{f\mp}^{(A)}\equiv\Theta(A\in\mathcal{A}_-)\hat{\rho}_-+\Theta(A\in\mathcal{A}_+)\hat{\rho}_+, \tag{4.10}$$

where

$$\mathcal{M}_-\cup\mathcal{M}_+=\{\{0,1,2,3,4,5,6,7,8,9\}\},\quad \mathcal{M}_-\cap\mathcal{M}_+=\phi, \tag{4.11}$$

$$\mathcal{A}_-\cup\mathcal{A}_+=\{\{1,2,5,6,9,10,13,14,19,20,23,24,27,28,31,32\}\},\quad \mathcal{A}_-\cap\mathcal{A}_+=\phi. \tag{4.12}$$

We find that Eq. (4.6) gives

$$(\overline{\epsilon}\Gamma_{M_-})_{A_+}=(\overline{\epsilon}\Gamma_{M_+})_{A_-}=0, \tag{4.13}$$

while Eq. (4.8) gives

$$(\Gamma^{M_-N_+}\epsilon)_{A_-}=0,$$
$$(\Gamma^{M_-N_-}\epsilon)_{A_+}=(\Gamma^{M_+N_+}\epsilon)_{A_+}=0. \tag{4.14}$$

Equation (4.9) gives

$$\xi_{A_-}=0. \tag{4.15}$$

As we consider the case of eight kinematical supersymmetries, the number of elements of the sets denoted by $\sharp(\mathcal{A}_\pm)$ must be

$$\sharp(\mathcal{A}_-)=8\quad\text{and}\quad\sharp(\mathcal{A}_+)=8. \tag{4.16}$$

Equations (4.13) and (4.14) are regarded as the ones which determine the anticommuting parameter ϵ, and the sets $\mathcal{A}_+$, $\mathcal{A}_-$, $\mathcal{M}_+$, and $\mathcal{M}_-$. In addition they must satisfy the conditions (4.11), (4.12), and (4.16).

We search for solutions by first trying out as an input an appropriate 32-component anticommuting parameter ϵ satisfying the Majorana-Weyl condition.

Given ϵ, we see if we can determine $\mathcal{A}_+$, $\mathcal{A}_-$, $\mathcal{M}_+$, and $\mathcal{M}_-$ successfully. Our strategy is as follows.

(i) Calculate $(\overline{\epsilon}\Gamma^M)_A$ and $(\Gamma^{MN}\epsilon)_A$ for all M, N, and A.

(ii) Calculate $\Sigma_A(\overline{\epsilon}\Gamma_{M_1})_A(\overline{\epsilon}\Gamma_{M_2})_A$. If this value is nonzero, then both indices M_1 and M_2 belong to either $\mathcal{M}_-$ or $\mathcal{M}_+$. We can, therefore, divide $\mathcal{M}_-\cup\mathcal{M}_+$ into two sets.

(iii) From Eq. (4.14) we see that if $(\Gamma^{M_-N_+}\epsilon)_A\neq0$, then $A\in\mathcal{A}_+$. If $(\Gamma^{M_-N_-}\epsilon)_A\neq0$ or $(\Gamma^{M_+N_+}\epsilon)_A\neq0$, then $A\in\mathcal{A}_-$. Use the results of (i) and (ii) to determine $\mathcal{A}_-$ and $\mathcal{A}_+$. We must then check if $\sharp(\mathcal{A}_-)=8$, $\sharp(\mathcal{A}_+)=8$ and $\mathcal{A}_-\cap\mathcal{A}_+=\phi$. If these are not satisfied, our original input ϵ is not a solution.

(iv) From Eq. (4.13) we see that if $(\overline{\epsilon}\Gamma_{M_-})_A\neq0$ then $A\in\mathcal{A}_-$, and if $(\overline{\epsilon}\Gamma^{M_+})_A\neq0$ then $A\in\mathcal{A}_+$. Determine $\mathcal{A}_-$ and $\mathcal{A}_+$. If $\mathcal{A}_-$ and $\mathcal{A}_+$ determined this way are consistent with the result from (iii), we obtain a solution to Eqs. (4.13) and (4.14). This also determines $\mathcal{M}_-$ and $\mathcal{M}_+$ as we have two ways of choosing them from (ii).

We have tried out many cases, some of which we will describe. The case leading to our model is

$$\epsilon=(\epsilon_0,0,\epsilon_1,0,0,0,0,0,0,\overline{\epsilon_0},0,\overline{\epsilon_1},0,0,0,0)^t. \tag{4.17}$$

Note that ϵ_0, ϵ_1, $\overline{\epsilon_0}$ and $\overline{\epsilon_1}$ are two-component anticommuting parameters. From step (ii), we see $\mathcal{M}_-\cup\mathcal{M}_+$ are divided into two sets:

$$\{\{0,1,2,3,4,7\}\}\ \text{ and }\ \{\{5,6,8,9\}\}. \tag{4.18}$$

From step (iii), we find

$$\mathcal{A}_-=\{\{1,2,5,6,19,20,23,24\}\},$$

$$\mathcal{A}_+ = \{\{9,10,13,14,27,28,31,32\}\}. \tag{4.19}$$

From step (iv), we obtain

$$\mathcal{M}_- = \{\{0,1,2,3,4,7\}\},$$
$$\mathcal{M}_+ = \{\{5,6,8,9\}\}. \tag{4.20}$$

We conclude that

$$\hat{\rho}_{b\mp} = \mathrm{diag}(\hat{\rho}_-, \hat{\rho}_-, \hat{\rho}_-, \hat{\rho}_-, \hat{\rho}_-, \hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_-, \hat{\rho}_+, \hat{\rho}_+),$$

$$\hat{\rho}_{f\mp} = \hat{\rho}_- 1_{(4)} \otimes \begin{pmatrix} 1_{(2)} & & & \\ & 0 & & \\ & & 1_{(2)} & \\ & & & 0 \end{pmatrix} + \hat{\rho}_+ 1_{(4)} \otimes \begin{pmatrix} 0 & & & \\ & 1_{(2)} & & \\ & & 0 & \\ & & & 1_{(2)} \end{pmatrix}, \tag{4.21}$$

which are the projectors of our model.

