Universality of low-energy scattering in 2+1 dimensions

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(Received 22 January 1998; published 24 June 1998)

For any relativistic quantum field theory in 2+1 dimensions, with no zero mass particles, and satisfying the standard axioms, we establish a remarkable low-energy theorem. The S-wave phase shift, $\delta_0(k)$, k being the c.m. momentum, vanishes as either $\delta_0 \to c/\ln(k/m)$ or $\delta_0 \to O(k^2)$ as $k \to 0$. The constant c is universal and $c = \pi/2$. This result follows only from the rigorously established analyticity and unitarity properties for 2-particle scattering. This kind of universality was first noted in non-relativistic potential scattering, albeit with an incomplete proof which missed, among other things, an exceptional class of potentials where $\delta_0(k)$ is $O(k^2)$ near $k=0$. We treat the potential scattering case with full generality and rigor, and explicitly define the exceptional class. Finally, we look at perturbation theory in ϕ_3^4 and study its relation to our non-perturbative result. The remarkable fact here is that in n-th order the perturbative amplitude diverges like $(\ln k)^n$ as $k \to 0$, while the full amplitude vanishes as $(\ln k)^{-1}$. We show how these two facts can be reconciled.
[S0556-2821(98)02714-3]

PACS number(s): 11.10.Jj, 11.10.Cd, 11.10.Kk, 11.80.−m

I. INTRODUCTION

Quantum field theories in 2+1 dimensions provide us with a useful field of investigation not only for theoretical and mathematical issues, but also in some cases for actual physical problems. In this paper we shall derive a general universality property for the low-energy scattering amplitudes in 2+1 massive field theories.

This universality property was first noted [1] in potential scattering. In three space dimensions, and for a very large class of spherically symmetrical potentials, the low-energy behavior of the S-wave phase shift is given by $\delta_0(k) \sim ak$ as $k \to 0$, where k is the momentum, and a is the scattering length which depends on the potential.

In two space dimensions, the situation is radically different. For a large class of rotationally symmetrical potentials, the behavior of $\delta_0(k)$ is $c/(\ln k)$ as $k \to 0$. More significantly, for a large class of such potentials, $c = \pi/2$, i.e.,

$$\delta_0(k) \sim \frac{\pi/2}{\ln k} \tag{1.1}$$

as $k \to 0$. In Ref. [1], the result (1.1) was shown to hold for the class of exponentially decreasing potentials, $V(r) = O(e^{-\mu r})$ with $\mu > 0$ as $r \to \infty$. This dependence of the phase shift $\delta_0(k)$ on $\ln k$ is also of practical importance. For example, it is the underlying reason why wire antennas are very efficient and widely used [2].

However, Eq. (1.1) clearly cannot be the entire story. In the special case $V(r) = 0$, $\delta_0(k) = 0$ identically—a trivial result that contradicts Eq. (1.1). From this case it is expected that there is a subspace of two-dimensional rotationally symmetrical potentials where Eq. (1.1) does not hold.

In this paper we consider both non-relativistic potential scattering in two space dimensions and relativistic quantum field theories in 2+1 dimensions. For potential scattering, we give a proof of Eq. (1.1) with the most general condition on the potential. We also identify the exceptional cases where $\delta_0(k) \to 0$ faster than $(\ln k)^{-1}$ as $k \to 0$. This is carried out in Sec. II.

In Sec. III we consider the case of a relativistic quantum field theory, more specifically one with the kinematics of ϕ_3^4, i.e., equal mass, spin zero, and neutral particles. Here we again show that analyticity, symmetry, and unitarity lead to the universal behavior $\delta_0(k) \to \pi/(2\ln k)$ as $k \to 0$, where k is the c.m. momentum. Again, there are exceptions with $\delta_0 = O(k^2)$. In the specific case of ϕ_3^4, it is shown that a leading-log summation of perturbation theory does indeed give us $\delta_0(k) \to (\pi/2)(\ln k)^{-1}$ as $k \to 0$. In other words, in this case, each order in perturbation theory for the n-th amplitude diverges like $(\ln k)^n$ as $k \to 0$, but the sum, the full amplitude, vanishes as $(\ln k)^{-1}$.

II. TWO-DIMENSIONAL POTENTIAL SCATTERING

In two dimensions the partial wave expansion of the scattering amplitude $T(k,\theta)$ is given by

$$T(k,\theta)=\frac{1}{\sqrt{k}}\sum_{n=0}^{\infty}\epsilon_n(e^{i\delta_n}\sin\,\delta_n)\cos\,n\theta, \tag{2.1}$$

where $\epsilon_0=1$, $\epsilon_n=2$ for $n\geqslant 1$. The phase shifts $\delta_n(k)$ are obtained in the standard way from the solutions of the Schrödinger equation. In this paper we are interested mainly in the term $n=0$.

The $n=0$ solutions, $u(k,r)$, satisfy

$$\left[\frac{d^2}{dr^2}+\frac{1}{4r^2}+k^2-gV(r)\right]u(k,r)=0. \tag{2.2}$$

Without the loss of generality, g is taken to be non-negative. Equation (2.2), under conditions on $V(r)$ to be specified below, has two independent solutions: behaving like $\sqrt{r}$ and $\sqrt{r}\ln r$ as $r\to 0$. We take as a regular solution

$$u(k,0)=0, u(k,r)\sim\sqrt{r}, \tag{2.3}$$

corresponding to a finite wave function at the origin. For a discussion of this choice, see Appendix A.

