

Local BRST cohomology of the gauged principal nonlinear sigma model

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The local BRST cohomology of the gauged nonlinear sigma model on a group manifold is worked out for any Lie group G . We consider both the case where the gauge field is dynamical and the case where it has no kinetic term (G/G topological theory). Our results shed novel light on the problem of gauging the WZW term as well as on the nature of the topological terms introduced a few years ago by DeWit, Hull, and Roček. We also consider the BRST cohomology of the rigid symmetries of the ungauged model and recover the results of D'Hoker and Weinberg on the most general effective actions compatible with the symmetries. [S0556-2821(98)06014-7]

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I. INTRODUCTION

A central theorem in the renormalization of Yang-Mills gauge models interacting with matter is that the most general solution of the Becchi-Rouet-Stora-Tyutin (BRST) invariance condition $s \int dx m = 0$ is given, up to trivial terms, by the integral of a gauge invariant polynomial in the field strengths, the matter fields and their covariant derivatives (in odd dimensions, there are also Chern-Simons terms). This theorem guarantees that all the divergences appearing in the quantum theory can be absorbed by counterterms that respect the original symmetry, making the theory renormalizable in the “modern sense” in any number of spacetime dimensions [1,2] (for related, but different ideas, see [3]). It also guarantees that gauge invariant operators can be renormalized in a gauge independent way.

The theorem, conjectured in [4], was recently proved in [5,6] through cohomological arguments (see [7] for earlier developments). Its interest transcends the question of renormalization since the BRST invariance condition also determines the allowed deformations of the action, i.e. the terms that can be consistently added to the classical, gauge invariant action while maintaining the number of (possibly deformed) gauge symmetries [8].

An important assumption made in [6] was, however, that the matter fields transform in linear representations of the symmetry. Now, nonlinear realizations are also important since nonlinear sigma models coupled to gauge fields occur in supergravity and string theory as well as in the effective description of low energy interactions among hadrons.

In this paper, we generalize the above theorem to the case of the gauged nonlinear sigma model on a group manifold G (gauged principal sigma model). For definiteness, the right action of G on itself will be gauged. To simplify the discussion, the spacetime manifold is assumed to be homeomor-

phic to R^n , leaving only the group manifold as a source for non-trivial topology.¹ We show that the most general deformation of the action is, up to trivial redefinitions, the integral of a strictly gauge invariant term plus winding number terms. [Winding number terms are characterized by two features: (i) They involve only group-valued fields, and (ii) they do not contribute to the field equations but are not exact in field space and hence cannot be eliminated globally by adding a total derivative. These terms are related to the De Rham cohomology of the group manifold—see Sec. V.] In particular, we recover from a different perspective the fact that there is no room in the principal case for the gauged Wess-Zumino-Witten term. Furthermore, we verify explicitly that the Chern-Simons terms actually differ from strictly gauge-invariant terms by (non-invariant) total derivatives plus, possibly, winding number terms, even when the Lie algebra cohomology of \mathcal{G} is non-trivial (\mathcal{G} denotes the Lie algebra of G). This property also holds for the topological terms described in [10], which are equivalent to winding number terms plus strictly gauge-invariant terms in the principal case considered here.

At the quantum level, our result implies that the coupled Yang-Mills–nonlinear- σ -model in any number of spacetime dimensions, even though generically not power-counting renormalizable, is renormalizable in the “modern sense” of [2]. Note that in perturbation theory, it is customary to restrict the fields to a neighborhood of the identity, so that the winding number terms, which are locally trivial, may be dropped.

We also compute the BRST cohomology for other values of the ghost degree. This is relevant for the problem of anomalies in Yang-Mills theory since the Wess-Zumino compensating field precisely transforms nonlinearly as a group element under gauge transformations [11,12]. Finally, the cohomology of the ungauged model is analyzed, which enables us to recover from a different angle the results of

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¹For a discussion on how to take into account the spacetime topology (restricted to product bundles), see [9].

D'Hoker and Weinberg on the most general effective actions compatible with the rigid G -symmetry of the σ -model [13].

II. MODEL

The starting point of our analysis is a general action of the form

$$S_0[A_\mu^a, g, y^i] = \int \mathcal{L}(A_\mu^a, g, y^i) dx, \quad (2.1)$$

where A_μ^a denotes a Yang-Mills connection ($a=1, \dots, N$) and g is an element of the corresponding Lie group G . We assume that g belongs to some faithful $k \times k$ matrix representation of G and adopt matrix notations throughout for g . Unless $G=GL(k)$, the matrix elements of g are not independent. One may express them in terms of local coordinates h^a on the group, $g=g(h^a)$, but because this can usually not be done globally, we shall avoid explicit parametrizations. The y^i stand for matter fields that transform linearly under some representation of G with generators $(Y_a)_j^i$. We shall also often adopt matrix notations for Y_a , viewing the y 's as column-vectors.

At this stage, we do not specify the exact form of the action but only assume that it is invariant under the following gauge transformations:

$$\delta_\epsilon A_\mu^a = \partial_\mu \epsilon^a + f^a_{bc} A_\mu^b \epsilon^c, \quad (2.2)$$

$$\delta_\epsilon g = g T_a \epsilon^a, \quad (2.3)$$

$$\delta_\epsilon y = Y_a y \epsilon^a, \quad (2.4)$$

and that these transformations form a complete set of gauge symmetries. The T_a are the generators of the Lie algebra of G and f^a_{bc} are the corresponding structure constants.

It is convenient to introduce the flat connection Θ_μ^a defined through

$$g^{-1} dg = \Theta_\mu^a T_a dx^\mu = \Theta^a T_a, \quad (2.5)$$

which in terms of local coordinates h^a on G reads

$$\Theta^a = \omega^a_b(h) dh^b. \quad (2.6)$$

The $\omega^a_b(h)$ are the components of the left-invariant forms Θ^a in the basis of the dh^a . The matrix $\omega^a_b(h)$ is invertible because the invariant forms Θ^a form a basis. We shall denote its inverse by $\Omega^a_b(h)$:

$$\omega^a_b(h) \Omega^b_c(h) = \delta^a_c. \quad (2.7)$$

The invariant forms Θ^a obey the Maurer-Cartan equation

$$d\Theta^a = -\frac{1}{2} f^a_{bc} \Theta^b \Theta^c. \quad (2.8)$$

It follows that the curvature of the connection (2.5) vanishes identically:

$$F^a_{\mu\nu}(\Theta) = \partial_{[\mu} \Theta^a_{\nu]} + f^a_{bc} \Theta^b_\mu \Theta^c_\nu = 0. \quad (2.9)$$

The quantity I_μ^a ,

$$I_\mu^a = \Theta^a_\mu - A^a_\mu, \quad (2.10)$$

transforms homogeneously since it is the difference between two connections,

$$\delta_\epsilon I_\mu^a = f^a_{bc} I_\mu^b \epsilon^c, \quad (2.11)$$

and it can be thought of as some sort of covariant derivative of the field g [the notation $I_\mu^a = \omega^a_b(h) D_\mu h^b$ is sometimes used in the literature]. Clearly, all the first-order derivatives $\partial_\mu h^a$ of h^a can be expressed in terms of I_μ^a . The connection Θ_μ^a can be used to define covariant derivatives of fields transforming linearly under the symmetry; to avoid confusion with the A -covariant derivative, we shall denote the corresponding covariant derivative by $D_\mu^{(\Theta)}$.

There exist two important choices for the action. One is the standard gauged model where both the group-valued field and the Yang-Mills field (as well as the matter fields y^i if any) have a kinetic term

$$\mathcal{L} = -\frac{1}{4} g_{ab} F^a_{\mu\nu} F^{b\mu\nu} - \frac{1}{2} g_{ab} I_\mu^a I^{b\mu} + \text{matter action}. \quad (2.12)$$

Here, g_{ab} is an invertible, invariant metric on \mathcal{G} , which we assume to exist. The field equations are (dropping the matter part)

$$D_\rho^{(A)} F^{a\rho\mu} + I^{a\mu} = 0, \quad (2.13)$$

$$D_\mu^{(\Theta)} I^{a\mu} = 0. \quad (2.14)$$

The first equation follows from varying the vector potential while the second equation is obtained by varying the group element [if one varies h^a , one really obtains Eq. (2.14) multiplied by the matrix ω^a_b , which is invertible]. In Eq. (2.14), the covariant derivative $D_\mu^{(\Theta)}$ may be replaced by $D_\mu^{(A)}$ since the I 's commute while the structure constants f^a_{bc} are anti-

symmetric in b and c . This leads to the alternative form of the equations of motion (2.14),

$$D_{\mu}^{(A)} I^{a\mu} = 0, \quad (2.15)$$

which clearly exhibits that the g -equations of motion are a consequence of the Yang-Mills equations of motion [Eq. (2.13) implies Eq. (2.15) by taking the covariant divergence with $D_{\mu}^{(A)}$]. Note the interesting feature that the Yang-Mills equations alone are independent even though the combined system (2.13) and (2.14) satisfies non-trivial Noether identities. Note also that in the gauge $g=1$, which is admissible, the action (2.12) reduces to the massive Yang-Mills action; the field g appears as a non-Abelian Stückelberg field.