Among other cases, we have tried the following one:

$$\epsilon = (\epsilon_0,0,\epsilon_1,0,\epsilon_2,0,\epsilon_3,0,0,\epsilon_0,0,\epsilon_1,0,\epsilon_2,0,\epsilon_3)^t. \tag{4.22}$$

From step (ii), we obtain

$$\{\{0,1,2,3,4,7\}\} \quad \text{and} \ \{\{5,6,8,9\}\}. \tag{4.23}$$

We find that $\mathcal{A}_-$ and $\mathcal{A}_+$ determined from step (iii) do not satisfy $\mathcal{A}_- \cap \mathcal{A}_+ = \phi$.

We have examined the following cases (and their permutations) as well with no success:

$$\epsilon = (\epsilon_0,0,\epsilon_1,0,\epsilon_2,0,\epsilon_3,0,0,-\epsilon_0,0,-\epsilon_1,0,-\epsilon_2,0,-\epsilon_3)^t,$$
$$\epsilon = (\epsilon_0,0,\epsilon_1,0,0,0,\epsilon_3,0,0,\overline{\epsilon_0},0,\epsilon_1,0,0,0,\epsilon_3)^t,$$
$$\epsilon = (\epsilon_0,0,\epsilon_1,0,0,0,\epsilon_3,0,0,\overline{\epsilon_0},0,\epsilon_1,0,0,0,-\epsilon_3)^t, \tag{4.24}$$
$$\epsilon = (\epsilon_0,0,\epsilon_1,0,0,0,\epsilon_3,0,0,\overline{\epsilon_0},0,-\epsilon_1,0,0,0,-\epsilon_3)^t,$$
$$\epsilon = (\epsilon_0,0,\epsilon_1,0,0,0,\epsilon_1,0,0,\overline{\epsilon_0},0,\overline{\epsilon_1},0,0,0,\overline{\epsilon_1})^t.$$

There is, however, another solution which we have found. Let

$$\epsilon = (\epsilon_0,0,\epsilon_1,0,0,0,0,0,0,0,0,0,0,\overline{\epsilon_2},0,\overline{\epsilon_3})^t. \tag{4.25}$$

The consistent sets

$$\mathcal{A}_- = \{\{1,2,5,6,27,28,31,32\}\},$$
$$\mathcal{A}_+ = \{\{9,10,13,14,19,20,23,24\}\}, \tag{4.26}$$

$$\mathcal{M}_- = \{\{4,7\}\},$$
$$\mathcal{M}_+ = \{\{0,1,2,3,5,6,8,9\}\}, \tag{4.27}$$

are obtained from steps (i), (ii), (iii), and (iv). The projectors (4.10) are

$$\hat{\rho}_{b\mp} = \mathrm{diag}(\hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_-, \hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_-, \hat{\rho}_+, \hat{\rho}_+),$$

$$\hat{\rho}_{f\mp} = \hat{\rho}_- 1_{(4)} \otimes \begin{pmatrix} 1_{(2)} & & & \\ & 0 & & \\ & & 0 & \\ & & & 1_{(2)} \end{pmatrix} + \hat{\rho}_+ 1_{(4)} \otimes \begin{pmatrix} 0 & & & \\ & 1_{(2)} & & \\ & & 1_{(2)} & \\ & & & 0 \end{pmatrix}. \tag{4.28}$$

This is the case considered in Refs. [20,21] in the context of M theory compactification to the lightcone heterotic strings [with ϵ_0, ϵ_1, $\overline{\epsilon_2}$ and $\overline{\epsilon_3}$ in Eq. (4.25) all real].

V. THE ROLE OF THE FUNDAMENTAL REPRESENTATION AND ANOMALY CANCELLATION OF WORLDVOLUME THEORY

So far, we have ignored the fields in the fundamental representation. These fields do not contribute to the diagrams in spherical topology. They are irrelevant to the questions concerning the spacetime coordinates. They create, however, disk diagrams and higher genera with boundaries and are responsible for creating an open string sector. This is in fact required, as nonorientable closed strings by themselves are not consistent. It is well known that the simplest way to establish the consistency is through the (global) cancellation of dilaton tadpoles between disk and RP^2 diagrams [22,23], leading to the $SO(32)$ Chan-Paton factor. This survives toroidal compactifications with or without discrete projection [24]. It should be that the sum of an infinite set of diagrams of the matrix model contributing to the disk/RP^2 geometry yields the string partition function of the disk/RP^2 diagram. The Chan-Paton trace at the boundary corresponds to the trace with respect to the flavor index. The n_f should therefore be fixed by the tadpole cancellation. The flavor symmetry of the model is the local gauge symmetry of strings.

