SU(2) calorons and magnetic monopoles

Kimyeong Lee* and Changhai Lu†
Physics Department, Columbia University, New York, New York 10027
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We investigate the self-dual Yang-Mills gauge configurations on $R^3\times S^1$ when the gauge symmetry SU(2) is broken to U(1) by the Wilson loop. We construct the explicit field configuration for a single instanton by the Nahm method and show that an instanton is composed of two self-dual monopoles of opposite magnetic charge. We normalize the moduli space metric of an instanton and study various limits of the field configuration and its moduli space metric. [S0556-2821(98)07014-3]

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I. INTRODUCTION

Recently there has been some interest in understanding the relation between calorons, or periodic instantons, and magnetic monopoles on $R^3\times S^1$, where the gauge symmetry is broken maximally by the Wilson loop. Especially it has been argued that instantons are composite objects made of magnetic monopoles [1–3]. Among self-dual configurations of a theory with a simple gauge group G of rank r, the configurations independent of the S^1 coordinate satisfy the ordinary Bogomol'nyi-Prasad-Sommerfield (BPS) equations for magnetic monopoles. On R^3 there exist r types of fundamental BPS magnetic monopoles associated with each simple root [4]. On $R^3\times S^1$, it was shown that there exists an additional type of fundamental monopole associated with the lowest negative root [1,2]. It was argued that a single instanton is made of a unique combination of $r+1$ different fundamental monopoles such that the net magnetic charge is zero [2,1]. Also the explicit moduli space metric of a single instanton in SU(n) theory has been obtained up to normalization [1].

In this paper we construct the explicit field configuration for a single SU(2) caloron on $R^3\times S^1$ with a nontrivial Wilson loop by using the Nahm construction and show that a single caloron is made of two distinct fundamental magnetic monopoles. We also examine various limits of the configuration, especially the trivial Wilson loop limit and the zero temperature limit. We also investigate the moduli space and its metric.

For convenience, we imagine a five dimensional theory with an additional time direction x_0. Thus our instantons and monopoles will appear as solitons in this theory. However, they may also play an important role in finite temperature Yang-Mills theory where x_4 is the Euclidean time. The caloron, or periodic instanton, solutions were found in the late 1970s [5–7]. The difference between those works and ours lies on the Wilson loop $W(\mathbf{x})=P\exp(\int dx_4 A_4)$. In all those cases [5] the Wilson loop is trivial and so magnetic monopole solutions appear only when the scale of the instanton is taken to be infinity [6]. In our case, the Wilson loop is nontrivial. In a chosen gauge the value of A_4 at spatial infinity

$$\langle A_4\rangle=-i\frac{u}{2}\sigma_3 \tag{1}$$

plays the role of the Higgs expectation value.

In the Feynman path integral, we can require that only the field configurations periodic in $x_4\in[0,\beta]$ contribute. The allowed local gauge transformations are the ones which leave the gauge field single-valued. For the gauge group SU(2), there is a group of topologically nontrivial (large) gauge transformations, for example,

$$U_L(x_4)=\exp\left(-i\frac{\pi x_4}{\beta}\sigma_3\right). \tag{2}$$

Even though it is not single-valued as $U_L(x_4+\beta)=-U_L(x_4)$, it is acceptable since the gauge fields remain single-valued. Using this large gauge transformation and the Weyl reflection $e^{i(\pi/4)\sigma_2}$, which sends $u\to -u$, we can choose the range of u to be

$$0\leqslant u\leqslant\frac{2\pi}{\beta}. \tag{3}$$

When $u\neq 0,2\pi/\beta$, one can see easily that the gauge symmetry is spontaneously broken from SU(2) to U(1). There is also an additional global U(1) symmetry corresponding to the translational symmetry on S^1 [1]. [Of course, one can gauge away the background field (1) once we impose the condition $A_\mu(x_4+\beta)=e^{i(u/2)\sigma_3}A_\mu(x_4)e^{-i(u/2)\sigma_3}$ for acceptable gauge configurations.]