The other choice is obtained by dropping the Yang-Mills kinetic term from Eq. (2.12), leading to the topological G/G -model with action

$$\mathcal{L} = -\frac{1}{2} g_{ab} I_{\mu}^a I^{b\mu}. \quad (2.16)$$

The equations of motion are

$$I^{a\mu} = 0, \quad (2.17)$$

$$D_{\mu}^{(\Theta)} I^{a\mu} = 0. \quad (2.18)$$

Again, the g -equation of motion is a consequence of the A -equation of motion. The model has no local degrees of freedom since in the gauge $g=1$, the connection A_{μ} vanishes.

We shall explicitly discuss below these two cases. However, our method also covers more general Lagrangians having the same set of fields and gauge symmetries. In fact, the explicit form of the Lagrangian is only used in Sec. X. The results of the following sections are manifestly independent of the dynamics and rely solely on the form of the gauge symmetries. And even the results of Sec. X are to a large extent independent of the Lagrangian.

III. PROBLEM

The BRST transformation [14,15] that incorporates the gauge symmetries can be constructed by following the general antifield procedure [16,17] described for instance in [18]. To write the BRST variations of the variables in a convenient form, it is useful to redefine appropriately the antifields conjugate to the group element g .

In local coordinates, the BRST transformation of the h^a reads

$$s h^a = \Omega^a_b(h) C^b, \quad (3.1)$$

as follows by replacing the gauge parameters ϵ^a by the ghosts C^a in the gauge variation of h^a . This term is generated by taking the antibracket of h^a with $\int d^n x h_a^* \Omega^a_b C^b$, which must thus be added to the Yang-Mills solution of the master equation. Here, the h_a^* are the antifields conjugate to h^a ,

$$(h^a(x), h_b^*(y)) = \delta_b^a \delta(x-y), \quad (3.2)$$

$$(h_a^*(x), h_b^*(y)) = 0. \quad (3.3)$$

This implies that the BRST variation of the antifields h_a^* is given by

$$s h_a^* = h_b^* \frac{\delta \Omega_c^b}{\delta h^a} C^c + \text{equation-of-motion terms}. \quad (3.4)$$

It is possible to replace the h_a^* by new variables g_a^* , with the same gradings, defined through

$$g_a^* = h_b^* \Omega_a^b(h), \quad (3.5)$$

which have much simpler BRST transformation rules,

$$s g_a^* = g_b^* f_{ac}^b C^c + \text{equation-of-motion terms}. \quad (3.6)$$

This equation indicates that the g_a^* transform according to the co-adjoint representation of \mathcal{G} . We shall work in the sequel with the antifields g_a^* rather than h_a^* , although they do not have canonical antibrackets,

$$(g_a^*, g_b^*) = -g_c^* f_{ab}^c \quad (3.7)$$

$$(g, g_a^*) = -g T_a. \quad (3.8)$$

Adopting the geometrical interpretation of the antifields given in [19], the h_a^* may be regarded as the vector fields tangent to the h^a -coordinate lines. Accordingly, they are defined only in the coordinate patch covered by the h^a . By contrast, the g_a^* are the left-invariant vector fields and are defined over the entire group manifold. In terms of g_a^* , the extra term in the solution of the master equation reads simply $\int d^n x g_a^* C^a$.

We can now write the BRST transformation of all the variables. Since the gauge transformations close off-shell, the BRST differential splits according to the antighost degree in the Koszul-Tate differential (δ) and the longitudinal differential along the gauge orbits (γ), $s = \delta + \gamma$, with no extra terms. The (left) action of these differentials on the fields explicitly reads

$$\begin{aligned}
 \gamma A_\mu^a &= \partial_\mu C^a + f_{bc}^a A_\mu^b C^c = D_\mu^{(A)} C^a, & \delta A_\mu^a &= 0, \\
 \gamma g &= g T_a C^a, & \delta g &= 0, \\
 \gamma y^i &= (Y_a)^i y^j C^a, & \delta y^i &= 0, \\
 \gamma C^a &= -\frac{1}{2} f_{bc}^a C^b C^c, & \delta C^a &= 0, \\
 \gamma A_a^{*\mu} &= A_b^{*\mu} f_{ac}^b C^c, & \delta A_a^{*\mu} &= \frac{\delta S_0}{\delta A_\mu^a}, \\
 \gamma g_a^* &= g_b^* f_{ac}^b C^c, & \delta g_a^* &= \frac{\delta S_0}{\delta h^b} \Omega^b{}_a(h), \\
 \gamma y_i^* &= y_j^* (Y_a)^i C^a, & \delta y_i^* &= \frac{\delta S_0}{\delta y^i}, \\
 \gamma C_a^* &= C_b^* f_{ac}^b C^c, & \delta C_a^* &= -D_\mu^{(A)} A_a^{*\mu} + g_a^* + y_i^* (Y_a)^i y^j.
 \end{aligned} \tag{3.9}$$

These relations imply

$$\gamma \Theta_\mu^a = D_\mu^{(\Theta)} C^a, \tag{3.10}$$

and enable us to express the ghost as $C = g^{-1} \gamma g$. In the usual abbreviations $C = C^a T_a$, one may rewrite the ghost transformation law as $\gamma C = -C^2$ since $C^2 = \frac{1}{2} T_a f_{bc}^a C^b C^c$.

Our goal is to compute the cohomological groups $H(s|d)$ of the BRST differential s modulo the spacetime exterior differential d , in the space of local forms. In ghost degree zero, these groups characterize the counterterms, while in ghost degree one, they classify the anomalies. In negative ghost number, they are related to the non-trivial conservation laws [20].

The longitudinal derivative γ is nilpotent off-shell. Therefore, we can proceed as in [6] and analyze first the γ -cohomology, $H(\gamma)$, and the γ -cohomology modulo the exterior derivative d , $H(\gamma|d)$, in the space of all fields and antifields. The De Rham cohomology of the group manifold will play an important role in this context. We shall then turn to $H(s|d)$.

IV. ANALYSIS OF $H(\gamma)$

The calculation of the cohomology is performed in the so-called ‘‘jet-space.’’ This space is simply the (infinite-dimensional) space coordinatized by the field and antifield components, as well as all their subsequent partial derivatives, $\mathcal{K} = \{A_\mu^a, g, y^i, C^a\}$, $\mathcal{K}^* = \{A_a^{*\mu}, g_a^*, y_i^*, C_a^*\}$, $\partial_\mu \mathcal{K}$, $\partial_\mu \mathcal{K}^*$ etc. Because the spacetime manifold is topologically R^n , these functions are actually globally defined (but note that they do not provide standard coordinates since the g 's are not independent). The differential γ anticommutes with the exterior derivative, so that the above transformation laws in Eq. (3.9) can be extended to the whole jet-space.

To describe the γ -cohomology, it is convenient to employ different jet-space coordinates. The construction of these new coordinates goes as follows. The quantity I_μ^a defined in

Eq. (2.10) can be used instead of the first derivatives of the field g . Indeed, I_μ can be expressed in terms of $\partial_\mu g$ and, conversely, $\partial_\mu g = g(I_\mu + A_\mu)$. Therefore, the jet-coordinates $\{g, \partial_\mu g, A_\mu^a\}$ may be reexpressed in terms of $\{g, I_\mu^a, A_\mu^a\}$. Trying to rearrange the jet-coordinates with two indices, one finds, for the second derivatives of the group element, $\partial_\mu \partial_\nu g = \frac{1}{2} g(D_{(\mu}^{(\Theta)} I_{\nu)}^a T_a + \partial_{(\mu} A_{\nu)}^a T_a + \Theta_{(\mu}^a \Theta_{\nu)}^b [T_a, T_b]) - \Theta_{(\mu}^a I_{\nu)}^b [T_a, T_b]$. The derivative of the connection can be split into a symmetric and an antisymmetric part, $\partial_\mu A_\nu = \frac{1}{2}(\partial_{(\mu} A_{\nu)} + \partial_{[\mu} A_{\nu]})$. But the curvature $F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + [A_\nu, A_\mu]$ is already contained in the antisymmetrized Θ -covariant derivatives of I_μ , $D_{[\mu}^{(\Theta)} I_{\nu]} = -F_{\mu\nu} + [I_\mu, I_\nu]$. Therefore, only the symmetrized derivatives of A_μ have to be kept. Furthermore, there are no relations between the new variables that could constrain the $D_\mu^{(\Theta)} I_\nu^a$. Thus, the coordinates with up to two indices, $\{g, \partial_\mu g, \partial_\mu \partial_\nu g, A_\mu^a, \partial_\mu A_\nu^a\}$, may be rearranged in the set $\{g, I_\mu^a, D_\mu^{(\Theta)} I_\nu^a, A_\mu^a, \partial_{(\mu} A_{\nu)}^a\}$.