The lack of the combinatorial argument and the absence of the vertex operator construction at this moment, however, prevent us from proceeding to such calculations via matrices. Instead, we will examine gauge anomalies of worldvolume theories by taking the T dual and subsequently the zero volume limit of $T^6/\mathcal{Z}^2$. In particular, let us do this for all six adjoint directions. The resulting theory is the six-dimensional worldvolume gauge theory obeying Eq. (3.21) with matter in the antisymmetric and fundamental representation. This is the type I superstrings in ten spacetime dimensions. This case is also the first nontrivial case of getting a potentially anomalous theory. In fact, by acting

$$\Gamma_{(6)} \equiv \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^7 \tag{5.1}$$

on Ψ, we see that the adjoint fermions λ and ψ_1 have chirality plus while $\psi_{2,3}$ have chirality minus. The fermions in the fundamental representation have chirality minus. The standard technology to compute non-Abelian anomalies is provided by the family's index theorem and the descent equations [11,25]. We find that the condition for the anomaly cancellation

$$\begin{aligned}\mathrm{tr}_{\mathrm{adj}}F^4 - \mathrm{tr}_{\mathrm{asym}}F^4 - n_f \mathrm{tr}F^4 &= (2k+8)\mathrm{tr}F^4 + 3(\mathrm{tr}F^2)^2 \\ &\quad - [(2k-8)\mathrm{tr}F^4 + 3(\mathrm{tr}F^2)^2] - n_f\mathrm{tr}F^4 \\ &= (16-n_f)\mathrm{tr}F^4 = 0,\end{aligned} \tag{5.2}$$

where we have indicated the traces in the respective representations. The case $n_f=16$ is selected by the consistency of the theory. In the case discussed in Eq. (4.28), we conclude from similar calculations that the anomaly cancellation of the worldvolume two-dimensional gauge theory selects 16 complex fermions.

VI. ONE-LOOP EFFECTIVE ACTION AND D-STRING SOLUTIONS

A. One-loop effective action

In this subsection, we will establish a formula for the one-loop effective action of the USp matrix model on a generic bosonic background.[2] Let us first find one-loop fluctuations on a generic classical solution of the $USp(2k)$ matrix model. We write

$$\begin{aligned} v_m = p_m + g a_m, \quad (m=0\sim 3), \quad \lambda = \chi_0 + g\phi_0, \\ v_I = p_I + g a_I, \quad (I=4\sim 9), \quad \psi_i = \chi_i + g\phi_i, \quad (i=1\sim 3)\end{aligned} \tag{6.1}$$

with $(p_m, p_i, \chi_0, \chi_i)$ a configuration satisfying equations of motion. In order to fix the gauge invariance we add the ghost and the gauge fixing term

$$S_{gfgh} = \frac{1}{2}\,\mathrm{Tr}([p_M, a^M]^2 - [p^K, b][p_K, c]), \tag{6.2}$$

where c and b are, respectively, the ghosts and the antighosts lying in the adjoint representation of $USp(2k)$. Denote by $S^{(2)}$ the part in $S_{\mathrm{adj+asym}}$ which is quadratic in a and ϕ. The one-loop effective action $W_{\text{one-loop}}$ is

$$W_{\text{one-loop}} = -i\,\log \int [da_m][da_I][d\phi_0][d\overline{\phi}_0][d\phi_i][d\overline{\phi}_i][dc][db]\exp[iS^{(2)} + iS_{gfgh}]. \tag{6.3}$$

Instead of resorting to the direct Gaussian integrations of the expression above, let us use Eqs. (2.26) and (2.27).

In the same way as Eq. (6.1), we decompose $\underline{v}_M$ and $\underline{\Psi}$ into the backgrounds and the quantum fluctuations. Let us denote the fluctuations by $\underline{v}_M^{(fl)}$ and $\underline{\Psi}^{(fl)}$. Then from Eq. (2.26) we have

$$S^{(2)} = S_{\mathcal{N}=1}^{d=10\;(2)}(\hat{\rho}_{b\mp}\underline{v}_M^{(fl)}, \hat{\rho}_{f\mp}\underline{\Psi}^{(fl)}), \tag{6.4}$$

where $S_{\mathcal{N}=1}^{d=10\;(2)}(\hat{\rho}_{b\mp}\underline{v}_M^{(fl)}, \hat{\rho}_{f\mp}\underline{\Psi}^{(fl)})$ is the part in the action of $d=10$, $\mathcal{N}=1$ super Yang-Mills theory which is quadratic in the fluctuations. As the variables are explicitly projected either onto $USp(2k)$ adjoint or onto antisymmetric matrices, we can safely replace the integration measure by that of the $u(2k)$ Lie algebra valued matrices. We obtain

$$\begin{aligned} W_{\text{one-loop}} &= -i\,\log \int [d\underline{v}_M^{(fl)}][d\underline{\Psi}^{(fl)}][d\underline{c}][d\underline{b}]\exp[iS_{\mathcal{N}=1}^{d=10\;(2)}(\hat{\rho}_{b\mp}\underline{v}_M^{(fl)}, \hat{\rho}_{f\mp}\underline{\Psi}^{(fl)}) + iS_{gfgh}(\hat{\rho}_-\underline{b}, \hat{\rho}_-\underline{c})] \\ &= \frac{1}{2}\log\det(\mathcal{O}_b\hat{\rho}_{b\mp}) - \frac{1}{2}\log\det\left(\mathcal{O}_f\hat{\rho}_{f\mp}\left(\frac{1+\Gamma_{11}}{2}\right)\right) - \log\det(\hat{P}_K\hat{P}^K\hat{\rho}_-), \end{aligned} \tag{6.5}$$

where

$$\mathcal{O}_{bL}^{M} = -\delta_L{}^M\hat{P}_K\hat{P}_K + 2i\hat{F}_L{}^M, \quad \mathcal{O}_f = -\Gamma_M\hat{P}^M, \tag{6.6}$$