In the normalization where the coupling constant $e^2=1$, the action, or four dimensional energy, is bounded from below, $S\geqslant 8\pi^2|k|$, by the topological index

$$\begin{aligned} k&=\frac{1}{64\pi^2}\int d^4x\ \epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\\ &=\frac{1}{16\pi^2}\int d^3S_i\ \epsilon_{ijk}(F^a_{jk}A^a_4-A^a_j\partial_4A^a_k). \end{aligned} \tag{4}$$

The boundary contributions can be nonzero near gauge singularities and spatial infinity. When $k>0$, the bound is saturated by the field configurations satisfying self-dual equations

*Electronic address: klee@phya.snu.ac.kr
†Electronic address: chlu@cuphy3.phys.columbia.edu

$$F_{ij}=\epsilon_{ijk}(D_kA_4-\partial_4A_k). \tag{5}$$

When the asymptotic value of A_4 lies in the interval (3), it was shown that there exist self-dual configurations for two kinds of fundamental magnetic monopoles of four zero modes [1,2]. One configuration is the ordinary BPS solution, which is independent of x_4. It describes monopoles of positive magnetic charge 4π and asymptotic Higgs value u. Another solution is an ordinary monopole with asymptotic Higgs value $2\pi/\beta-u$. We need to apply a large gauge transformation (2) and a Weyl reflection to this solution to get the right boundary condition. It describes monopoles of negative magnetic charge -4π. The topological charges of both type of monopoles are positive and are given, respectively, by

$$k_1=\frac{\beta u}{2\pi},\quad k_2=1-\frac{\beta u}{2\pi}. \tag{6}$$

The masses of magnetic monopoles in a conventional sense are the magnetic charge times the length scale, and so

$$m_1=4\pi u,\quad m_2=4\pi\left(\frac{2\pi}{\beta}-u\right). \tag{7}$$

As five dimensional solitons, the monopoles really carry mass βm_1 and βm_2. Each type of monopole can carry electric charge q_i, which is integer quantized as they arise from W boson excitations.

The reason for the opposite charge of these two monopoles can be seen easily in the unitary gauge. For the first monopole, A_4 increases from zero to u as one moves from monopole core to spatial infinity. For the second monopole, the value of A_4 decreases from $2\pi/\beta$ to u as one moves from monopole core to spatial infinity. The magnetic field is proportional to the spatial derivative of A_4 and so the two monopoles carry opposite charge. However, there is no static force between them because the Higgs interaction is now repulsive, as one can see from the mass formula, and it cancels the magnetic attraction exactly. That is why in principle two solutions can be superposed. The configurations for two superposed distinct fundamental monopoles will satisfy the self-dual equations and have zero total magnetic charge, unit topological charge, and eight zero modes. Those are exactly the field configurations for a single instanton.

Another interesting question is to find the moduli space metric. The moduli space of a single instanton on $R^3\times S^1$ is found up to right coefficients. Especially, the relative moduli space for a single instanton was argued to be Taub-NUT (Newman-Unti-Tamburino) with Z_2 singularity [1]. We find the exact moduli space metric and the moduli space by using the constituent monopole picture [8,9].

The plan of this paper is as follows. In Sec. II, we briefly review the Nahm formalism and use it to construct the field configuration for a single instanton on $R^3\times S^1$ with the nontrivial Wilson loop. In Sec. III, we show that the field configuration approaches the single monopole configuration at the expected positions of monopoles. This shows that a single instanton solution is a complicated superposition of two monopole configurations. In Sec. IV, we study the field configuration outside the monopole core region. In Sec. V, we show that our solution has a gauge singularity at center-of-mass, which leads to the unit topological charge. In Sec. VI, we take the limit where one of the monopoles becomes massless and show that our solution becomes the well-known periodic instanton. In Sec. VII, we take the zero temperature limit and obtain the instanton solution in R^4. In Sec. VIII, we find the moduli space and its metric. In Sec. IX, we conclude with some remarks.