The claim is now that g , A_μ^a and all their derivatives can be replaced by g , A_μ^a with its symmetrized derivatives, and I_μ^a with its successive Θ -covariant derivatives ($k=1,2,\dots$):

$$\begin{aligned}
 &\{g, \partial_{\alpha_1 \dots \alpha_k} g, \partial_{\alpha_1 \dots \alpha_{k-1}} A_{\alpha_k}^a\} \\
 &\rightarrow \{g, \partial_{(\alpha_1 \dots \alpha_{k-1}} A_{\alpha_k)}^a, D_{\alpha_1 \dots \alpha_{k-1}}^{(\Theta)} I_{\alpha_k}^a\}.
 \end{aligned} \tag{4.1}$$

A good way of checking the equivalence of the two sets of coordinates is to compare their size. Indeed, remembering that the index a takes N values and that there are only N independent g 's, it is easy to see that each set contains $N + N \sum_{l=1}^k n(n+1) \dots (n+l)/l! + Nn \sum_{l=1}^{k-1} n(n+1) \dots (n+l)/l!$ independent coordinates, as should be the case (n is the spacetime dimension). The explicit proof that the two sets of coordinates are equivalent may be obtained by induction. Assume the above statement to be true up to derivatives of order k for g and of order $k-1$ for A_μ^a (i.e. for coordinates with k spacetime indices). The derivatives of order k of A_μ^a

can be expressed in terms of symmetrized derivatives of A_μ^a and derivatives of order $k-1$ of $F_{\mu\nu}^a$. But terms of the form $\partial_{\alpha_1 \dots \alpha_{k-1}} F_{\alpha_k \alpha_{k+1}}^a$ are contained in $D_{\alpha_1 \dots \alpha_{k-1}}^{(\Theta)} D_{[\alpha_k}^{(\Theta)} I_{\alpha_{k+1}]}$. The derivatives of order $k+1$ for g are generated by taking k symmetrized Θ -covariant derivatives of I_μ : $D_{(\alpha_1 \dots \alpha_k}^{(\Theta)} I_{\alpha_{k+1})} \sim \partial_{(\alpha_1 \dots \alpha_k} \Theta_{\alpha_{k+1})} + \text{‘‘lower order’’} \sim g^{-1} \partial_{(\alpha_1 \dots \alpha_{k+1})} g + \text{‘‘l.o.’’}$, which completes the proof that the above change of coordinates is indeed invertible.

As new basis of jet-coordinates, we can thus choose the following combinations of fields and derivatives:

- the group element g and the ghost C without derivatives,
- the I_μ^a with all subsequent Θ -covariant derivatives,
- the matter fields y^i with Θ -covariant derivatives,
- the antifields \mathcal{K}^* with Θ -covariant derivatives,
- the Yang-Mills connection A_μ^a and its symmetrized derivatives, and the derivatives of the ghost C .

The vector potential A_μ^a with its symmetrized derivatives and the derivatives of C^a form contractible pairs, as observed in [9]. Accordingly, they do not contribute to the γ -cohomology.

The fields $\chi^A := \{I_\mu^a, y^i, \mathcal{K}^*\}$ all transform linearly under the action of γ , $\gamma \chi^A \sim (Z_a)^A_B \chi^B C^a$ [see Eq. (3.9)]. The $(Z_a)^A_B$ are the generators of some representation of G , for instance of the adjoint representation in the case of I_μ^a . It is possible to combine these fields with the group element g to form invariant quantities $\tilde{\chi}^A = U(g)^A_B \chi^B$, $\gamma \tilde{\chi}^A = 0$. Here, $U(g)$ stands for the representative of the group element g in the relevant representation (generated by Z_a). Since $U(g)$ transforms contragrediently to the corresponding fields or antifields,

$$\gamma U(g) = -(-)^{\epsilon_\chi} U(g) Z_a C^a, \quad (4.2)$$

the variables $\tilde{\chi}^A$ are invariant, $\gamma \tilde{\chi}^A = 0$, i.e., one may replace covariant fields by invariant fields (the ϵ_χ denote the parity of the field χ). Furthermore, a short calculation shows that $\partial_\mu \tilde{\chi}^A = U(g) D_\mu^{(\Theta)} \chi^A$. It is therefore possible to replace the jet-variables χ^A and their Θ -covariant derivatives by the quantities $\tilde{\chi}^A$ and their ordinary derivatives. The introduction of the tilde variables follows the pattern of [21] (see also [2]).

In the new basis of jet-coordinates and after elimination of the trivial pairs, the action of the longitudinal derivative γ reduces to the simple form

$$\gamma g = g C, \quad (4.3)$$

$$\gamma C = -C^2, \quad (4.4)$$

$$\gamma[\tilde{\chi}^A] = 0, \quad (4.5)$$

which fits with the general conditions on ‘‘good’’ jet-coordinates given in [22]. The square brackets around $\tilde{\chi}^A$ stand for $\tilde{\chi}^A$ and all the subsequent ordinary derivatives. It follows from Eq. (4.5) that the most general solution of the cocycle condition $\gamma m = 0$ is, up to trivial terms, a linear combination of polynomials in the gauge-invariant variables

$[\tilde{\chi}^A]$ times a solution of $\gamma n = 0$ involving only the g 's and the C 's. To complete the analysis of the cohomology of γ , we thus need to compute the cohomology defined by

$$\gamma g = g C, \quad (4.6)$$

$$\gamma C = -C^2. \quad (4.7)$$

This is done by relating Eqs. (4.6) and (4.7) to the De Rham cohomology of the group manifold.

It is the identification $\gamma \rightarrow d$ and $C \rightarrow \Theta$ that establishes the link. Here, the exterior derivative d acts in the space of g and Θ in the same way as γ acts in the space of g and C . Thus the BRST complex involving the group element and the ghost is identified with the De Rham complex of the group manifold. The relevant identities are now $dg = g \Theta$ and $d\Theta = -\Theta^2$, where the second equation is recognized to be the Maurer-Cartan structure equation for left-invariant forms on the group, which we used already above. Let $\omega_I = \omega_I(\Theta, g)$ form a basis of $H_{DR}(G)$, and let $\omega_I(C, g)$ be the function of C and g obtained after replacing Θ by C in $\omega_I(\Theta, g)$. Then, a general cocycle solving the equations $\gamma m = 0$ has the form

$$m = \sum_I P^I([\tilde{\chi}^A], dx) \omega_I(C, g) + \gamma n, \quad (4.8)$$

where the P^I are arbitrary polynomials in the variables $[\tilde{\chi}^A]$ and the differentials dx^μ (we assume no explicit x -dependence). Furthermore, m is trivial if and only if $P^I = 0$ (for each I).

Note that the invariant polynomials in the covariantly transforming quantities χ^A , which are related to the Casimir invariants of the corresponding representation, form a subset of all $P^I([\tilde{\chi}^A], dx)$.

V. TOPOLOGICAL TERMS

Consider the pullbacks to the spacetime manifold of the forms $\omega_I(\Theta, g)$. These are just given by $\omega_I(\Theta, g)$ where Θ is viewed as the spacetime form $\Theta_\mu dx^\mu$ rather than a 1-form on the group manifold (and d is the spacetime d rather than the exterior derivative on the group manifold). For this reason, we shall denote these pullbacks by the same symbol $\omega_I(\Theta, g)$. The spacetime exterior forms $\omega_I(\Theta, g)$ are related to the γ -cocycles $\omega_I(C, g)$ through the descent equation [20].

Indeed, expanding $\tilde{\omega}_I \equiv \omega_I(\Theta + C, g)$ according to the ghost number yields

$$\tilde{\omega}_I = \omega_I^{0,p} + \omega_I^{1,p-1} + \dots + \omega_I^{p,0} \quad (5.1)$$

where p is the form degree of $\omega_I(\Theta, g)$ and where, in $\omega_I^{k,l}$, the first superscript k stands for the ghost number while the second superscript l stands for the form degree ($k+l=p$). Of course, $\omega_I^{0,p} = \omega_I(\Theta, g)$ and $\omega_I^{p,0} = \omega_I(C, g)$. Now, $\tilde{\omega}_I$ is annihilated by $\tilde{\gamma} = \gamma + d$ by construction,

$$\tilde{\gamma} \tilde{\omega}_I = 0 \quad (5.2)$$

(the previous equation is usually referred to as ‘‘Russian formula’’ [12,23]). If one also expands this equation according to the ghost number, one finds a tower of ‘‘descent equations’’ that read explicitly

$$\begin{aligned}
 d\omega^{(0,p)} &= 0 \\
 \gamma\omega^{(0,p)} + d\omega^{(1,p-1)} &= 0 \\
 \gamma\omega^{(1,p-1)} + d\omega^{(2,p-2)} &= 0 \\
 &\vdots \\
 \gamma\omega^{(p-1,1)} + d\omega^{(p,0)} &= 0 \\
 \gamma\omega^{(p,0)} &= 0.
 \end{aligned} \tag{5.3}$$

It follows from the Poincaré lemma on the group manifold that $\omega_I^{0,p}$ is locally exact, $\omega_I^{0,p} = dK^{0,p-1}$. This implies that all the forms $\omega^{k,g-k}$ occurring in the descent are also locally trivial, $\omega^{k,p-k} = dK^{k,p-k-1} + \gamma K^{k-1,p-k}$, where $K^{l,p-1}$ is the component of $\tilde{K}(\Theta + C, g)$ of ghost number l . These relations, however, hold only locally. Globally, it is not possible to bring the ω 's to the trivial form. For this reason, the $\omega^{k,g-k}$ will be referred to in the sequel as the ‘‘topological terms.’’