The phase shift, $\delta_0(k)$, is defined by

$$u(k,r)\underset{r\to\infty}{\to}c\sqrt{r}[\cos\,\delta_0 J_0(kr)-\sin\,\delta_0 Y_0(kr)]. \tag{2.4}$$

The sign of the second term is chosen to correspond to the definition of δ_0 in the 3-dimensional case, i.e., $u\to c\sqrt{\pi/2}\cos(kr-\pi/4+\delta_0)$ as $r\to\infty$. By rearranging terms in Eq. (2.4), we get

$$u(k,r)\underset{r\to\infty}{\to}c\sqrt{r}e^{-i\delta_0}[H_0^{(2)}(kr)+e^{2i\delta_0}H_0^{(1)}(kr)]. \tag{2.5}$$

We can always choose $u(k,r)$ such that

$$u(k,r)\underset{r\to\infty}{\to}-\frac{1}{2i\sqrt{k}}[e^{-i(kr-\pi/4)}+S(k)e^{+i(kr-\pi/4)}], \tag{2.6}$$

where we have used the asymptotic formulas for $H_0^{(1),(2)}(z)$ for large $|z|$, and

$$S(k)\equiv e^{2i\delta_0(k)}. \tag{2.7}$$

The Jost functions in this case are solutions of Eq. (2.2), finite at $r=0$, which we denote as $f_\pm(k,r)$ with the asymptotic behavior

$$f_\pm(k,r)\underset{r\to\infty}{\to}e^{\mp i(kr-\pi/4)}. \tag{2.8}$$

We can thus write

$$u(k,r)=-\frac{1}{2i\sqrt{k}}[f_+(k,r)+S(k)f_-(k,r)]. \tag{2.9}$$

It is convenient to follow a method of treating singular potentials [3]. We shall see below how this simplifies the task of taking the limit $k\to 0$. Following Ref. [3], we define $g(k,r)$ as

$$g(k,r)\equiv\frac{1}{2i\sqrt{k}}[f_+(k,r)+f_-(k,r)]. \tag{2.10}$$

The sign here is different from that in the 3-dimensional case. From Eq. (2.9) we now have

$$u(k,r)=-[g(k,r)+A(k)f_-(k,r)], \tag{2.11}$$

where $A(k)$ is the $n=0$ scattering amplitude

$$A(k)\equiv\frac{1}{2i\sqrt{k}}[S(k)-1]\equiv\frac{1}{\sqrt{k}}e^{i\delta_0}\sin\,\delta_0. \tag{2.12}$$

The condition $u(k,r)\to 0$ as $r\to 0$ gives us

$$A(k)=-\lim_{r\to 0}[g(k,r)/f_-(k,r)]. \tag{2.13}$$

Notice that this limit is always finite. This is because f_-, being a combination of Re f_- and Im f_-, i.e., of two linearly independent solutions of Eq. (2.2), has to behave as $f_-\sim\sqrt{r}\,\ln r$ as $r\to 0$.

The asymptotic behavior of $u(k,r)$ can be written as

$$u(k,r)\underset{r\to\infty}{\to}\frac{i\,\cos(kr-\pi/4)}{\sqrt{k}}-A(k)e^{i(kr-\pi/4)}. \tag{2.14}$$

This follows from Eqs. (2.8) and (2.9).

Following Ref. [3], we introduce a Green's function $G(r,r')$ for $r,r'>0$, defined by

$$\left[\frac{d^2}{dr^2}+\frac{1}{4r^2}+k^2\right]G(r,r')\equiv\delta(r-r'). \tag{2.15}$$

This G is given explicitly by

$$G(r,r')=\frac{\pi}{2}\sqrt{rr'}[J_0(kr)Y_0(kr')-J_0(kr')Y_0(kr)]\theta(r'-r), \tag{2.16}$$

where J_0 and Y_0 are the standard Bessel functions of the first and second kind.

The next step is to introduce a $u_0(k,r)$ which is a solution of the free, $V=0$, Schrödinger equation. We set

$$u_0(k,r)\equiv u(k,r)-g\int_0^\infty dr'G(r,r')V(r')u(k,r'). \tag{2.17}$$

From Eq. (2.15) it is now obvious that

$$\left[\frac{d^2}{dr^2}+\frac{1}{4r^2}+k^2\right]u_0(k,r)=0. \tag{2.18}$$

As $r\to\infty$, $u_0\to u$, and from Eq. (2.14) it is clear that u_0 is given by

$$u_0(k,r)=\sqrt{\frac{\pi}{2}}i\sqrt{r}J_0(kr)-\sqrt{\frac{\pi}{2}}A(k)\sqrt{kr}H_0^{(1)}(kr). \tag{2.19}$$

The integral equation for u can now be written as

$$u(k,r)=u_0(k,r)+g\int_r^\infty dr'\tilde{G}(k;r,r')V(r')u(k,r'), \tag{2.20}$$

with

$$\tilde{G}(k;r,r')=\frac{\pi}{2}\sqrt{rr'}[J_0(kr)Y_0(kr')-J_0(kr')Y_0(kr)]. \tag{2.21}$$

Using Eqs. (2.11) and (2.19), we can get from Eq. (2.20) two separate integral equations for $g(k,r)$ and $f_-(k,r)$. These are

$$g(k,r)=-i\sqrt{\frac{\pi}{2}}\sqrt{r}J_0(kr)+g\int_r^\infty dr'\tilde{G}(k;r,r')V(r')g(k,r') \tag{2.22}$$

and

$$f_-(k,r)=\sqrt{\frac{\pi}{2}}\sqrt{kr}H_0^{(1)}(kr)+g\int_r^\infty dr'\tilde{G}(k;r,r')V(r')f_-(k,r'). \tag{2.23}$$

These last two equations are the same except for the inhomogeneous term. We are interested in studying them in the limit of small k. Before we can do that, it is convenient to remove a $\sqrt{k}$ factor from f_- and define $\tilde{f}_-(k,r)$ as

$$\tilde{f}_-(k,r)\equiv\frac{1}{\sqrt{k}}f_-(k,r). \tag{2.24}$$

With this definition, Eq. (2.13) becomes

$$e^{i\delta_0(k)}\sin\delta_0(k)=-\lim_{r\to 0}[g(k,r)/\tilde{f}_-(k,r)]. \tag{2.25}$$

We now take the $k\to 0$ limit of Eq. (2.22) and the equation corresponding to Eq. (2.23) for $\tilde{f}_-$. Using

$$\frac{\pi}{2}[J_0(kr)Y_0(kr')-J_0(kr')Y_0(kr)]=\ln\frac{r'}{r}+O(k^2) \tag{2.26}$$

for small k, we get

$$g(k,r)=-i\sqrt{\frac{\pi}{2}}\sqrt{r}+g\int_r^\infty dr'\sqrt{rr'}\left(\ln\frac{r'}{r}\right)V(r')g(k,r')+O(k^2) \tag{2.27}$$

and

$$\tilde{f}_-(k,r)=i\sqrt{\frac{2}{\pi}}\left(\ln k+\ln r-\ln 2+\gamma-i\frac{\pi}{2}\right)\sqrt{r}+g\int_r^\infty dr'\sqrt{rr'}\left(\ln\frac{r'}{r}\right)V(r')\tilde{f}_-(k,r')+O(k^2), \tag{2.28}$$

where γ is Euler's constant. For $r>0$, taking the $k\to 0$ limit under the integral sign is allowed if we assume

$$\int_a^\infty r'dr'(1+|\ln r'|^2)|V(r')|<\infty, \quad a>0. \tag{2.29}$$

We shall discuss this condition in more detail later.