II. THE NAHM CONSTRUCTION

The Nahm construction uses the Nahm data and the solution of the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) equations to construct the self-dual magnetic monopole configurations [10,11]. In addition, by studying the moduli space of the Nahm data, one can construct the moduli space metric of the corresponding magnetic monopole configurations. Especially in the SU(2) gauge theory, there has been considerable work in the Nahm construction of magnetic monopoles [10].

For an SU(2) gauge theory on $R^3\times S^1$, there are three relevant time intervals for the Nahm equation:

$$-\frac{\pi}{\beta}<t<-\frac{u}{2},\quad -\frac{u}{2}<t<\frac{u}{2},\quad \frac{u}{2}<t<\frac{\pi}{\beta}. \tag{8}$$

Because we are considering calorons, we should require the Nahm data to be periodic in the time variable t in the Nahm equations [3]. The first and last intervals correspond to monopoles of topological charge k_2 and the second interval corresponds to monopoles of topological charge k_1. Since a single instanton is made of two distinct monopoles, we need to introduce the jumping condition at the boundary $t=\pm u/2$.

In general, the Nahm data consist of a family of triple Hermitian matrix functions $\mathbf{T}(t)$, of dimensions $l(t)\times l(t)$, defined in every interval, together with triple matrices $\boldsymbol{\alpha}_p$ of dimension $l(t_p)\times l(t_p)$, defined at each point t_p where $l(t)$ does not jump. The value $l(t)$ in each interval is the number of corresponding monopoles. These should satisfy the Nahm equations

$$\frac{dT_i}{dt}-i[T_4,T_i]=-i\epsilon_{ijk}T_jT_k+\sum_p(\boldsymbol{\alpha}_P)_i\delta(t-t_P). \tag{9}$$

When $l(t_P-\epsilon)\neq l(t_P+\epsilon)$, there is a usual boundary condition on Nahm data on both sides of t_P. Since the time-interval is periodic in $2\pi/\beta$, $\mathbf{T}(-\pi/\beta)=\mathbf{T}(\pi/\beta)$. Associated with $\boldsymbol{\alpha}_P$, there exists a $2l(t_P)$-component row vector a_P satisfying

$$a_P^\dagger a_P=\boldsymbol{\alpha}_P\cdot\boldsymbol{\sigma}-i(\alpha_P)_0 I_{2l(t_P)\times 2l(t_P)}. \tag{10}$$

The next step is to find $2l(t)\times 2$ matrix functions $v(t)$ and 2-component row vectors S_p obeying the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) equation

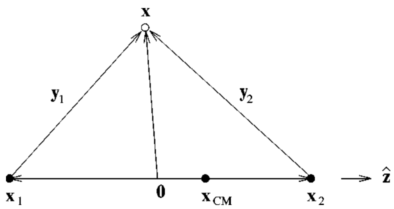

FIG. 1. The position vectors for two magnetic monopoles.

$$0=\left[-\frac{d}{dt}+(\mathbf{T}+\mathbf{x})\cdot\boldsymbol{\sigma}+ix_4\right]v+\sum_P a_P^\dagger S_P\delta(t-t_P). \tag{11}$$

The matrix $v(t)$ is periodic, $v(-\pi/\beta)=v(\pi/\beta)$. These matrices should satisfy the normalization condition

$$I_{2\times 2}=\int_{-\pi/\beta}^{\pi/\beta}dt\; v^\dagger v+\sum_P S_P^\dagger S_P. \tag{12}$$