The descent equations (5.3) will be exploited in the next section. A particularly important case arises when $p=n$. In that case, one sees from Eqs. (5.3) that the spacetime integral

$$\int_{R^n} \omega_I^{0,n} \tag{5.4}$$

is gauge-invariant since its integrand is gauge-invariant up to a total derivative. It can thus be added to the action without breaking gauge invariance. However, because $\omega_I^{0,n}$ is locally exact, the topological term (5.4) does not modify the equations of motion. Terms of the form (5.4) are called ‘‘winding number terms.’’ Although locally trivial, they cannot be eliminated globally. Also, their integrands do not differ from a strictly gauge-invariant integrand up to the exterior derivative of a (globally defined) $(n-1)$ -form, since this would imply that the last element in the corresponding descent is trivial.

For instance in three spacetime dimensions and for a compact, simple gauge group such as $SU(3)$, the non-trivial γ -cocycle $\text{Tr}(g^{-1}\gamma g g^{-1}\gamma g g^{-1}\gamma g) = \text{Tr} C^3$ corresponds to the three-form winding number term $\text{Tr}(g^{-1}dg g^{-1}dg g^{-1}dg) = \text{Tr} \Theta^3$. Varying the field g in this expression yields a total derivative, which indicates that the winding number terms do not contribute to the equations of motion for g . On the other hand, they cannot be globally written as a total derivative in the space of fields and accordingly, they cannot be dropped from the action. Locally, it is of course always possible to express them as total derivatives.

Finally, we note that the forms $\omega_I^{p,0}$ are the only non-trivial cocycles of the exterior derivative d acting in the algebra \mathcal{A} of local forms on the jet-space of the fields, ghosts and antifields as described in Sec. IV. This follows from the

generalization of the so-called ‘‘algebraic Poincaré lemma’’ to the case where some fields (here g) belong to a cohomologically non-trivial manifold (here the group manifold G) [24,25] (see also [20]).

Algebraic Poincaré lemma. The cohomology $H^p(d, \mathcal{A})$ of d in the algebra of local p -forms is isomorphic to the De Rham cohomology of G in the same form degree for $p < n$,

$$H^p(d, \mathcal{A}) \simeq H_{DR}^p(G), \quad p < n. \tag{5.5}$$

In maximal form degree, $H_{DR}^n(G)$ is isomorphic to the quotient of the variationally closed n -forms by the d -exact n -forms. An n -form $\mathcal{L}d^n x$ is said to be variationally closed if and only if the Euler-Lagrange derivatives of \mathcal{L} with respect to all the fields, ghosts and antifields vanish.

For later purposes, we also quote the covariant Poincaré lemma, which describes $H(d)_{inv}$, i.e. the cohomology of d in the space of invariant polynomials.

Covariant Poincaré lemma. Let $P^k([\tilde{\chi}])$ be a d -closed invariant polynomial of form degree k . Then, P may be assumed to be d -exact in the space of invariant polynomials, i.e.

$$dP^k([\tilde{\chi}]) = 0, \quad \gamma P = 0 \Rightarrow P^k = dQ^{(k-1)}([\tilde{\chi}]) + \alpha^k, \quad \gamma Q = 0, \tag{5.6}$$

where α^k is a constant form.

Thus, $H(d)_{inv}$ vanishes in the setting considered here, contrary to the case without the nonlinearly transforming field g , where the obstructions to choosing Q invariant in Eq. (5.6) were identified to be the invariant polynomials in the curvature form F [26,9]. In the presence of the group valued field g , it is, however, possible to replace covariant quantities (χ) by invariant ones ($\tilde{\chi}$) and covariant derivatives by ordinary derivatives (see Sec. IV). Thus, any polynomial in $[\tilde{\chi}]$ is automatically invariant, and the action of d obviously does not introduce any new variables. The vanishing of $H(d)_{inv}$ then follows because the invariants $\tilde{\chi}^A$ and all their derivatives are independent jet-variables (subject to no identity). The effect of the group-valued field is particularly striking in the Abelian case, where the curvature itself becomes d -trivial in the space of invariants, $F = dI$. Here, I is the gauge invariant quantity $d\phi - A$, and $g = \exp\{\phi\}$.

VI. ANALYSIS OF $H(\gamma)d$

The next step towards a complete description of the BRST cohomology modulo d is the calculation of $H^{(*,*)}(\gamma)d$. The bi-grading ‘‘ $(*,*)$ ’’ refers as before to the ghost degree and the form degree respectively. Via standard descent equations one can, again as before, relate $H(\gamma)d$ to the cohomology $H(\gamma)$ which is known from the above analysis.

A representative $a^{(g,p)}$ of some class in $H^{(g,p)}(\gamma)d$ has to satisfy

$$\gamma a^{(g,p)} + da^{(g+1,p-1)} = 0. \tag{6.1}$$

If $a^{(g+1,p-1)}$ happens to be trivial in $H(\gamma|d)$, then it can be eliminated through trivial redefinitions and $a^{(g,p)} \in H^g(\gamma)$. If $a^{(g+1,p-1)}$ is not in the trivial class of $H(\gamma|d)$, then it cannot be trivially absorbed. In this case, the algebraic Poincaré lemma ensures the existence of a descent

$$\gamma a^{(g+k,p-k)} + da^{(g+k+1,p-k-1)} = 0, \quad k=1,2,\dots, \quad (6.2)$$

which ends when $a^{(g+k+1,p-k-1)} = 0$ for some value of k , which happens at the latest when zero-forms are produced.² Any bottom $a^{g+k,p-k}$ of a descent is a cocycle of γ , $\gamma a^{g+k,p-k} = 0$. Therefore, the last term in the descent takes the form, Eq. (4.8),

$$a^{g+k,p-k} = \sum_I P^I([\tilde{\chi}^A], dx) \omega_I(C, g) + \gamma n. \quad (6.3)$$

The γ -trivial part can be absorbed through redefinitions of the previous terms and may be assumed to be absent.

It turns out that some non-trivial γ -cocycles are actually trivial in $H(\gamma|d)$ and accordingly must also be discarded. More precisely, if P^I is d -trivial,

$$P^I([\tilde{\chi}^A], dx) = d\rho^I([\tilde{\chi}^A], dx), \quad \gamma\rho^I = 0, \quad (6.4)$$

for some invariant polynomial ρ^I , then the corresponding cocycle in $H(\gamma)$ is γ -trivial modulo d ,

$$a^{(g+k,p-k)} = P^I([\tilde{\chi}^A], dx) \omega_I(C, g) = d(\rho^I \omega_I) - \gamma(\rho^I \hat{\omega}_I), \quad (6.5)$$

where $\hat{\omega}_I$ is the second to last term in the descent, Eq. (5.3), associated with the De Rham cohomology of G analyzed in the previous section, $d\omega_I = \gamma\hat{\omega}_I$.

If the descent is non-trivial, so that $a^{g+k,p-k}$ can be lifted at least once, then P^I must be constant up to terms that are d -exact in the space of invariants. Indeed, one finds from $da^{g+k,p-k} + \gamma a^{g+k-1,p-k+1} = 0$ that $(dP^I)\omega_I + \gamma\mu^I = 0$, which yields $dP^I = 0$ since the ω_I are independent in cohomology. The equality $P^I = \alpha^I + d\rho^I$, where the α^I are constant forms, then follows from the covariant Poincaré lemma. As we have just seen, the $d\rho^I$ component of P^I can be discarded.

The only elements of $H(\gamma)$ that could serve as bottom of a non-trivial descent are therefore the basis elements of the De Rham cohomology, ω_I , multiplied by constant p -forms. These constant forms may be eliminated from the analysis by imposing Lorentz invariance, which leaves only the zero-forms ω_I as interesting bottoms. Furthermore, there is no obstruction to lifting ω_I up to maximal form degree, as follows immediately from Eq. (5.3). Therefore, any bottom $\alpha^I \omega_I$ is admissible.

One can summarize the results as follows. The solutions a of the cocycle condition $\gamma a + db = 0$ fall into two classes.