[2]The solutions we will construct in the next subsection and in Sec. VII are relevant only in the large k limit. We will, therefore, ignore the fields lying in the fundamental representation.

$$\hat{P}_K \bullet = [p_K, \bullet], \quad \hat{F}_{KL} \bullet = i[[p_K, p_L], \bullet]. \tag{6.7}$$

In obtaining Eq. (6.6), we have set all fermionic backgrounds χ_0 and χ_i to zero. As a consequence, the one-loop effective action on a generic bosonic background is given by[3]

$$W_{\text{one-loop}} = \left(\frac{6}{2} - \frac{4}{2} - 1\right) \mathrm{Tr}\,\log(\hat{P}_K \hat{P}^K \hat{\rho}_-) + \left(\frac{4}{2} - \frac{4}{2}\right) \mathrm{Tr}\,\log(\hat{P}_K \hat{P}^K \hat{\rho}_+) + W_b + W_f, \tag{6.8}$$

$$W_b = \frac{1}{4}\,\mathrm{Tr}\,\log\left[\left(\delta_L{}^M + \frac{4}{(\hat{P}_K \hat{P}^K)^2}\,\hat{F}_L{}^N \hat{F}_N{}^M\right)\hat{\rho}_{b\mp}\right], \tag{6.9}$$

$$W_f = -\frac{1}{4}\,\mathrm{Tr}\,\log\left[\left(1 + \frac{i}{2\hat{P}_K \hat{P}^K}\,\Gamma^{MN}\hat{F}_{MN}\right) \times \hat{\rho}_{f\mp}\left(\frac{1+\Gamma_{11}}{2}\right)\right]. \tag{6.10}$$

We put the matrix $\hat{F}_{MN}$ into the following form with respect to the Lorentz indices:

$$\hat{F}_{MN} = \begin{pmatrix} 0 & -\hat{B}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{B}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hat{B}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{B}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\hat{B}_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\hat{B}_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\hat{B}_5 \\ 0 & 0 & 0 & 0 & \hat{B}_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{B}_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_5 & 0 & 0 & 0 \end{pmatrix}. \tag{6.11}$$

When the classical configuration is BPS saturated, $\hat{F}_{MN}=0$ and $W_{\text{one-loop}}$ vanishes.

B. D-string solution

Let us construct a few particular classical bosonic solutions of the model. We set the fields lying in the fundamental representation of $USp(2k)$ to zero. The equation of motion is

$$[p_N, [p^M, p^N]] = 0. \tag{6.12}$$

There are three cases of solutions representing a D-string configuration, depending upon which two directions the worldsheet extends to infinity. When both of the directions are the adjoint directions, say v_0 and v_1, the nonvanishing components are

$$p_0 = \left(\frac{1+\sigma^3}{2}\right) \otimes \mathbf{x} + \left(\frac{1-\sigma^3}{2}\right) \otimes (-\mathbf{x}^t),$$
$$p_1 = \left(\frac{1+\sigma^3}{2}\right) \otimes \boldsymbol{\pi} + \left(\frac{1-\sigma^3}{2}\right) \otimes (-\boldsymbol{\pi}^t). \tag{6.13}$$

When both are in the antisymmetric directions, say v_5 and v_8, the nonvanishing components are

$$p_5 = \left(\frac{1+\sigma^3}{2}\right) \otimes \mathbf{x} + \left(\frac{1-\sigma^3}{2}\right) \otimes \mathbf{x}^t,$$
$$p_8 = \left(\frac{1+\sigma^3}{2}\right) \otimes \boldsymbol{\pi} + \left(\frac{1-\sigma^3}{2}\right) \otimes \boldsymbol{\pi}^t. \tag{6.14}$$

When one is in the adjoint direction, say v_0, and the other is in the antisymmetric direction, say v_8,

$$p_0 = \left(\frac{1+\sigma^3}{2}\right) \otimes \mathbf{x} + \left(\frac{1-\sigma^3}{2}\right) \otimes (-\mathbf{x}^t),$$
$$p_8 = \left(\frac{1+\sigma^3}{2}\right) \otimes \boldsymbol{\pi} + \left(\frac{1-\sigma^3}{2}\right) \otimes \boldsymbol{\pi}^t. \tag{6.15}$$

In above expressions, $\mathbf{x}$ and $\boldsymbol{\pi}$ are infinite size matrices with the commutator $[\boldsymbol{\pi}, \mathbf{x}] = -i$.

Let us now turn to the solutions representing two parallel D-strings and two antiparallel D-strings. We will illustrate

[3] The calculation in what follows parallels those of Refs. [5,26].

this by the most interesting case that the two D strings are extended in the two directions (v_5 and v_8) of antisymmetric representations separated by d in the v_4 direction which is the adjoint direction. The nonvanishing components are

$$p_5=\left(\frac{1+\sigma^3}{2}\right)\otimes\begin{pmatrix}\mathbf{x} & 0\\ 0 & \mathbf{x}\end{pmatrix}+\left(\frac{1-\sigma^3}{2}\right)\otimes\begin{pmatrix}\mathbf{x}^t & 0\\ 0 & \mathbf{x}^t\end{pmatrix},$$

$$p_8=\left(\frac{1+\sigma^3}{2}\right)\otimes\begin{pmatrix}\boldsymbol{\pi} & 0\\ 0 & \boldsymbol{\pi}\end{pmatrix}+\left(\frac{1-\sigma^3}{2}\right)\otimes\begin{pmatrix}\boldsymbol{\pi}^t & 0\\ 0 & \boldsymbol{\pi}^t\end{pmatrix}, \tag{6.16}$$