At this stage, we introduce two functions, $A(r)$ and $B(r)$, defined by the following integral equations:

$$A(r)=1+g\int_r^\infty r'dr'\left(\ln\frac{r'}{r}\right)V(r')A(r') \tag{2.30}$$

and

$$B(r)=\ln r+g\int_r^\infty r'dr'\left(\ln\frac{r'}{r}\right)V(r')B(r'). \tag{2.31}$$

It is clear from inspecting Eqs. (2.27) and (2.28) that

$$A(r)\equiv\lim_{k\to 0}\left[\frac{ig(k,r)}{\sqrt{\pi/2}\sqrt{r}}\right] \tag{2.32}$$

and

$$\left[\frac{-i\tilde{f}_-(k,r)}{\sqrt{2/\pi}\sqrt{r}}\right]\equiv\left[A(r)\left(\ln k-\ln 2+\gamma-i\frac{\pi}{2}\right)+B(r)\right]+O(k^2). \tag{2.33}$$

Thus, for small k we have

$$-\left[\frac{g(k,r)}{\tilde{f}_-(k,r)}\right]=\frac{(\pi/2)A(r)}{A(r)\left(\ln k-\ln 2+\gamma-i\frac{\pi}{2}\right)+B(r)}+O(k^2). \tag{2.34}$$

Our task is now to study the existence of solutions $A(r)$ and $B(r)$ of the two integral equations (2.30) and (2.31), and more specifically, to study the behavior of A and B for small r.

In Appendix B, we shall prove that for the general class of potentials, $V(r)$, satisfying

$$\text{(A)} \quad \int_0^\infty r'dr'|V(r')|(|\ln r'|+1)<\infty \tag{2.35}$$

and

$$\text{(B)} \quad \int_a^\infty r'dr'|V(r')|(\ln r')^2<\infty, \quad a>1, \tag{2.36}$$

the solutions $A(r)$ and $B(r)$ exist for all $r>0$, and furthermore, near $r=0$ one has the behavior

$$A(r)=[-gC_a(g)+o(1)]\ln r \tag{2.37}$$

and

$$B(r)=[1-gC_b(g)+o(1)]\ln r. \tag{2.38}$$

Here,

$$C_a(g)=\int_0^\infty rdrV(r)A(r) \tag{2.39}$$

and

$$C_b(g)=\int_0^\infty rdrV(r)B(r). \tag{2.40}$$

Both integrals for C_a and C_b are absolutely convergent since one can easily show that, as $r\to\infty$, A and B have the bounds

$$|A(r)|<\text{const}, \quad |B(r)|<\text{const}\times|\ln r|, \tag{2.41}$$

for $r>r_0>1$. The convergence of Eqs. (2.39) and (2.40) at $r=0$ is guaranteed by Eqs. (2.35), (2.37), and (2.38).

Going back to Eq. (2.34), we write, for the neighborhood of $r\approx 0$,

$$-\frac{g(k,r)}{\tilde{f}_-(k,r)}=\frac{(\pi/2)gC_a(g)\ln r+O(1)}{gC_a(g)\ln r(\ln k-\ln 2+\gamma-i\pi/2)+[gC_b(g)-1]\ln r+O(1)}+O(k^2). \tag{2.42}$$

This result leads to

$$e^{i\delta_0(k)}\sin\delta_0(k)=\frac{\pi}{2}\left[\frac{gC_a(g)}{gC_a(g)(\ln k-\ln 2+\gamma-i\pi/2)+[gC_b(g)-1]}\right]+O(k^2). \tag{2.43}$$

There are now two cases to consider, $C_a(g)\neq 0$ and $C_a(g)=0$. For $C_a(g)\neq 0$, we have the universal result as $k\to 0$:

$$\delta_0(k)=\frac{\pi}{2\ln k}+O\left[\frac{1}{(\ln k)^2}\right]. \tag{2.44}$$

One should note that $C_b(g)$ is finite. A somewhat stronger form of Eq. (2.44) is that, as $k\to 0$,

$$e^{i\delta_0(k)}\sin\delta_0(k)=\frac{\pi}{2\ln k-i\pi}+O\left[\frac{1}{(\ln k)^{2,3}}\right], \tag{2.45}$$

meaning that the real part of the first term is accurate to the order $(\ln k)^{-2}$ while the imaginary part is accurate to $(\ln k)^{-3}$.

The second case, $C_a(g)=0$, is clearly exceptional. If $C_a(g)=0$ for any interval $g_1<g<g_2$, then $V\equiv 0$. For $V\equiv 0$, $C_a(g)$ can only vanish for discrete values of g. In this case, because of Eqs. (2.38) and (2.39), $(1-gC_b)$ cannot vanish. Hence, it follows from Eq. (2.42) that, as $k\to 0$,

$$\delta_0(k)=O(k^2). \tag{2.46}$$

Equation (2.43) also implies the uniform formula

$$\delta_0(k)=\frac{\xi}{\xi+1}\,\frac{\pi}{2\ln k-i\pi}+O\left[\frac{1}{(\ln k)^{2,3}}\right] \tag{2.47}$$

in the same sense as Eq. (2.45), where

$$\xi=\frac{gC_a(g)\big(\ln k-i\pi/2\big)}{gC_b(g)-1}. \tag{2.48}$$

III. THRESHOLD BEHAVIOR IN 2+1 DIMENSIONS: THE FIELD THEORETICAL CASE

We take as our starting point the axiomatic local field theory with a minimum non-zero mass. There is then very little difference between $2+1$and $3+1$ dimensions. In both cases, the on-shell scattering amplitude depends on two variables. The analyticity domain of the scattering amplitude is obtained, in both cases, in two steps: (i) the analytic continuation of the off-shell amplitude [4], and (ii) the use of the positivity of the absorptive part to enlarge the analyticity domain [5]. The partial wave expansion in the (2+1)-dimensional case is given in terms of Chebyshev polynomials and not Legendre polynomials. Indeed, for the $(2+1)$-dimensional case, we have

$$T(s,\cos\theta)=16\sum_{n=0}^{\infty}\epsilon_n f_n(s)\cos n\theta. \tag{3.1}$$

Here, s is the square of the center-of-mass energy, and θ is the scattering angle. In the elastic region, $f_n(s)$ is related to the phase shifts by

$$f_n(s)=\sqrt{s}e^{i\delta_n}\sin\ \delta_n. \tag{3.2}$$

This and the factor of 16 in Eq. (3.1) are chosen to give $T(s,\cos\theta)=-g+O(g^2)$ in a ϕ_3^4 perturbative field theory with a $(g/4!)\phi^4$ interaction.