First, there are the solutions that lead to no (non-trivial) descent, i.e. that are strictly annihilated by γ (no d -exact term occurs),

$$\gamma a = 0. \quad (6.6)$$

These solutions can be expressed in terms of the invariants $[\tilde{\chi}^A]$ and the De Rham forms $\omega_I(C, g)$ as in Eq. (6.3). Second, there are the solutions leading to a non-trivial descent. These are the lifts of the De Rham forms $\omega_I(C, g)$, up to trivial terms and terms strictly annihilated by γ . These solutions are locally trivial (locally in field space) but not globally so. There are no other solutions associated with non-trivial descents besides these topological terms.

VII. COMPARISON WITH THE PURE YANG-MILLS CASE

It is interesting to compare the results obtained here with those of the pure Yang-Mills case analyzed in [27,26,9,6] (or the case of Yang-Mills fields coupled to matter fields transforming according to some linear representation of G). Since the analysis in those works was carried out for reductive algebras, we shall assume throughout this section that \mathcal{G} is reductive.

In the linear case (by which we mean ‘‘only linear representations’’), the γ -cohomology is represented by products of elements of the Lie algebra cohomology with invariant polynomials in the curvature, the matter fields, the antifields and the corresponding covariant derivatives with respect to the gauge connection (denoted by $[]_c$), $\gamma m = 0 \Rightarrow m = P^I_{inv}([F_{\mu\nu}]_c, [y^i]_c, [\Omega^*]_c) \omega_I^{Lie}(C) + \gamma n$. Thus, the γ -cohomologies in the nonlinear (g present) and linear (g absent) cases have a similar structure, except that it is the De Rham cohomology that is relevant in one case, while it is the Lie algebra cohomology in the other case. Of course, for compact groups, the two cohomologies are isomorphic. But this is not true in general.

We turn now to the cohomology $H(\gamma|d)$ and assume that the Lie algebra cohomology and the De Rham cohomology are isomorphic, to emphasize the differences that arise when working ‘‘modulo d .’’ The elements of $H(\gamma|d)$ that are not equivalent to elements of $H(\gamma)$ can be characterized by the bottom of their associated non-trivial descent, which is a γ -cocycle. So we have to compare the bottoms that can be lifted in both cases. We have seen that in the nonlinear case, the only non-trivial bottoms involve only the ghosts, but no other fields. This is not true in the linear case, where one may have bottoms that contain the curvature forms. Moreover, while there may be obstructions to lifting bottoms more than once in the linear case [27], this is not true in the nonlinear case, where any bottom can be lifted to maximum form degree. For instance, only the primitive elements of the Lie algebra cohomology can be lifted all the way up to maximum form degree in the linear case [27]. An example is given by the product $(\text{Tr } C^3)(\text{Tr } C^5)$ [in $SU(5)$, say] which cannot be lifted all the way up to form degree 8 because one encounters the obstruction $\text{Tr } F^2$ at form degree 4. Clearly, $\text{Tr } F^2$ is non-trivial in $H(\gamma)$. When the group element g is

²For more details on this procedure when the cohomology of d is non-trivial, as here, see [20].

present, then any combination $\lambda^l \omega_l$ can be lifted up to maximal form degree. The product $\text{Tr}(C^3)\text{Tr}(C^5)$ lifts for example to $\text{Tr}(g^{-1}dg)^3\text{Tr}(g^{-1}dg)^5$. A way to understand the removal of the obstruction in the nonlinear case is to observe that one may lift the ghosts using the flat connection Θ . The obstructions are known to involve the curvatures [27]. They are absent here because the curvature of Θ identically vanishes.

Finally, we note that in the linear case, the Chern-Simons forms cannot be replaced by strictly gauge-invariant terms. By contrast, in the nonlinear case, the Chern-Simons forms differ from strictly gauge-invariant terms by total derivatives and winding number terms that are locally trivial and do not contribute to the equations of motion. For instance, the Chern-Simons term $3\text{Tr}(AdA + \frac{2}{3}A^3)$ and the winding number term $\text{Tr} \Theta^3$ descend on the same cocycle $\text{Tr} C^3$. Thus, their difference descends on zero and hence is equivalent to a strictly gauge invariant term modulo a total derivative. An explicit calculation yields indeed $\text{Tr}(AdA + \frac{2}{3}A^3) = \text{Tr}\{ID^\Theta I - \frac{2}{3}I^3\} - \frac{1}{3}\text{Tr} \Theta^3 - d\text{Tr} \Theta A$ with $I = I_\mu^a T_a dx^\mu$.

VIII. RELATION TO THE WORK OF De WIT, HULL AND ROČEK

The same conclusions apply to the topological terms considered in [10]. These again differ from strictly gauge invariant terms by total derivatives (and locally trivial winding number terms if the De Rham cohomology of the group manifold in form degree n does not vanish). From this point of view, the interesting construction of [10] does not bring in new terms in the principal case, even when the group is not semi-simple.

To illustrate this point, we recall the construction of [10], specializing to the principal case and considering four space-time dimensions for definiteness.

When trying to construct gauge theories with a non-compact gauge group G , it is natural to consider actions involving integrands of the form [28]

$$\mathcal{T} = S_{ij}(g) F^i \wedge F^j. \quad (8.1)$$

This term is strictly gauge invariant, $\delta_\epsilon(S_{ij}(\phi) F^i \wedge F^j) = 0$, if $S_{ij}(g)$ transforms in the following way:

$$\delta_\epsilon S_{ij} = -S_{lj} f_{ik}^l \epsilon^k - S_{li} f_{jk}^l \epsilon^k = -2S_{l(ij)k} f_{ik}^l \epsilon^k. \quad (8.2)$$

When S_{ij} is an invariant symmetric tensor, the term $S_{ij} F^i \wedge F^j$ defines a characteristic class and is a topological invariant.

In [10], De Wit, Hull and Roček generalize the above setting through modifications of the action. As in [29], they modify the above term by adding to it an appropriate non-gauge-invariant term,

$$\mathcal{T}_{mod} = S_{jk}(g) F^j \wedge F^k + \frac{2}{3} C_{i,jk} A^i \wedge A^j \left(dA^k + \frac{3}{8} f_{lm}^k A^l A^m \right), \quad (8.3)$$

and observe that Eq. (8.3) is invariant up to a total derivative if at the same time the transformation law for S_{ij} , Eq. (8.2), is modified to

$$\delta_\epsilon S_{ij} = (-2S_{l(ij)k} f_{ik}^l + C_{k,ij}) \epsilon^k, \quad (8.4)$$

where the constants $C_{k,ij} = C_{k,ji}$ are subject (i) to obey $C_{(k,ij)} = 0$ and (ii) to satisfy the 1-cocycle condition of the Lie algebra cohomology of \mathcal{G} in the symmetric tensor product of the adjoint representation space with itself,³ $\frac{1}{2} C_{m,ij} f_{lk}^m + f_{il}^m C_{l,mj} + f_{jl}^m C_{l,mi} = 0$. The term involving bare A 's in Eq. (8.3) is reminiscent of a Chern-Simons term.

As the authors of [10] also observe, any exact contribution to $C_{k,ij}$ of the form $C_{k,ij} = f_{k(i}^m S_{j)m}$ can be absorbed through a constant shift of S_{ij} . Thus, if $C_{k,ij}$ is a coboundary, the term (8.3) can be brought back to the form (8.1) by redefinition of S_{ij} and the addition of a total derivative and therefore is not a true generalization of Eq. (8.1).

Our point is that even when $C_{k,ij}$ is a non-trivial 1-cocycle of the Lie algebra cohomology of \mathcal{G} in the symmetric tensor product of the adjoint representation space with itself (which can only occur when \mathcal{G} is non-semi-simple), one can redefine Eq. (8.3) [or, for that matter, even Eq. (8.1)] by adding a total derivative so that this term is strictly gauge invariant and involves only the manifestly invariant variables $[\tilde{\chi}^A]$ constructed above, up to possible locally trivial winding number terms.

This is an immediate consequence of our general analysis and is particularly striking when the gauge group is R^k , which is a non-compact, Abelian group. We denote its generators by T_a , $a = 1, \dots, k$, $[T_a, T_b] = 0$. The field g is then $\exp[\phi^a T_a]$ where ϕ^a is a vector in R^k . The relevant transformation laws simply read

$$s \phi^a = C^a, \quad (8.5)$$

$$s C^a = 0, \quad (8.6)$$

$$s A_\mu^a = \partial_\mu C^a. \quad (8.7)$$

As usual, $F^a = dA^a$ and $dF^a = \delta_\epsilon F^a = 0$. The De Rham cohomology of G is trivial except for the constants, $H_{DR}^k(G) = 0$ for $k \neq 0$, $H_{DR}^0(G) = R$, while the Lie algebra cohomology of \mathcal{G} consists of the polynomials in the ghosts C , $H(\mathcal{G}) = P(C)$. In particular, there is no winding number term. Furthermore, since the structure constants are zero, any constant $C_{k,ij}$ with $C_{k,ij} = C_{k,ji}$ defines a non-trivial 1-cocycle with a value in the symmetric product of the adjoint representation. We assume $C_{(k,ij)} = 0$ in the sequel so as to satisfy the first condition (i) above.