$$p_4=\left(\frac{1+\sigma^3}{2}\right)\otimes\begin{pmatrix}-d/2 & 0\\ 0 & d/2\end{pmatrix}+\left(\frac{1-\sigma^3}{2}\right)\otimes\begin{pmatrix}d/2 & 0\\ 0 & -d/2\end{pmatrix},$$

for two parallel D-strings, and

$$p_5=\left(\frac{1+\sigma^3}{2}\right)\otimes\begin{pmatrix}\mathbf{x} & 0\\ 0 & \mathbf{x}\end{pmatrix}+\left(\frac{1-\sigma^3}{2}\right)\otimes\begin{pmatrix}\mathbf{x}^t & 0\\ 0 & \mathbf{x}^t\end{pmatrix},$$

$$p_8=\left(\frac{1+\sigma^3}{2}\right)\otimes\begin{pmatrix}\boldsymbol{\pi} & 0\\ 0 & -\boldsymbol{\pi}\end{pmatrix}+\left(\frac{1-\sigma^3}{2}\right)\otimes\begin{pmatrix}\boldsymbol{\pi}^t & 0\\ 0 & -\boldsymbol{\pi}^t\end{pmatrix}, \tag{6.17}$$

$$p_4=\left(\frac{1+\sigma^3}{2}\right)\otimes\begin{pmatrix}-d/2 & 0\\ 0 & d/2\end{pmatrix}+\left(\frac{1-\sigma^3}{2}\right)\otimes\begin{pmatrix}d/2 & 0\\ 0 & -d/2\end{pmatrix},$$

for two antiparallel D-strings.

C. Force between antiparallel D-strings

We would like to determine the scale of our spacetime given by the model. This can be done by computing the force mediating two classical objects which are by themselves a non-Bogomoln'yi-Prasad-Sommerfield (BPS) configuration. We will evaluate the W_b and the W_f in the case of the two antiparallel D-strings separated by distance d, which have been constructed in the preceding subsection. We compute the force exerting with each other. From Eq. (6.17) we have $\hat{P}^0=\hat{P}^1=\hat{P}^2=\hat{P}^3=\hat{P}^6=\hat{P}^7=\hat{P}^9=0$, $\hat{B}_1=\hat{B}_2=\hat{B}_3=\hat{B}_5=0$, $\hat{P}_K\hat{P}^K=(\hat{P}^4)^2+(\hat{P}^5)^2+(\hat{P}^8)^2$, $\hat{P}^4=(d/2)\hat{B}_4$ and, after some algebra, we obtain

$$[\hat{P}^5,\hat{P}^8]=-i\hat{B}_4,\quad [\hat{P}^4,\hat{P}^5]=0,\quad [\hat{P}^4,\hat{P}^8]=0. \tag{6.18}$$

When we take trace with Lorentz indices in Eq. (6.9) and with spinor indices in Eq. (6.10), we arrive at the following expressions:

$$W_b=\frac{1}{2}\,\mathrm{Tr}\left[\log\left(1-\frac{4\hat{B}_4\hat{B}_4}{(\hat{P}_K\hat{P}^K)^2}\right)\hat{\rho}_+\right], \tag{6.19}$$

$$W_f=-\mathrm{Tr}\left[\log\left(1-\frac{1}{(\hat{P}_K\hat{P}^K)^2}\,\hat{B}_4\hat{B}_4\right)\hat{\rho}_- +\log\left(1-\frac{1}{(\hat{P}_K\hat{P}^K)^2}\,\hat{B}_4\hat{B}_4\right)\hat{\rho}_+\right]. \tag{6.20}$$

In the Appendix, the eigenvalues of $\hat{B}_4\hat{B}_4$, their degeneracies and the eigenmatrices are determined. We compile the results in Table I for the antisymmetric eigenmatrices and in Table II for the adjoint eigenmatrices. (See tables in the Appendix.)

Using these tables, we obtain

$$W_b=\frac{k}{2}\sum_{n=0}^{\infty}\log\left(1-\frac{16}{(d^2+4n+2)^2}\right), \tag{6.21}$$

$$W_f=-2k\sum_{n=0}^{\infty}\log\left(1-\frac{4}{(d^2+4n+2)^2}\right). \tag{6.22}$$

Putting all these together, we find

$$W_{\text{one-loop}}=-\frac{k}{2}\log\left[\left(\frac{d^2}{4}\right)^{-4}\frac{d^2/4+1/2}{d^2/4-1/2}\left(\frac{\Gamma\left(\frac{d^2}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{d^2}{4}\right)}\right)^8\right]=-\frac{k}{2}\left\{\frac{8}{d^6}+\mathcal{O}(d^{-8})\right\}. \tag{6.23}$$

This potential provides the asymptotic behavior of the force mediating two antiparallel D-strings. From this we conclude that the dimension of spacetime is ten at least in this naive large k limit.

VII. CONSTRUCTION OF D3-BRANE SOLUTIONS

It is not difficult to extend the construction of the D-string solutions in the previous section to general Dp-brane solutions. We will illustrate this by a D3-brane, two parallel D3-branes and multiple D3-branes which are parallel to one another.