The absorptive part of T is

$$A_s(s,\cos\ \theta)=16\sum_{n=0}^{\infty}\ \epsilon_n\times\text{Im}\ f_n(s)\cos\ n\theta, \tag{3.3}$$

with Im $f_n(s)\geqslant 0$, from the unitarity condition. From Eq. (3.3), it is easy to obtain

$$\left|\left(\frac{d}{d\ \cos\ \theta}\right)^n A_s(s,\cos\ \theta)\right| \leqslant \left(\frac{d}{d\ \cos\ \theta}\right)^n A_s(s,\cos\ \theta)\bigg|_{\cos\ \theta=1};\quad s\geqslant 4m^2 \tag{3.4}$$

for all θ such that $-1\leqslant\cos\ \theta\leqslant+1$. This last inequality is precisely what made the enlargement of the analyticity domain in the $3+1$ case possible [5]. Therefore, one gets the same enlargement in $2+1$ dimensions.

For simplicity, we consider a case with the kinematics and symmetry of pion-pion scatterings although our results are much more general. We use the Mandelstam variables

$$s=4(k^2+m^2),$$
$$t=2k^2(\cos\ \theta-1),$$
$$u=4m^2-s-t. \tag{3.5}$$

For any fixed t, $|t|<4m^2$, $T(s,t)$ is analytic in the doubly cut s-plane with cuts along

$$s=4m^2+\lambda,$$
$$u=4m^2+\lambda;\quad \lambda>0. \tag{3.6}$$

For fixed s, the absorptive part, $A_s(s,\cos\ \theta)$, is analytic inside an ellipse in the $\cos\theta$-plane, which is an enlargement of the Lehmann ellipse [6]. The foci are at $\cos\ \theta=\pm 1$ and the right extremity is at $\cos\ \theta=1+4m^2/2k^2$.

The partial wave amplitudes, $f_n(s)$, are defined as

$$f_n(s)=\frac{1}{16\pi}\int_{-1}^{+1}T(s,\cos\theta)\cos n\theta\frac{d(\cos\theta)}{\sin\theta}. \tag{3.7}$$

The f_n's are analytic in a region that contains

$$|s-4m^2|<4m^2, \tag{3.8}$$

excluding a cut along $4m^2\leqslant s\leqslant 8m^2$. A major difference with the $(3+1)$-dimensional case is the kinematical factor $\sqrt{s}$ which comes from the unitarity as explicitly shown in Eq. (3.2), a point clarified with the help of Stora [7].

Thus, the unitarity condition in $2+1$ dimensions is

$$\text{Im}\ f_n(s)\geqslant\frac{1}{\sqrt{s}}|f_n(s)|^2,\quad \forall s>4m^2. \tag{3.9}$$

In the elastic region, $4m^2\leqslant s<16m^2$,

$$\text{Im}\ f_n(s)=\frac{1}{\sqrt{s}}|f_n(s)|^2. \tag{3.10}$$

This slightly changed form of the unitarity condition given in Eq. (3.9) gives a different Froissart bound [8] in the $2+1$ case. The number of partial waves effectively contributing to the scattering amplitude is still bounded by

$$L=C\sqrt{s}\ \ln\ s, \tag{3.11}$$

for large s. However, the Froissart bound in $2+1$ dimensions is

$$|F(s,\cos\ \theta)|<Cs\ \ln\ s,\quad -1\leqslant\cos\ \theta\leqslant+1. \tag{3.12}$$

This is instead of the $s\ \ln^2 s$ in the $3+1$ case. The number of subtractions in the dispersion relations, for $|t|<4m^2$, is still at most 2, as in the $3+1$ case [9].

The general properties outlined so far are sufficient to determine the singularity of $f_n(s)$ at $k=0$. For simplicity, we restrict ourselves to the S-wave case, although our method applies to the higher waves. It is convenient to change variables and define

$$f_0(s)=F_0(k). \tag{3.13}$$

We also set the mass $m=1$. In the variable k, the analyticity domain of $F_0(k)$ contains the half circle Γ:

$$\Gamma:\{|k|<1,\quad \text{and}\quad \text{Im}\ k>0\}. \tag{3.14}$$

A very important property of $T(s,t)$ is the reality property: T is real for $s<4$, $t<4$, $u<4$. From this property, it follows that $f_0(s)$ is real for $0<s<4$, and hence $F_0(k)$ is real for $k=i\kappa$, $0<\kappa<1$. By Schwarz's reflection principle, for $k\in\Gamma$, we have

$$F_0(k)=F_0^*(-k^*). \tag{3.15}$$

The unitarity condition, Eq. (3.10), can be written in a form suitable for analytic continuation. With initially $k=k^*$, we write

$$F_0(k)-F_0^*(k^*)=\frac{2i}{\sqrt{s}}F_0(k)F_0^*(k^*). \tag{3.16}$$

This gives

$$F_0(k)=\frac{F_0^*(k^*)}{1-(2i/\sqrt{s})\,F_0^*(k^*)}, \tag{3.17}$$

and defines a function analytic in the second sheet. This function will be the continuation to the semicircle, $|k|<1$, Im $k<0$, through the line $0<k<1$. The only thing to prevent that would be an accumulation of zeros of [1

$-(2i/\sqrt{s})F_0^*(k^*)]$ along this line, giving a natural boundary. There is nothing in the general axioms to prevent that [10]. However, it is sufficient to assume that $F_0(k)$ is continuous on $0<k<1$ in order to avoid this catastrophe. We thus get the continuation of $F_0(k)$ to the second sheet [11], which, using the reality condition (3.15), can be written as

$$F_0(k)=\frac{F_0(-k)}{1-(2i/\sqrt{s})\,F_0(-k)}. \quad (3.18)$$

Hence, $F_0(k)$ is meromorphic for $|k|<1$, outside the origin.