Equations (8.3) and (8.4) simplify to

$$\mathcal{T}_{mod} = S_{jk}(\phi) F^j F^k + \frac{2}{3} C_{i,jk} A^i A^j F^k, \quad (8.8)$$

$$s S_{ij} = C_{k,ij} C^k. \quad (8.9)$$

³For useful information on Lie algebra cohomology, see [30].

The transformation law of S_{jk} implies $S_{jk}(\phi) = C_{i,jk}\phi^i$ up to an irrelevant constant. The above term \mathcal{T}_{mod} is gauge invariant up to a total derivative, $\gamma\mathcal{T}_{mod} = -\frac{2}{3}dC_{i,jk}(C^iA^jF^k + A^iC^jF^k)$. From the point of view of [10], the expression in Eq. (8.8) represents a non-trivial extension of the strictly gauge invariant theory, since $C_{k,ij}$ is a non-trivial Lie algebra cocycle. However, by adding an appropriate total derivative to it, one straightforwardly verifies that \mathcal{T}_{mod} is equivalent to the strongly gauge invariant expression $\frac{2}{3}C_{i,jk}(d\phi^i - A^i) \times (d\phi^j - A^j)F^k$,

$$\mathcal{T}_{mod} \equiv \frac{2}{3}C_{i,jk}\nabla\phi^i\nabla\phi^jF^k + \frac{2}{3}C_{i,jk}d\left(\phi^iA^jF^k - \frac{1}{2}\phi^id\phi^jF^k\right), \quad (8.10)$$

where $\nabla\phi^i = d\phi^i - A^i$ may be regarded as the exterior covariant derivative of ϕ and is just the invariant I^i introduced above, $\nabla_\mu\phi^i = I^i_\mu$.

This shows that in the principal case it is not the Lie algebra cohomology that controls the ‘‘novelty’’ of Eq. (8.3). This term is always equivalent to a strictly gauge invariant term [plus winding number terms if $H^4_{DR}(G)$ happens to be non-trivial]. It would be interesting to extend the analysis of this issue to scalar fields taking values in quotient spaces G/H , for which the general construction of [10] was devised.

IX. GAUGED WESS-ZUMINO-WITTEN TERM

The above calculation of $H(\gamma|d)$ sheds also a new light on the problem of gauging the Wess-Zumino-Witten term [11,31–33]. The Wess-Zumino-Witten term $\mathcal{L}_{WZW}(g)$ is a term that can be added to the Lagrangian of the (ungauged) nonlinear σ -model without breaking its rigid symmetries. Its characteristic property (which may be used as its definition) is that it is not strictly invariant under the rigid symmetries of the model, but only invariant up to a surface term. Furthermore, one cannot ‘‘improve’’ it by a surface term such that the sum is strictly invariant (even locally in field space).

Because the Wess-Zumino-Witten term is invariant only up to a non-trivial surface term, its gauging raises difficulties. These have been analyzed in [32,33], with the conclusion that in the principal case in which one gauges the right action (as here), there are unremovable obstructions to gauging the Wess-Zumino-Witten term. These obstructions have been related in [33] to the equivariant cohomology. The impossibility of gauging the Wess-Zumino-Witten term is also a direct consequence of our analysis.

Indeed, suppose that one has found a functional $\mathcal{L}_{WZW}([g],[A^a_\mu])$ that (i) reduces to the Wess-Zumino-Witten term $\mathcal{L}_{WZW}(g)$ when the gauge field is set to zero and (ii) is gauge-invariant up to a surface term,

$$\gamma\mathcal{L}_{WZW}([g],[A^a_\mu]) + da^{(0,n-1)}([g],[A^a_\mu],C) = 0. \quad (9.1)$$

Such a term would provide a ‘‘gauging’’ of the WZW term. But our results indicate that such a term would necessarily be equivalent to a strictly invariant term, modulo winding number terms that do not contribute to the equations of motion,

$$\begin{aligned} \mathcal{L}_{WZW}([g],[A^a_\mu]) &= \mathcal{L}^{inv}([g],[A^a_\mu]) + dm \\ &+ \text{‘‘winding number terms’’} \end{aligned} \quad (9.2)$$

for some m . This would imply, upon setting A^a_μ and its subsequent derivatives equal to zero, that the original Wess-Zumino term is (locally) equivalent to the strictly invariant term $\mathcal{L}^{inv}([g],[A^a_\mu]=0)$, which we know cannot be true. (The strict invariance of $\mathcal{L}^{inv}([g],[A^a_\mu]=0)$ under rigid transformations follows from the strict invariance of $\mathcal{L}^{inv}([g],[A^a_\mu])$ under gauge transformations.) This means that there simply is no room for a gauged Wess-Zumino-Witten term.

Our approach is less explicit than the analysis of [33] since it does not identify the nature of the obstruction (it just indicates that there is an obstruction). At the same time, it is more complete because we show that the obstruction exists even if one allows \mathcal{L}_{WZW} to depend on the individual field components and all their derivatives. As pointed out very clearly in [33], the previous calculations were performed only in the ‘‘universal’’ algebra generated by g , the 1-form A and their exterior derivatives dg , dA (but not in the algebra generated by all the separate individual components of the fields and their higher order derivatives). So these calculations excluded only gauged Wess-Zumino-Witten terms \mathcal{L}_{WZW} depending on g , A and their exterior derivatives but still left open the possibility of gauging \mathcal{L}_{WZW} in the ‘‘big algebra’’ containing all the field components and their derivatives individually [33].

X. ANALYSIS OF $H(s|d)$

In order to characterize the cohomology modulo d of the complete BRST operator in the space of fields and antifields, it is necessary to specify the dynamics of the theory. Indeed, s contains information on the equations of motion through the Koszul-Tate differential δ , and the BRST cohomology will in general depend on the dynamics although the gauge transformations are not affected. We shall first develop the analysis in the case of the usual action (2.12), which is, if one reinstates explicitly the coupling constants,

$$\mathcal{L} = -\frac{1}{4}g_{ab}F^a_{\mu\nu}F^{b\mu\nu} - m^2\frac{1}{2}g_{ab}I^a_\mu I^{b\mu} + \text{matter action} \quad (10.1)$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \alpha f^a_{bc}A^b_\mu A^c_\nu \quad (10.2)$$

$$I^a_\mu = (\omega^a_b(h)\partial_\mu h^b - \alpha A^b_\mu) \quad (10.3)$$

where α is the Yang-Mills coupling constant. We shall then explain how the results extend to more general actions.

The idea follows the pattern developed in [6,8]. One controls the antifield dependence of the BRST cocycles through expansion of the condition

$$sa + db = 0 \quad (10.4)$$

according to the antighost number, $a = a_0 + \dots + a_k$ and $b = b_0 + \dots + b_m$. Only the case where the highest antighost degree of a is equal to that of b ($k=m$) shall be described here because the other cases can be easily reduced to this one. At highest antighost number k —which we take to be >0 since otherwise there is no antifield—the above cocycle condition reads

$$\gamma a_k + db_k = 0. \quad (10.5)$$

This implies $\gamma b_k + dc_k = 0$ and hence, according to our analysis of $H(\gamma|d)$, b_k must be trivial (it must be liftable at least once but it contains the antifields and therefore cannot be a pure topological term). We can thus assume $\gamma a_k = 0$; i.e., up to trivial redefinitions, $a_k = P^I \omega_I$. Next, the subleading equation in the above decomposition of the cocycle condition has to be used:

$$\delta a_k + \gamma a_{k-1} + db_{k-1} = 0. \quad (10.6)$$

Acting with γ on this equation produces $d\gamma b_{k-1} = 0$ and thus $\gamma b_{k-1} + dc_{k-1} = 0$. By the same reasoning as above, one finds that c_{k-1} is trivial if $k > 1$, and thus one may assume $b_{k-1} = Q^I \omega_I$. If $k = 1$, $b_0 = b_0^{inv} + b_0^{top}$ may have a non-trivial, topological component. The resulting equation $\gamma b_0^{top} + dc_0 = 0$ may be lifted to $\gamma a_0^{top} + db_0^{top} = 0$. By subtracting, if necessary, the topological term a_0^{top} from a_0 , it is possible to eliminate the non-invariant component of b_0 and to assume $b_{k-1} = Q^I \omega_I$ also for $k = 1$.

Upon inserting the explicit forms of a_k and b_{k-1} in Eq. (10.6), it is straightforward to derive that $\delta P^I + dQ^I = 0$. Furthermore, if P^I is in the trivial class of $H_k^n(\delta|d)$, i.e. if $P^I = \delta M^I + dN^I$, then a_k can be absorbed through trivial redefinitions. The antighost dependence of a is thus controlled by $H_k^n(\delta|d)$. It is through these cohomological groups that the dynamics enter.