Then, the corresponding self-dual gauge field configuration is given by

$$\begin{aligned}A_\mu&=\int_{-\pi/\beta}^{\pi/\beta}dt\; v^\dagger(t)\partial_\mu v(t)+\sum_P S_P^\dagger\partial_\mu S_P\\&=\frac{1}{2}\int_{-\pi/\beta}^{\pi/\beta}dt\;[v^\dagger(t)\partial_\mu v(t)-\partial_\mu v^\dagger(t)v(t)]\\&\quad+\frac{1}{2}\sum_P[S_P^\dagger\partial_\mu S_P-\partial_\mu S_P^\dagger S_P].\end{aligned} \tag{13}$$

In this paper we will concern ourselves with the field configuration for two distinct fundamental monopoles, so that $l(t)=1$ for the entire interval $[-\pi/\beta,\pi/\beta]$. The solutions of the Nahm equations at each interval are trivial. We rotate and translate the field configuration so that two massive monopoles lie on the z axis. The corresponding Nahm data are

$$\begin{aligned}\mathbf{T}_0=\mathbf{T}_2&=-\mathbf{x}_2=-(0,0,(\mathbf{x}_2)_3),\\ \mathbf{T}_1&=-\mathbf{x}_1=-(0,0,(\mathbf{x}_1)_3),\end{aligned} \tag{14}$$

where $\mathbf{x}_1,\mathbf{x}_2$ are the positions of two massive monopoles. In our choice, the distance between two monopoles is $D=(\mathbf{x}_2-\mathbf{x}_1)_3>0$. For a given coordinate point $\mathbf{x}$, we introduce its relative positions with respect to two monopoles, as shown in Fig. 1,

$$\mathbf{y}_1=\mathbf{x}-\mathbf{x}_1,\quad \mathbf{y}_2=\mathbf{x}-\mathbf{x}_2, \tag{15}$$

and weighted relative positions

$$\mathbf{s}_1=u\mathbf{y}_1,\quad \mathbf{s}_2=\left(\frac{2\pi}{\beta}-u\right)\mathbf{y}_2. \tag{16}$$

The center-of-mass position is

$$\mathbf{x}_{\rm cm}=k_1\mathbf{x}_1+k_2\mathbf{x}_2. \tag{17}$$

From Eq. (9) we get the jumping functions $(\boldsymbol{\alpha}_1)_i=D\delta_{i3}$ and $(\boldsymbol{\alpha}_2)_i=-D\delta_{i3}$. Their corresponding two row vectors in Eq. (10) are then

$$a_1=(\sqrt{2D},0),\quad a_2=(0,\sqrt{2D}). \tag{18}$$

The solutions of the ADHMN equation (11) at each interval can be expressed as

$$\begin{aligned}v_0(t)&=\frac{1}{\sqrt{N_2}}e^{[ix_4+\sigma\cdot\mathbf{y}_2](t+\pi/\beta)}\,C_2\quad\text{for}\quad t\in\left[-\frac{\pi}{\beta},-\frac{u}{2}\right),\\ v_1(t)&=\frac{1}{\sqrt{N_1}}e^{[ix_4+\sigma\cdot\mathbf{y}_1]t}\,C_1\quad\text{for}\quad t\in\left(-\frac{u}{2},\frac{u}{2}\right),\\ v_2(t)&=\frac{1}{\sqrt{N_2}}e^{[ix_4+\sigma\cdot\mathbf{y}_2](t-\pi/\beta)}\,C_2\quad\text{for}\quad t\in\left(\frac{u}{2},\frac{\pi}{\beta}\right],\end{aligned} \tag{19}$$

where C_i are 2×2 matrices and

$$N_i=\frac{1}{y_i}\sinh s_i(i=1,2). \tag{20}$$

The periodic condition $v_0(-T/2)=v_2(T/2)$ is automatically satisfied. For this solution, the normalization condition (12) becomes

$$I_{2\times 2}=C_1^\dagger C_1+C_2^\dagger C_2+S_1^\dagger S_1+S_2^\dagger S_2. \tag{21}$$