Let us introduce $G_0(k)$ as

$$G_0(k)=\frac{1}{F_0(k)}. \quad (3.19)$$

We get

$$G_0(k)=G_0(-k)-\frac{2i}{\sqrt{s}}. \quad (3.20)$$

Next, we define $H_0(k)$ as

$$H_0(k)\equiv G_0(k)-\frac{2}{\pi\sqrt{s}}\left(\ln k-i\frac{\pi}{2}\right). \quad (3.21)$$

$H_0(k)$ is again real for $k=i\kappa$, $0<\kappa<1$. Using Eq. (3.21), we get

$$H_0(k)=H_0(-k). \quad (3.22)$$

H_0 is therefore an even function of k, i.e.,

$$H_0(k)\equiv K_0(k^2). \quad (3.23)$$

$K_0(k^2)$ is a meromorphic function of k^2, and the S-wave amplitude can be written as

$$F_0(k)=\frac{1}{K_0(k^2)+(2/\pi\sqrt{s})(\ln k-i\,\pi/2)}. \quad (3.24)$$

If $K_0(k^2)$ has no pole at the origin, the $\ln k$ dominates the denominator as $k\to 0$, and we get

$$F_0(k)\simeq\frac{\pi}{2}\sqrt{s}\left(\frac{1}{\ln k}\right). \quad (3.25)$$

The phase shift then behaves as

$$\delta_0(k)\simeq\frac{\pi}{2\ln k}, \quad (3.26)$$

which is precisely the behavior obtained in the potential case. As in the potential case, the existence of a pole of $K_0(k^2)$ at $k^2=0$ cannot be excluded.

The derivation we presented above also applies to higher waves, but it can be proved that what is hopefully an exception for $n=0$ turns out to be the rule for $n\geqslant 1$. $K_n(k^2)$ has a pole, and we shall show in a future publication that $\delta_n\sim k^{2n}$ for $n\geqslant 1$.

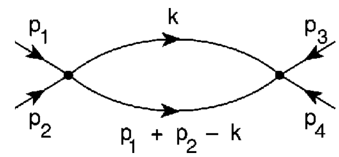

FIG. 1. A second-order diagram.

For the restricted class of potentials such that

$$\int_0^\infty r dr|1+|\ln r||\,|V(r)|\exp\,\mu r<\infty,$$

the derivation of the dispersion relations for $|t|<\mu^2$ obtained first by one of us [12] in the $3+1$ case also holds in $2+1$ dimensions. It implies that the partial wave amplitude is analytic in $|k|<\mu/2$, Im $k>0$, and therefore the derivation presented in this section applies also to this potential case.

Equation (3.17) was also obtained by Bros and Iaglonitzer in Ref. [13], Eq. (5), and in Ref. [14], Eq. (19), in a more general but less elementary approach based on a postulated analyticity of the S-matrix. These authors emphasize the Riemann sheet structure at the threshold rather than the actual behavior of the physical scattering amplitude.

IV. PERTURBATION THEORY FOR ϕ_3^4

It is of importance to compare our result with the perturbation theory. We are fortunate that in 2+1 dimensions we have a rigorously defined super-renormalizable theory [15] with a mass gap, namely, ϕ_3^4.

Taking

$$\mathcal{L}_{\rm int}(\phi)=\frac{-g}{4!}:\phi^4(x):$$

we obtain up to the order g^2 for $T(p_1,p_2;-p_3,-p_4)$

$$T(s,t)=-g+g^2[f(s)+f(t)+f(u)]+O(g^3), \quad (4.1)$$

where $f(s)$, $s=(p_1+p_2)^2$, is given by the Feynman diagram shown in Fig. 1,

$$f(s)=\left(\frac{-i}{2}\right)\int\frac{d^3k}{(2\pi)^3}\times\frac{1}{(k^2-\mu^2+i\epsilon)[(p_1+p_2-k)^2-\mu^2+i\epsilon]}. \quad (4.2)$$

The factor $(\frac{1}{2})$ is for identical outgoing particles, and the $(-i)$ follows from $S=1+iT$, S being the S-matrix.

This last integral can be easily evaluated in the Euclidean region, $s<4\mu^2$, by carrying out a Wick rotation, and the result is

$$f(s)=-\frac{1}{16\pi\sqrt{s}}\ln\left(\frac{2\mu-\sqrt{s}}{2\mu+\sqrt{s}}\right),\quad 0<s<4\mu^2. \quad (4.3)$$

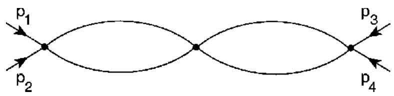

FIG. 2. A third-order diagram behaving as $(\ln k)^2$ as $k \to 0$.

The normalization of T is chosen such that the elastic unitarity is given by

$$\frac{1}{2i}(T-T^*)=\frac{1}{16\sqrt{s}}\int_0^{2\pi}\frac{d\theta}{2\pi}|T(s,\theta)|^2, \quad 4\mu^2 \leqslant s < 16\mu^2. \tag{4.4}$$

The partial wave expansion is then

$$T(s,\theta)=16\sqrt{s}\sum_n \epsilon_n \cos n\theta e^{i\delta_n}\sin\delta_n. \tag{4.5}$$

As $s \to 4\mu^2$, $k \to 0$, then for the physical θ, $t \to 0$, $u \to 0$, and the leading log term comes from Eq. (4.3), since $f(0)$ is finite.

We get for $k \to 0$

$$T=-g-\frac{g^2}{32\pi\mu}\ln\frac{k^2}{\mu^2}+O(1)g^2+O(g^3). \tag{4.6}$$

The first thing to notice is that at the order g^2, T diverges as $k \to 0$. This is just the opposite of the full result we obtained in the previous section where $T \to 0$ as $k \to 0$.