Let us first consider a D3-brane solution. When the worldvolume extends in v_5, v_8, v_6, and v_9 directions, the nonvanishing components are given by

$$\begin{aligned} p_5 &= \left(\frac{1+\sigma^3}{2}\right)\otimes \mathbf{x_1} + \left(\frac{1-\sigma^3}{2}\right)\otimes \mathbf{x_1^t},\\ p_8 &= \left(\frac{1+\sigma^3}{2}\right)\otimes \boldsymbol{\pi}_1 + \left(\frac{1-\sigma^3}{2}\right)\otimes \boldsymbol{\pi}_1^{\mathbf{t}},\\ p_6 &= \left(\frac{1+\sigma^3}{2}\right)\otimes \mathbf{x_2} + \left(\frac{1-\sigma^3}{2}\right)\otimes \mathbf{x_2^t},\\ p_9 &= \left(\frac{1+\sigma^3}{2}\right)\otimes \boldsymbol{\pi}_2 + \left(\frac{1-\sigma^3}{2}\right)\otimes \boldsymbol{\pi}_2^{\mathbf{t}}. \end{aligned} \tag{7.1}$$

It is straightforward to check that this configuration satisfies the equation of motion. In the above expression, $\mathbf{x_1}$, $\mathbf{x_2}$, $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are operators (infinite matrices) with the commutators

$$[\boldsymbol{\pi}_1,\mathbf{x_1}] = -i\sqrt{\frac{V_4}{k}}, \quad [\boldsymbol{\pi}_2,\mathbf{x_2}] = -i\sqrt{\frac{V_4}{k}}. \tag{7.2}$$

Here we must take the limit of $k\to\infty$ with V_4/k fixed to $(\alpha')^2$.

Now let us calculate the value of the action. We have

$$[p^5,p^8] = \sigma^3\otimes i\alpha' 1_k, \quad [p^6,p^9] = \sigma^3\otimes i\alpha' 1_k. \tag{7.3}$$

When we substitute these into the action,

$$\begin{aligned} S &= \frac{1}{g^2(\alpha')^2}\,\mathrm{Tr}\left(\frac{1}{2}[p^5,p^8][p_5,p_8] + \frac{1}{2}[p^6,p^9][p_6,p_9]\right)\\ &\sim \frac{1}{g^2(\alpha')^2}\,V_4 = T_{3\text{-brane}}V_4. \end{aligned} \tag{7.4}$$

Here g^2 is regarded as string coupling g_{st}. This is consistent with the D-brane action which is given by the tension times the volume of the D-brane. Therefore it is appropriate to think of the above solution as a D3-brane solution.

Next, take two parallel D3-branes which are separated by distance d in the v_4 direction. The nonvanishing components are

$$\begin{aligned} p_5 &= \left(\frac{1+\sigma^3}{2}\right)\otimes \begin{pmatrix}\mathbf{x_1} & 0\\ 0 & \mathbf{x_1}\end{pmatrix} + \left(\frac{1-\sigma^3}{2}\right)\otimes \begin{pmatrix}\mathbf{x_1^t} & 0\\ 0 & \mathbf{x_1^t}\end{pmatrix},\\ p_8 &= \left(\frac{1+\sigma^3}{2}\right)\otimes \begin{pmatrix}\boldsymbol{\pi}_1 & 0\\ 0 & \boldsymbol{\pi}_1\end{pmatrix} + \left(\frac{1-\sigma^3}{2}\right)\otimes \begin{pmatrix}\boldsymbol{\pi}_1^{\mathbf{t}} & 0\\ 0 & \boldsymbol{\pi}_1^{\mathbf{t}}\end{pmatrix},\\ p_6 &= \left(\frac{1+\sigma^3}{2}\right)\otimes \begin{pmatrix}\mathbf{x_2} & 0\\ 0 & \mathbf{x_2}\end{pmatrix} + \left(\frac{1-\sigma^3}{2}\right)\otimes \begin{pmatrix}\mathbf{x_2^t} & 0\\ 0 & \mathbf{x_2^t}\end{pmatrix},\\ p_9 &= \left(\frac{1+\sigma^3}{2}\right)\otimes \begin{pmatrix}\boldsymbol{\pi}_2 & 0\\ 0 & \boldsymbol{\pi}_2\end{pmatrix} + \left(\frac{1-\sigma^3}{2}\right)\otimes \begin{pmatrix}\boldsymbol{\pi}_2^{\mathbf{t}} & 0\\ 0 & \boldsymbol{\pi}_2^{\mathbf{t}}\end{pmatrix},\\ p_4 &= \left(\frac{1+\sigma^3}{2}\right)\otimes \begin{pmatrix}-d/2 & 0\\ 0 & d/2\end{pmatrix} + \left(\frac{1-\sigma^3}{2}\right)\otimes \begin{pmatrix}d/2 & 0\\ 0 & -d/2\end{pmatrix}. \end{aligned} \tag{7.5}$$

Finally let us consider N parallel D3-branes which are separated in the v_4 and v_7 directions. We denote the position of the ith D3-brane by $v_4 = d_4^{(i)}$ and $v_7 = d_7^{(i)}$. The worldvolume extends in the v_5, v_8, v_6 and v_9 directions. The nonvanishing components are