The group $H_k^n(\delta|d)$ has been shown in [34] to be isomorphic to the characteristic cohomology $H^{n-k}(d|\delta)$ of antifield-independent $(n-k)$ -forms that are weakly closed (i.e. closed modulo the equations of motion) but not weakly exact. Thus, $H_1^n(\delta|d)$ is isomorphic to the space of non-trivial weakly conserved currents. It does not vanish for the above theory, which is Poincaré invariant. For higher antighost degree $k > 1$, the groups $H_k^n(\delta|d)$ turn out, however, to be trivial [34].

The triviality of $H_k^n(\delta|d)$ for $k > 2$ follows from the general theorems of [34,24,35]. The triviality of $H_2^n(\delta|d)$ is demonstrated by following the perturbative argument of [34]: the theory obtained by taking the limit $\alpha = 0$ in the action (10.1) describes a set of $U(1)$ gauge fields together with a nonlinear g -field with rigid G -symmetry, which does not interact with the gauge fields.⁴ In that limit, the only

non-trivial cocycles of $H_2^n(\delta|d)$ are known to be $\lambda^a C_a^*$ up to trivial terms. These terms cannot be deformed to cocycles of the full theory when $\alpha \neq 0$, even in the Abelian case (the undifferentiated term g_a^* in δC_a^* prevents it) and thus we may conclude that $H_2^n(\delta|d)$ vanishes (we refer the reader to [34] for a detailed explanation of the method).

Note also that if $P_k^n([\tilde{\chi}^A])$ is a trivial invariant polynomial, $P_k^n([\tilde{\chi}^A]) = \delta M_{k+1}^n + dN_k^{n-1}$, then it is also trivial in the space of *invariant* polynomials, as one can see by setting all variables equal to zero in M_{k+1}^n and N_k^{n-1} but the gauge invariant $[\tilde{\chi}^A]$. In contrast to the situation analyzed in [6], the invariance of M_{k+1}^n and N_k^{n-1} is thus not an issue here.

Let us come back to the analysis of the cocycle condition $sa + db = 0$. The fact that the only non-vanishing cohomology group $H_k^n(\delta|d)$ is $H_1^n(\delta|d)$ implies that the BRST cocycles may be assumed to have an expansion that stops after the second summand, $a = a_0 + a_1$, where a_1 may be chosen to be invariant, $a_1 = P^I \omega_I$ and $P^I \in H_1^n(\delta|d)$. If $gh(a) < 0$, then of course $a = a_1$ [and $gh(a)$ is actually equal to -1]. There are thus two types of cocycles in $H(s|d)$: those for which a_1 does not vanish (they involve non-trivially the antifields) and those for which $a_1 = 0$. We shall call the first class ‘‘type I,’’ while solutions in the second class will be of ‘‘type II.’’

The analysis of the BRST cohomology for other gauge-invariant Lagrangians proceeds in exactly the same fashion. If these gauge invariant Lagrangians satisfy the rather mild ‘‘normality condition’’ given in [34], the groups $H_k^n(\delta|d)$ are also zero for $k > 1$. Thus, the solutions of the BRST cocycle condition $sa + db = 0$ fall again in two classes, just as for the specific Lagrangian (2.12).

A. Type I

Let $\{j_A^\mu\}$ be a complete set of gauge-invariant conserved currents and let c_A be such that

$$\partial_\mu j_A^\mu + \delta c_A = 0, \quad \gamma c_A = 0, \quad \text{antigh}(c_A) = 1 \quad (10.7)$$

(the c_A 's define the rigid symmetries associated with the conserved current j_A^μ [34]). The solutions of type I take the form

$$k_I^A (j_A^\mu \hat{\omega}_\mu^I + c_A \omega^I), \quad (10.8)$$

where the $\omega^I(g, C)$ are the De Rham cocycles and $\gamma \hat{\omega}_\mu^I + \partial_\mu \omega^I = 0$. In order to completely list all the independent solutions of type I, it is necessary to know all the local conserved currents. This is a question that depends on the detailed form of the Lagrangian and that will not be pursued here. Two remarks should however be made: (i) Potential anomalies are classified through $H^{(1,n)}(s|d)$. The above results indicate that there is no anomaly of type I, i.e. that the antifield dependence of anomalies may be eliminated through trivial redefinitions if $H_{DR}^2(G) = 0$, no matter what the conserved currents are. In a similar manner, one can get rid of the antifields in $H^{(0,n)}(s|d)$ if $H_{DR}^1(G) = 0$. $H^{(0,n)}(s|d)$ classifies the observables of the theory, and is relevant for renormalization and deformation issues. (ii) The

⁴The corresponding equations of motion are obtained by keeping in the original equations of motion the terms with the maximum number of derivatives. Thus, the perturbative method of [34] indeed applies.

solutions of type I become trivial upon restricting the fields to lie in a neighborhood of the identity since the forms ω^I are then trivial. Again, this is true independently of the form of the conserved currents.

B. Type II

The solutions of type II do not involve the antifields. The BRST cocycles $sa+db=0$ are then γ -cocycles, $\gamma a+db=0$ ($sa=\gamma a$). As we have seen, the solutions of this latter equation also fall into two classes: those that are strictly invariant and those that are invariant only modulo a total derivative, the so-called topological terms (see Sec. V). Although the cocycle condition of the s mod d cohomology reduces to the cocycle condition of the γ mod d cohomology when a does not contain the antifields, the coboundary condition is different. Some classes of $H(\gamma|d)$ are trivial in $H(s|d)$, namely, those that are zero when the equations of motion hold (or, more generally, are δ -exact). The dynamics plays thus a central role for determining the explicit form of the most general coboundary of type I. This is particularly obvious in the topological G/G -model, to which we now turn.

XI. G/G TOPOLOGICAL THEORY

For the topological action, Eq. (2.16) [36–41], the local BRST cohomology $H(s|d)$ reduces to the topological terms of Sec. V. There is no other cohomological class. The most expedient way to see this is to redefine the gauge-invariant $[\tilde{\chi}]$ variables [see Eq. (4.2)] in such a way that they form contractible pairs.

With the definition

$$\tilde{g}'_a^* = \tilde{g}_a^* - \partial_\mu \tilde{A}_a^{*\mu}, \quad (11.1)$$

the s -variations of the new tilde variables simply become

$$s\tilde{A}_a^{*\mu} = \tilde{I}_a^\mu, \quad s\tilde{I}_a^\mu = 0 \quad (11.2)$$

$$s\tilde{C}_a^* = \tilde{g}'_a^*, \quad s\tilde{g}'_a^* = 0. \quad (11.3)$$

For deriving the previous equations, one has to take into account Eqs. (2.17), (2.18), as well as the interchangeability of $D_\mu^{(A)}$ and $D_\mu^{(\Theta)}$ acting on I^μ . Thus, the gauge-invariant variables $\tilde{\chi}$ and their derivatives all drop out from the BRST cohomology, leaving only the undifferentiated group element g and the ghost C^a , the BRST transformations of which are

$$sg = gC, \quad sC = -C^2. \quad (11.4)$$

Accordingly, only the cocycles $\omega^I(g, C)$ and their lift appear in the BRST cohomology $H(s|d)$. In particular, the only non-trivial local observables are the winding numbers.

XII. PERTURBATION THEORY

The De Rham cohomology detects the global properties of the group manifold G . It is customary, in the context of perturbation theory, to restrict the fields g to a neighborhood

of the identity of G homeomorphic to R^k (denoted by \tilde{G} in the sequel). Then, $H_{DR}^k(\tilde{G})=0$ for $k \neq 0$ and $H_{DR}^0(\tilde{G})=R$. This greatly simplifies the analysis.

First, one finds that the ghosts C^a drop out from the cohomology. Indeed, one may redefine in \tilde{G} the ghosts as $C^a \rightarrow D^a = \Omega_b^a(h)C^b$. These new variables form contractible pairs with the h^a ,

$$\gamma h^a = D^a, \quad \gamma D^a = 0 \quad (12.1)$$

and also

$$sh^a = D^a, \quad sD^a = 0. \quad (12.2)$$

Thus, $H(\gamma)$ is given by the functions of the $\tilde{\chi}$ and their derivatives as observed in [42]. This implies that $H^k(\gamma)$ vanishes unless $k < 1$. The De Rham cocycles $\omega^I(g, C)$ are trivial in \tilde{G} .

Second, because the ghosts drop out from $H(\gamma)$, only the cocycles of one type survive in $H(\gamma|d)$, namely those that lead to trivial descents and that can be redefined so as to be strictly annihilated by γ . The topological cocycles disappear. At ghost number zero, the terms that are gauge invariant only up to a total derivative can thus be replaced by strictly gauge invariant terms involving only the $\tilde{\chi}$ -variables and their derivatives.

Finally, only the cohomological groups $H^{(0,n)}(s|d)$ and $H^{(-1,n)}(s|d)$ are different from zero. This is again because the ghosts drop out from the cohomology. Hence, in the expansion of the BRST cocycles a ($sa+db=0$) according to the antighost number, one may assume that there is only one term, $a = a_k$, with $gh(a) = -k = -\text{antigh}(a)$, $\gamma a_k = 0$, $\delta a_k + db_{k-1} = 0$. Non-trivial solutions are obtained only for $k=0,1$. The solutions with $k=1$ correspond to the gauge invariant conserved currents considered above. The solutions with $k=0$ are the observables and can be assumed to be strictly gauge invariant, i.e. to involve only the $[\tilde{\chi}]$ (note again that the condition $\delta a_k + db_{k-1} = 0$ is empty for $k=0$ since a contains then no antifield, but that the coboundary condition is non-trivial and eliminates the on-shell vanishing observables).