In the third order, the leading $\ln k$ behavior comes from the two-bubble diagram shown in Fig. 2. The triangle diagram in Fig. 3 is only of the order $(\ln k)$. We conjecture that this continues in higher orders, and the leading $(\ln k)$ approximation is given by

$$T \simeq -g\sum_{n=0}^{\infty}\left(\frac{g\ \ln(k/\mu)}{16\pi\mu}\right)^n, \quad k \to 0. \tag{4.7}$$

This sum is divergent for $k < \mu\exp(-16\pi\mu/g)$. Thus the present perturbation calculation does not give a meaningful result. If we ignore this divergence and sum the geometric series formally, the result is

$$T \simeq -g\left\{\frac{1}{1-[g\ \ln(k/\mu)/16\pi\mu]}\right\}. \tag{4.8}$$

However, as $k \to 0$, $s \to 4\mu^2$,

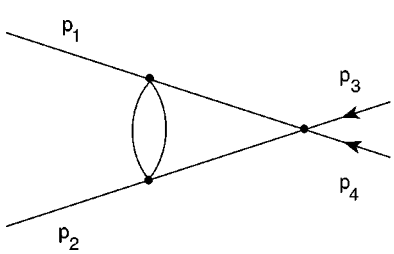

FIG. 3. A third-order diagram behaving as $\ln k$ as $k \to 0$.

$$\frac{T}{16\sqrt{s}} \simeq e^{i\delta_0}\sin\delta_0. \tag{4.9}$$

We thus recover the potential scattering result as $k \to 0$,

$$e^{i\delta_0(k)}\sin\delta_0(k) \sim \frac{-g}{32\mu}\left(\frac{1}{-g\ \ln(k/\mu)/16\pi\mu}\right) \tag{4.10}$$

and

$$\delta_0(k) \sim \frac{\pi}{2}\frac{1}{\ln(k/\mu)}, \quad k \to 0. \tag{4.11}$$

ϕ_3^4 is a well-defined theory, both perturbatively and nonperturbatively, and it is clear from our results that as $k \to 0$ the perturbation theory gives the wrong answer. It is perhaps interesting to note that ϕ_3^4 is asymptotically free. If our conjecture on the higher-order $(\ln k)$ behavior is correct, then this would be the first completely rigorous demonstration of how the perturbation theory order by order could be extremely misleading.

V. REMARKS AND CONCLUSIONS

We close this paper with three significant remarks.

(i) The power of elastic unitarity together with analyticity is clearly demonstrated by the following remark stressed to us by Porrati [16]. Once we are given a phase-shift behavior such that

$$a_0(k)=e^{i\delta_0(k)}\sin\delta_0(k)=\frac{c}{\ln k - i\pi/2}+O\left(\frac{1}{(\ln k)^{1+\epsilon}}\right), \quad k \to 0, \tag{5.1}$$

then the unitarity alone fixes c to be $c=\pi/2$, since

$$a_0^*(k)=a_0(-k)=\frac{c}{\ln|k|-i\pi/2}+O\left(\frac{1}{(\ln -k)^{1+\epsilon}}\right). \tag{5.2}$$

The factors $(i\pi/2)$ are necessary to keep $a_0(k)$ real for k, purely imaginary, and $\text{Im}k>0$. Hence we get

$$\text{Im } a_0(k)=\frac{\pi}{2}\frac{c}{(\ln k)^2}+O\left(\frac{1}{(\ln k)^{2+\epsilon}}\right). \tag{5.3}$$

From $\text{Im } a_0=|a_0|^2$, we obtain, when $c \neq 0$,

$$c=\frac{\pi}{2}. \tag{5.4}$$

It should be pointed out, however, that this argument requires analyticity in k in a semicircle in $\text{Im } k>0$, and hence only applies to exponentially decreasing potentials.

(ii) In one dimension, the simplest potential is the δ-function potential. In two or three dimensions, the corresponding simplest potential is the so-called point interaction, which is the same as the Fermi pseudopotential. There is a vast literature on the Fermi pseudopotential.

Recently, Jackiw [17] obtained the phase shift $\delta_0(k)$ for the point interaction in two dimensions. Although this potential does not belong to the class considered in Sec. II, his result for $k \to 0$ agrees with that of Ref. [1] and ours; see Eq. (3.26) in his paper. It should be stressed, however, that our relativistic result holds for any $2+1$ field theory with the standard analyticity and without zero-mass particles; we are not restricted to ϕ_3^4.

(iii) In a ϕ^4-type field theory, the renormalized coupling constant is defined by the value of the $2 \to 2$ scattering amplitude, $T(s,t,u)$, evaluated at some Euclidean point $(s,t,u) < 4\mu^2$, often for convenience taken to be the symmetric point $s=t=u=4\mu^2/3$. In 3+1 dimensions, given the well-established analyticity and unitarity properties of T, it has been shown in many papers [18] that the coupling constant is bounded. Some of these bounds are surprisingly strong. In ϕ_3^4, Glimm and Jaffe [15] obtained bounds directly from the constructive field theory, but their results are weaker than what can be obtained from the analyticity and unitarity.

The general methods used in the papers cited in Ref. [18] for the 3+1 case can be easily modified to apply to 2+1 dimensions. Only the kinematic factor outside the partial wave expansion is different. The results of this paper thus present us with a new and significant challenge. We have now a new piece of information on the scattering amplitude which is exact. Namely, we know that

$$T(s,t,u) \ln \frac{\sqrt{s-4\mu^2}}{2\mu} \to 16\pi\mu$$

$$\text{as } s \to 4\mu^2, \quad t \to 0, \quad u \to 0,$$

i.e., at certain points on the Mandelstam triangle. Given the power of unitarity and analyticity, we are quite confident that this new input will improve the bounds on the coupling constant. Only the magnitude of the improvement is in question. Work on this problem is in progress.

Note Added in Proof. The work of Bros and Iaglonitzer [13,14] precedes the potential scattering results of Ref. [1]. We only learned of the existence of Refs. [13,14] after a first draft of this paper had been completed. The relevant sentence in our abstract was not amended when the main text was.

ACKNOWLEDGMENTS

We thank our colleagues H. Grosse, H. Lehmann, M. Porrati, and H. C. Ren for their useful comments, and R. Stora for his crucial help. We also thank J. Bros for pointing out to us his previous work. Two of us, N.N.K. and T.T.W., are grateful to the CERN Theory Division for its kind hospitality. This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-91ER40651, Task B, and under Grant No. DE-FG02-84ER40158.