$$
\begin{aligned}
p_5&=\left(\frac{1+\sigma^3}{2}\right)\otimes\begin{pmatrix}\mathbf{x_1}&&\\&\ddots&\\&&\mathbf{x_1}\end{pmatrix}+\left(\frac{1-\sigma^3}{2}\right)\otimes\begin{pmatrix}\mathbf{x_t^t}&&\\&\ddots&\\&&\mathbf{x_t^t}\end{pmatrix},\\
p_8&=\left(\frac{1+\sigma^3}{2}\right)\otimes\begin{pmatrix}\boldsymbol{\pi}_1&&\\&\ddots&\\&&\boldsymbol{\pi}_1\end{pmatrix}+\left(\frac{1-\sigma^3}{2}\right)\otimes\begin{pmatrix}\boldsymbol{\pi}_1^t&&\\&\ddots&\\&&\boldsymbol{\pi}_1^t\end{pmatrix},\\
p_6&=\left(\frac{1+\sigma^3}{2}\right)\otimes\begin{pmatrix}\mathbf{x_2}&&\\&\ddots&\\&&\mathbf{x_2}\end{pmatrix}+\left(\frac{1-\sigma^3}{2}\right)\otimes\begin{pmatrix}\mathbf{x_2^t}&&\\&\ddots&\\&&\mathbf{x_2^t}\end{pmatrix},\\
p_9&=\left(\frac{1+\sigma^3}{2}\right)\otimes\begin{pmatrix}\boldsymbol{\pi}_2&&\\&\ddots&\\&&\boldsymbol{\pi}_2\end{pmatrix}+\left(\frac{1-\sigma^3}{2}\right)\otimes\begin{pmatrix}\boldsymbol{\pi}_2^t&&\\&\ddots&\\&&\boldsymbol{\pi}_2^t\end{pmatrix},\\
p_4&=\left(\frac{1+\sigma^3}{2}\right)\otimes\begin{pmatrix}d_4^{(1)}&&\\&\ddots&\\&&d_4^{(N)}\end{pmatrix}+\left(\frac{1-\sigma^3}{2}\right)\otimes\begin{pmatrix}-d_4^{(1)}&&\\&\ddots&\\&&-d_4^{(N)}\end{pmatrix},\\
p_7&=\left(\frac{1+\sigma^3}{2}\right)\otimes\begin{pmatrix}d_7^{(1)}&&\\&\ddots&\\&&d_7^{(N)}\end{pmatrix}+\left(\frac{1-\sigma^3}{2}\right)\otimes\begin{pmatrix}-d_7^{(1)}&&\\&\ddots&\\&&-d_7^{(N)}\end{pmatrix}.
\end{aligned}
\tag{7.6}
$$

VIII. *F* THEORY ON AN ELLIPTIC FIBERED *K*3

We will now show that the model is able to describe the F theory compactification on an elliptic fibered $K3$ [14,27]. Our objective here is to demonstrate that the matrix model in fact derives one of the very few exact results in critical string theory. While the original construction of Vafa is purely geometrical in nature, our model provides an action principle and path integrals to the F theory compactification.

In Secs. IV, V, and VI, we have seen that our model is the matrix model of type IIB superstrings on a large $T^6/\mathcal{Z}^2$ orientifold. The coupling constant has no spacetime dependence and is a *bona fide* parameter. One can make the coupling space-dependent by taking the matrix T dual in various ways to go to higher dimensional worldvolume gauge theories as we have already discussed in the previous sections. The coupling constant then starts running with the coordinates labelling the quantum moduli space, i.e., VEV, which is denoted by $\vec{u}$. This is in accordance with the marginal scalar deformation of the original action to a type of nonlinear σ model. The background field appearing through this procedure is a massless axion-dilaton field. The running coupling constant is, therefore, identified as the space-dependent axion-dilaton background field $\lambda(\vec{u})$.

Let $\vec{u}$ be the complex coordinates on a complex n-dimensional base space B_n. F theory compactification of an elliptically fibered $C-Y$ $(n+1)$ fold M_{n+1} on the base B_n is defined by saying that the u-dependent axion-dilaton background field of type IIB superstrings on $B_n\times R^{9-2n.1}$ is the modular parameter of the fiber T^2 as a function of $\vec{u}$. We would like to show that this is in fact the case in our matrix model. To provide F theory setup as a reduced model for the case $n=1$, we are going to send the period R of the four out of the six adjoint directions v_0, v_1, v_2, v_3 to zero and to take the matrix T dual. The resulting model in the limit of vanishing mass parameters is type IIB on a large $T^2/\mathcal{Z}^2$ orientifold, namely on CP^1, equipped with sixteen D7-branes. Coupling starts running as we turn on the mass parameters. Following Sen [27], we would now like to take the scaling limit

$$
\begin{aligned}
&\tilde{R}\to\infty,\\
&m_i\tilde{R}\to\text{finite}\quad i=1,\sim 4,\\
&m_i\tilde{R}\to\infty\quad i=5,\sim 16,
\end{aligned}
\tag{8.1}
$$

simultaneously taking the matrix T dual. The second and the third lines of this equation come from the consistency with the RR charge counting. The resulting worldvolume theory around one of the four $O7$'s is the $d=4$, $\mathcal{N}=2$ supersymmetric $USp(2k)$ gauge theory with one massless antisymmetric hypermultiplet and four fundamental hypermultiplets with masses m_i. The special properties of this theory valid

TABLE I. Results for the antisymmetric eigenmatrices.

The eigenvalue of $\hat{B}_4\hat{B}_4$	The degeneracy
4	k^2-k
0	k^2
The eigenvalue of $\hat{P}_K\hat{P}^K$	**The degeneracy**
d^2+4n+2	k

TABLE II. Results for the adjoint eigenmatrices.

The eigenvalues of $\hat{B}_4\hat{B}_4$	The degeneracy
4	k^2+k
0	k^2
The eigenvalue of $\hat{P}_K\hat{P}^K$	**The degeneracy**
d^2+4n+2	k

for all k are that it is UV finite and that at least low energy physics is the same for all k [28]. One can, therefore, deduce the u dependence of the coupling of the model in the large k limit by simply looking at the $k=2$ case, namely, the $SU(2)$ SUSY gauge theory with four flavors. The u dependence of the coupling λ is supplied by the work of Seiberg-Witten (SW) [29]. The work of Sen [27] shows that the way the modular parameter of the bare torus in the massless limit is dressed by the four mass parameters in the SW curve of the massive four flavor case is mathematically identical to the description of F theory in the neighborhood of the constant coupling. One can therefore safely conclude that the coupling $\lambda(u)$ of the model is in fact the modular parameter of the spectral torus. This is what we wanted to show.