In particular, there is no perturbative anomaly. This provides a cohomological interpretation of the Wess-Zumino anomaly cancellation mechanism [11,12,43]. By enlarging the original field space with the group elements g (if the complete gauge group is broken), the anomaly becomes trivial, i.e. eliminable through a local counterterm. In the antifield language, this means that there exists a local counterterm M_1 which trivializes the anomaly ΔS , $\gamma M_1 = (M_1, S) = i\Delta S$ [44–46].

XIII. GLOBAL GHOSTS

Finally, the situation of a non-gauged sigma model shall be considered. The theory contains only the group elements $g \in G$ and is invariant under the global transformation $\delta_\epsilon g = gT_a \epsilon^a$, where ϵ^a are constant parameters. The main interest of this setting lies in the construction of effective actions,

where it is crucial to have an exhaustive list of all operators that are compatible with the rigid symmetries (see e.g. [2]).

The incorporation of rigid symmetries in the antifield formalism has been analyzed in [47,48] in the context of the sigma model. Further developments may be found in [49]. The symmetry parameters ϵ^a are promoted to anticommuting constant ghosts C^a and the relevant transformation laws read

$$\hat{\gamma}g = gT_a C^a, \quad (13.1)$$

$$\hat{\gamma}C^a = -\frac{1}{2}f^a{}_{bc}C^b C^c. \quad (13.2)$$

The aim is now to compute the cohomology of $\hat{\gamma}$ in the set of fields

$$\{C^a, g, \partial_\mu g, \partial_\mu \partial_\nu g, \dots\}. \quad (13.3)$$

Derivatives of the global ghosts obviously cannot occur since they are zero. As before, all the derivatives of g may be reexpressed through the variables $\Theta = g^{-1}dg$ and their subsequent derivatives, yielding as new coordinates of the jet-space the set

$$\{C^a, g, \Theta_\mu, \partial_\mu \Theta_\nu, \dots\}, \quad (13.4)$$

or equivalently, using the invariant tilde variables $\tilde{\Theta}^a = U(g)_b^a \Theta^b$,

$$\{C^a, g, \tilde{\Theta}_\mu, \partial_\mu \tilde{\Theta}_\nu, \dots\}. \quad (13.5)$$

In these variables, the action of $\hat{\gamma}$ takes the simple form

$$\hat{\gamma}g = gT_a C^a, \quad (13.6)$$

$$\hat{\gamma}C^a = -\frac{1}{2}f^a{}_{bc}C^b C^c, \quad (13.7)$$

$$\hat{\gamma}[\tilde{\Theta}^a] = 0. \quad (13.8)$$

The first two equations can again be identified with the De Rham complex while the last equation states that the $[\tilde{\Theta}]$ are invariant. The representatives of the $\hat{\gamma}$ -cohomology have thus the form found above,

$$m = P^I([\tilde{\Theta}])\omega_I(C, g), \quad (13.9)$$

where the P^I are arbitrary polynomials and the ω_I form a basis of the De Rham cohomology of G .

Apart from the strictly invariant terms, which are exhaustively classified by Eq. (13.9), also the invariant terms that are invariant only up to a total divergence play an important role in various physical models. These terms can be analyzed via descent equations in almost the same way as in Sec. VI. A non-trivial solution of the mod d cocycle condition at form degree n , $\hat{\gamma}a^{(g,n)} + da^{(g+1,n-1)} = 0$, necessarily descends all the way down to zero-forms as in Eq. (6.2). But now, in the last step of the descent, the constants cannot be discarded any more. Indeed, one gets at the last step the condition

$d\hat{\gamma}a^{(g+n,0)} = 0$. It follows from the algebraic Poincaré lemma that $\hat{\gamma}a^{(g+n,0)}$ has to be equal to a constant, which must be of ghost degree $g+n+1$. In the gauged case, there was no such constant since the ghosts are fields. Here, however, the ghosts are constant and so $\hat{\gamma}a^{(g+n,0)}$ may be a polynomial in C^a :

$$\hat{\gamma}a^{(g+n,0)}(g, C) = \alpha(C). \quad (13.10)$$

This phenomenon was previously observed in a similar context in [50]. By applying $\hat{\gamma}$ to Eq. (13.10) it follows that $\hat{\gamma}\alpha(C) = 0$. If in addition $\alpha(C)$ is $\hat{\gamma}$ -trivial in the space of constants, $\alpha(C) = \hat{\gamma}\beta(C) = 0$, then it may be absorbed by trivial redefinitions of the preceding descent equations. In that case, the bottom $a^{(g+n,0)}$ satisfies $\hat{\gamma}a^{(g+n,0)}(g, C) = 0$ and thus, as we have seen in Sec. VI, is equivalent to a De Rham cocycle $\omega_I(\Theta, g) \in H_{DR}^{g+n}(G)$. For $g=0$, these cocycles lift up to winding number terms in form degree n .

Upon restricting the g -field to a neighborhood \tilde{G} of the identity, the De Rham cocycles become trivial and accordingly can be absorbed through redefinitions. Thus, if $\alpha(C)$ vanishes in Eq. (13.10), or is $\hat{\gamma}$ -trivial in the space of constants, the original $a^{(g,n)}$ differs from a term strictly annihilated by γ by a total divergence. The obstruction to replacing a term invariant up to a total divergence in the Lagrangian by a term strictly invariant is thus an element of the Lie algebra cohomology $H(\mathcal{G})$: if one hits a non-trivial Lie cocycle $\alpha(C)$ in the descent, there is no way to redefine the Lagrangian so that it is strictly invariant.

Furthermore, any Lie algebra cocycle can be written, in the neighborhood of the identity, as $\hat{\gamma}a^{(g+n,0)}(g, C)$ for some $a^{(g+n,0)}$ that involves explicitly the σ -field g . This is because the De Rham cohomology of \tilde{G} is trivial. Replacing C by $g^{-1}(d + \hat{\gamma})g$ in $a^{(g+n,0)}(g, C)$ and keeping the term of form degree n yields the top of a descent generating $\alpha(C)$ at the bottom. Thus, any Lie algebra cocycle $\alpha(C)$ can be lifted all the way up to form degree n . [On the full group manifold G , the term $a^{(g+n,0)}(g, C)$ will in general not be globally defined. This leads to a multiply valued Lagrangian with a quantization condition on the corresponding coupling constant.] It follows that the local n -forms with vanishing ghost degree that are invariant up to a divergence are classified by the Lie algebra cohomology at ghost degree $n+1$, $H^{n+1}(\mathcal{G})$.

For compact groups, the Lie algebra cohomology is isomorphic to the De Rham cohomology of the (complete) group manifold G , which establishes the link to the results of D'Hoker and Weinberg [13] (see also [51]).

XIV. CONCLUSIONS

In this paper, we have investigated the local cohomology of the gauged principal nonlinear sigma-model.

The analysis has been pursued by taking due account of the topology of the group manifold. We have shown that the most general local BRST cocycle a is, up to trivial contributions, the sum of terms of three different kinds:

$$sa + db = 0 \Leftrightarrow a = A_1 + A_2 + A_3 + sm + dn. \quad (14.1)$$

The cocycle A_1 has been called of “type I” and involves the antifields linearly, as well as the conserved currents. The cocycles A_2 and A_3 do not involve the antifields and are of “type II.” A_2 is strictly annihilated by γ and involves therefore only the gauge invariant variables $\tilde{\chi}^A$ and their derivatives. The cocycle A_3 depends on g and the ghosts. It is a solution of $\gamma A_3 + db = 0$ and is related to the De Rham cohomology of the group manifold. A_1 is also related to $H_{DR}^k(G)$, so that both A_1 and A_3 may be regarded as “generalized winding number terms.”

At ghost number 0 and form degree n (observables), A_3 exists if and only if $H_{DR}^n(G) \neq 0$. Similarly, A_1 exists if and only if $H_{DR}^1(G) \neq 0$ and if there are non-trivial conserved currents. A_3 defines a term which is gauge invariant up to total derivatives, and A_1 defines a term which is gauge in-

variant up to field equations and total derivatives, while A_2 defines a term which is strictly gauge invariant.

In perturbation theory, it is customary to replace G by a topologically trivial neighborhood \tilde{G} of the identity. In doing so, both A_1 and A_3 disappear at ghost number 0, and only the strictly gauge invariant terms are left. Furthermore, there is no cohomology at positive ghost number. In particular, there is no non-trivial anomaly.

It would be interesting to extend the analysis to coset models built on a homogeneous space G/H . Work in this direction is in progress.

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