APPENDIX A

In this appendix, we study briefly Eq. (2.2), together with the Dirichlet boundary condition (2.3). We start with the free equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{4r^2} + k^2\right) u(k,r) = 0, \tag{A1}$$

with $u(k,0)=0$. Because of the presence of the attractive singular potential $-1/(4r^2)$, one must be careful in the extension of the differential operator $-(d^2/dr^2) - (1/4r^2)$, to a self-adjoint operator on $L^2(0,\infty)$. This has been thoroughly studied in the literature [19,20]. We quote the result here. The two independent fundamental solutions of Eq. (A1) are $\sqrt{r} J_0(kr)$ and $\sqrt{r} Y_0(kr)$. Both vanish at the origin. Every other solution, being a linear combination of these two, also vanishes at $r=0$. Therefore, we are in the limit-circle case for the differential operator with a Dirichlet boundary condition at $r=0$. There exist an infinite number of self-adjoint extensions of the symmetric differential operator, depending on one (real) parameter. Each self-adjoint extension is defined by the amount of mixing of the two fundamental solutions. Among all these extensions, there exists a ''distinguished'' one, which corresponds to taking the pure Bessel solution $\sqrt{r} J_0(kr)$. These generalized eigenfunctions are less singular, behaving like $\sqrt{r}$ at the origin, as compared to the eigenfunctions of all other extensions, which behave like $\sqrt{r} \ln r$ as $r \to 0$. Moreover, it can be shown that the ''distinguished'' extension corresponds to the Friedrichs extension [20,21]. But, for the physicist, the more important fact is this: in all the other self-adjoint extensions, there exists, besides the continuum, a negative energy eigenvalue. In other words, there exists always a real bound state with negative energy, $E_0 = k_0^2 < 0$ [19,20].

The extension H_λ is defined by taking the behavior, as $r \to 0$,

$$u(r) \to \sqrt{r} + \lambda \sqrt{r} \ \ln r; \quad \lambda \ \text{ real.} \tag{A2}$$

It is then easy to check that if we define a solution such that

$$\sqrt{r}[J_0(kr) + Y_0(kr)] \underset{r \to 0}{\to} \sqrt{r} + \lambda \sqrt{r} \ \ln r, \tag{A3}$$

then an elementary calculation shows that, by setting $k = +i\kappa_0$, we get

$$\ln \kappa_0 = \frac{1 - \lambda(\gamma - \ln 2)}{\lambda}, \tag{A4}$$

where γ is the Euler constant. Thus, for any real λ, $\lambda > 0$, we have a bound state at $E = -\kappa_0^2(\lambda)$.

There is no such bound state in the ''distinguished'' extension. However, in this case we are just at the threshold of having a bound state. More precisely, in the ''distinguished'' extension, if we add to the free Hamiltonian a purely attractive (negative) potential, no matter how weak it happens to be, there appears a true bound state. This fact is well established in the literature using a variational argument.

As an aside here we give the upper bound of Setô [22] on the number N of bound states for dimension $=2$, and $l=0$. This is the 2-dimensional version of the old Bargmann inequality for $d=3$. The Setô bound is

$$N_2^0 \leqslant 1 + \frac{\frac{1}{2}\int_0^\infty dr \int_0^\infty dr' \left|\ln \frac{r}{r'}\right| V(r)V(r')}{-\int_0^\infty rV(r)dr}, \qquad \text{(A5)}$$

where, given our assumptions on $V(r)$, all the integrals are finite. The fact that there is always a bound state, regardless of how weak an attractive potential V may be, is somehow reflected by the presence of 1 in the right-hand side of Eq. (A5). This cannot be improved.

In any case, this last property of the ''distinguished'' extension of the free differential operator to a self-adjoint operator *without a bound state* is the most important criterion by which we must choose this extension, and discard all others. As physicists, we do not have the freedom to start with a ''free Hamiltonian'' that binds a free particle. Mathematicians have this luxury.

We finally come to Eq. (2.2) itself. Starting from the ''distinguished'' extension of the free Hamiltonian, and adding to it a potential V, does not alter the self-adjointness, provided V is ''weak'' in the sense of Kato and others [21,23]. The condition defining this ''weak'' class is expressed precisely in the following integrability condition on the potential:

$$\int_0^\infty r dr(1+|\ln r|)|V(r)| < \infty. \qquad \text{(A6)}$$

This ensures the semi-boundedness of the total Hamiltonian, and the finiteness of the number of bound states. Note that Eq. (A6) is precisely the condition (2.35) which we had to use in Sec. II. We shall need it in Appendix B to establish the existence and study the properties of the solutions of the two integral equations (2.30) and (2.31).

To conclude this appendix, let us point out that an extension different from the ''distinguished'' one can be used to simulate a renormalized delta-function interaction, as was done by Jackiw [17].

APPENDIX B

In this appendix we study the integral equations (2.30) and (2.31). For the class of potentials satisfying Eqs. (2.35) and (2.36), we first prove that the solutions $A(r)$ and $B(r)$ exist and are bounded, as $r \to \infty$, as in Eq. (2.41). Next, we prove that the behavior of $A(r)$ and $B(r)$ as $r \to 0$ is given by Eqs. (2.37) and (2.38), respectively. We will only give the details for Eq. (2.31). The procedure for Eq. (2.30) is easier and very similar.