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APPENDIX

In this appendix we will determine the eigenvalues of the operators $\hat{B}_4\hat{B}_4$ and $\hat{P}_K\hat{P}_K$. We consider both cases that the eigenmatrices are in the adjoint and the antisymmetric representations in $USp(2k)$. These eigenvalues and their degeneracy are needed in order to calculate the one-loop effective action.

Suppose that an operator $\hat{O}$ has an adjoint action on a $2k\times 2k$ matrix a:

$$\hat{O}a=[o,a]. \tag{A1}$$

Here o is the $2k\times 2k$ matrix. Let us first consider the case that the matrix a is given by Eq. (2.6). Note that the operator $\hat{B}_4=i[\hat{P}_5,\hat{P}_8]$ is represented by the matrix $b_4=-\sigma^3\otimes\sigma^3\otimes\mathbf{1}_{(\mathbf{k/2})}$.

It is not difficult to see that the eigenvalues of $\hat{B}_4\hat{B}_4$ are either 0 or 4. For the 0 eigenvalue we simply solve $\hat{B}_4 a^{(0)}_{(\mathrm{asym})}=0$ and the eigenmatrices are

$$\begin{aligned}&\left(\frac{1+\sigma^3}{2}\right)\otimes\mathbf{1}_{(2)}\otimes H_0+\left(\frac{1-\sigma^3}{2}\right)\otimes(\mathbf{1}_{(2)}\otimes H_0)^t,\\&\left(\frac{1+\sigma^3}{2}\right)\otimes\sigma^3\otimes H_3+\left(\frac{1-\sigma^3}{2}\right)\otimes(\sigma^3\otimes H_3)^t,\\&\sigma^+\otimes\sigma^1\otimes A_1+\sigma^-\otimes\{-(\sigma^1\otimes A_1)^*\},\\&\sigma^+\otimes\sigma^2\otimes A_2+\sigma^-\otimes\{-(\sigma^2\otimes A_2)^*\}.\end{aligned} \tag{A2}$$

Since the $(k/2)\times(k/2)$ matrices satisfy $H^\dagger_{0,3}=H_{0,3}$, $A_1^t=-A_1$ and $A_2^t=A_2$, the degeneracy is k^2. As for the eigenvalue 4, the solution is

$$\begin{aligned}&\left(\frac{1+\sigma^3}{2}\right)\otimes(\sigma^1\otimes H_1+\sigma^2\otimes H_1)+\left(\frac{1-\sigma^3}{2}\right)\otimes(\sigma^1\otimes H_1+\sigma^2\otimes H_1)^t,\\&\left(\frac{1+\sigma^3}{2}\right)\otimes(\sigma^1\otimes H_2-\sigma^2\otimes H_2)+\left(\frac{1-\sigma^3}{2}\right)\otimes(\sigma^1\otimes H_2-\sigma^2\otimes H_2)^t,\\&\sigma^+\otimes(\mathbf{1}_{(2)}\otimes A_0+\sigma^3\otimes A_0)+\sigma^-\otimes(-(1_{(2)}\otimes A_0+\sigma^3\otimes A_0)^*),\\&\sigma^+\otimes(\mathbf{1}_{(2)}\otimes A_3-\sigma^3\otimes A_3)+\sigma^-\otimes(-(\mathbf{1}_{(2)}\otimes A_3-\sigma^3\otimes A_3)^*),\end{aligned} \tag{A3}$$

and the degeneracy is k^2-k because of $H^\dagger_{1,2}=H_{1,2}$ and $A^t_{0,3}=-A_{0,3}$.

Let us now calculate the eigenvalues of the operator $\hat{P}_K\hat{P}^K=\frac{1}{4}\hat{B}_4\hat{B}^4+\hat{P}_5\hat{P}^5+\hat{P}_8\hat{P}^8$. Clearly $\hat{B}_4\hat{B}_4$ and $\hat{P}_5\hat{P}^5+\hat{P}_8\hat{P}^8$ are simultaneously diagonalized. When $\hat{P}_5\hat{P}^5+\hat{P}_8\hat{P}^8$ acts on the eigenstates with eigenvalue 4 of $\hat{B}_4\hat{B}_4$, we replace $\hat{B}_4\hat{B}_4$ by its eigenvalue. Let $\hat{P}\equiv\hat{P}_5\hat{B}_4/2\sqrt{2}$ and $\hat{Q}\equiv\hat{P}_8/\sqrt{2}$. We obtain

$$[\hat{P},\hat{Q}]=-i.$$

The eigenvalues of $\hat{P}_5\hat{P}^5+\hat{P}_8\hat{P}^8=2(\hat{P}\hat{P}+\hat{Q}\hat{Q})$ are those of the harmonic oscillator and are given by $4n+2$ with integer n. The degeneracy is k for large k. We summarize the results in Table I. Our calculation of the effective action does not require the case in which the eigenvalue of $\hat{B}_4\hat{B}_4$ is zero.

Similarly, the eigenmatrices lying in the adjoint representation [Eq. (2.4)] can be determined. The difference is the off-diagonal degrees of freedom, which change the degeneracy of $\hat{B}_4\hat{B}_4$ eigenvalues. The degeneracy of the $\hat{P}_K\hat{P}^K$ eigenvalues is the same as in the previous case. Summing up the adjoint case, we obtain Table II.

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