Our starting point is the integral equation

$$B(r) = \ln r + g\int_r^\infty r' dr' \left(\ln \frac{r'}{r}\right) V(r') B(r'). \qquad \text{(B1)}$$

We can first consider the case $r' \geqslant r \geqslant 1$, where we have the inequality

$$0 \leqslant \ln \frac{r'}{r} \leqslant \ln r'. \qquad \text{(B2)}$$

Therefore, an upper bound $\bar{B}$ is obtained for B by replacing the integral equation (B1) by

$$\bar{B}(r) = \ln r + g\int_r^\infty r' dr' |V(r')| \ln r' \bar{B}(r'), \quad r > 1. \qquad \text{(B3)}$$

The solution of Eq. (B3) can be obtained by standard methods and is given by

$$\bar{B}(r) = \left[\int_1^r \frac{dt}{t} \exp\left(-g\int_t^\infty u|V(u)|\ln u du\right) + C\right] \times \exp\left(g\int_r^\infty t|V(t)|\ln t dt\right). \qquad \text{(B4)}$$

The constant C is given by

$$C = \int_1^\infty \frac{1}{r}\left[1 - \exp\left(-g\int_r^\infty t|V(t)|\ln t dt\right)\right] dr, \qquad \text{(B5)}$$

which is finite given Eq. (2.35). Using this result in Eq. (B4), we find that

$$\bar{B}(r) = [1 + o(1)]\ln r, \quad \text{as } r \to \infty. \qquad \text{(B6)}$$

This establishes the bound on $B(r)$ for $r \geqslant 1$,

$$|B(r)| \leqslant C_1 \ln r + D_1, \qquad \text{(B7)}$$

where C_1 and D_1 are positive constants depending on g.

By the same technique, we arrive at similar conclusions for $A(r)$. This time, the bounding condition for $\bar{A}(r)$ is $\bar{A}(\infty) = 1$. We obtain

$$\bar{A}(r) = 1 + o(1), \quad \text{as } r \to \infty \qquad \text{(B8)}$$

and

$$|A(r)| \leqslant \bar{A}(r) \leqslant D_2, \quad r \geqslant 1, \qquad \text{(B9)}$$

where D_2 is a positive constant.

From these bounds one can easily get, as $r \to \infty$,

$$A(r) = 1 + o(1); \quad B(r) = [1 + o(1)]\ln r. \qquad \text{(B10)}$$

It is important to note that for the first estimate we need only the condition (2.35), whereas for the second we need Eq. (2.36).

Finally, we consider the region $r < 1$ for both $A(r)$ and $B(r)$. The case for $B(r)$ is more delicate (singular), and we treat it first.

We can write Eq. (B1) as

$$B(r)=\ln r+g\int_r^1 r'\left(\ln\frac{r'}{r}\right)V(r')B(r')dr' +g\int_1^\infty r'\left(\ln\frac{r'}{r}\right)V(r')B(r')dr'. \quad \text{(B11)}$$

In the second integral, since $r'\geqslant 1$ and $r<1$, we can use the bound Eq. (B7) and get, using condition (2.29),

$$\left|\int_1^\infty r\,\ln\frac{r'}{r}V(r')B(r')dr'\right|<C+D\left(\ln\frac{1}{r}\right), \quad \text{(B12)}$$

where C and D are positive constants. In the first integral, we have

$$\left|\ln\frac{r'}{r}\right|\leqslant|\ln r|, \quad r<r'\leqslant 1. \quad \text{(B13)}$$

An upper bound, $\bar{\bar{B}}(r)$, for $B(r)$ in $r\leqslant 1$ is now obtained by substituting Eqs. (B12) and (B13) in Eq. (B11). We obtain the integral equation

$$\bar{\bar{B}}(r)=C_2+D_2|\ln r|+g|\ln r|\left[\int_r^1 r'|V(r')|\bar{\bar{B}}(r')dr'\right], \quad \text{(B14)}$$

with some positive constants C_2 and D_2.

The solution of Eq. (B14) can be obtained by elementary methods. It is

$$\bar{\bar{B}}(r)=Z(r)g|\ln r|\left\{C_3+\int_r^1 r'|V(r')|[C_2+D_2|\ln r'|]\times[Z^{-1}(r')]dr'+C_2+D_2|\ln r|\right\}, \quad \text{(B15)}$$

where

$$Z(r)=\exp\left[\int_r^1 dr'gr'|\ln r'||V(r')|\right]. \quad \text{(B16)}$$

Noting that $Z(r)$ is bounded for $0\leqslant r\leqslant 1$, from the condition (2.35), we get

$$|B(r)|\leqslant\bar{\bar{B}}(r)<\lambda+\mu|\ln r|. \quad \text{(B17)}$$

In the same way, we can analyze the integral equation (2.30) for $A(r)$. We again find that, for $r\to 0$,

$$|A(r)|\leqslant\lambda_1|\ln r|+\mu_1. \quad \text{(B18)}$$

Using these two bounds, we can now prove the asymptotic estimates Eqs. (2.37) and (2.38). From Eq. (2.30), we get, as $r\to 0$,

$$A(r)=-gC_a\ln r+g\int_r^\infty r'\ln r'V(r')A(r')dr'+1. \quad \text{(B19)}$$

This can be written as

$$A(r)=-gC_a\ln r+g\int_r^1 r'\ln r'V(r')A(r')dr'+O(1). \quad \text{(B20)}$$

The integral in Eq. (B20) could diverge as $r\to 0$. However, setting

$$I(r)=g\int_r^1 r'\ln r'V(r')A(r')dr', \quad \text{(B21)}$$

and using Eq. (B18), we get

$$|I(r)|<g\lambda_1\int_r^1 r'|\ln r'|^2|V(r')|dr' +g\mu_1\int_r^1 r'|\ln r'||V(r')|dr' <g\lambda_1\int_r^1 r'|\ln r'|^2|V(r')|dr'+O(1). \quad \text{(B22)}$$

Next we define

$$F(r)\equiv r^2|\ln r|^2|V(r)|. \quad \text{(B23)}$$

From the condition (2.35), we have

$$\int_0^1 dr'r'|\ln r'|V(r)=\int_0^1\frac{dr'}{r'|\ln r'|}\times F(r')<\text{const.} \quad \text{(B24)}$$

This implies that $F(r)\to 0$ as $r\to 0$. From Eqs. (B22) and (B23), we get

$$|I(r)|\leqslant g\lambda_1\int_r^1\frac{dr'}{r'}F(r')+O(1), \quad \text{(B25)}$$

and, hence, since $F(r')$ vanishes as $r'\to 0$,

$$|I(r)|=|\ln r|o(1). \quad \text{(B26)}$$

This establishes Eq. (2.37). For Eq. (2.38), the derivation is similar.

It is important to notice that, if $A(r)/\ln r\to 0$ as $r\to 0$, then $B(r)/\ln r$ cannot approach zero as $r\to 0$. This is because A and B are solutions of the *same* differential equation,

$$\frac{d}{dr}\left(r\frac{dX}{dr}\right)=-grV(r)X(r),$$

and are thus linearly independent.

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