To find C_1,C_2 and S_1,S_2, we use the normalization condition (21) and the discontinuity equations derived from Eq. (11). In addition, we require the gauge field (14) to be single-valued. Then, C_i and S_i are determined uniquely up to acceptable gauge transformations. The S_1 and S_2 can be regarded as the first and second row vectors of a 2×2 matrix S, which takes the explicit form

$$S=\frac{1}{\sqrt{\mathcal{N}}}e^{-i(u/2)x_4\sigma_3}, \tag{22}$$

where

$$\begin{aligned}\mathcal{N}=1+\frac{2D}{\mathcal{M}}\{&N_1(\cosh s_2-(\hat{\mathbf{y}}_2)_3\sinh s_2)\\&+N_2(\cosh s_1+(\hat{\mathbf{y}}_1)_3\sinh s_1)\}.\end{aligned} \tag{23}$$

The two matrices C_i are more complicated. It is useful first to introduce two 2×2 matrices,

$$\begin{aligned}B_1&=e^{i(\pi/\beta)x_4}e^{-(\sigma/2)\cdot\mathbf{s}_1}e^{-(\sigma/2)\cdot\mathbf{s}_2}-e^{-i(\pi/\beta)x_4}e^{(\sigma/2)\cdot\mathbf{s}_1}e^{(\sigma/2)\cdot\mathbf{s}_2},\\ B_2&=e^{i(\pi/\beta)x_4}e^{-(\sigma/2)\cdot\mathbf{s}_2}e^{-(\sigma/2)\cdot\mathbf{s}_1}-e^{-i(\pi/\beta)x_4}e^{(\sigma/2)\cdot\mathbf{s}_2}e^{(\sigma/2)\cdot\mathbf{s}_1},\end{aligned} \tag{24}$$

and a scalar quantity

$$\mathcal{M}=2\left\{\cosh s_1 \cosh s_2+\hat{\mathbf{y}}_1\cdot\hat{\mathbf{y}}_2 \sinh s_1 \sinh s_2 -\cos\left(\frac{2\pi}{\beta}x_4\right)\right\}, \tag{25}$$

where $\mathcal{M}=B_1B_1^\dagger=B_2B_2^\dagger$. Then the desired expression for C_i's are given as

$$C_1=\sqrt{\frac{2DN_1}{\mathcal{N}}}\frac{B_1^\dagger}{\mathcal{M}}[e^{-(\sigma/2)\cdot\mathbf{s}_2}Q_+ + e^{(\sigma/2)\cdot\mathbf{s}_2}Q_-]e^{-i(\pi/\beta)x_4\sigma_3},$$

$$C_2=\sqrt{\frac{2DN_2}{\mathcal{N}}}\frac{B_2^\dagger}{\mathcal{M}}[e^{(\sigma/2)\cdot\mathbf{s}_1}Q_+ + e^{-(\sigma/2)\cdot\mathbf{s}_1}Q_-] \tag{26}$$

with projection operators

$$Q_\pm=\frac{1\pm\sigma_3}{2}. \tag{27}$$

The gauge field (14) becomes

$$A_\mu(\mathbf{x},x_4)=C_1^\dagger V_\mu(\mathbf{y}_1;u)C_1 + C_2^\dagger V_\mu\left(\mathbf{y}_2;\frac{2\pi}{\beta}-u\right)C_2+C_1^\dagger\partial_\mu C_1+C_2^\dagger\partial_\mu C_2 + S^\dagger\partial_\mu S, \tag{28}$$

where $V_\mu(\mathbf{x};u)$ is the ordinary BPS monopole solution,

$$V_4(\mathbf{x};u)=\frac{\sigma_a}{2i}\hat{x}_a\left[\frac{1}{|\mathbf{x}|}-\frac{u}{\coth(u|\mathbf{x}|)}\right],$$

$$V_i(\mathbf{x};u)=\frac{\sigma_a}{2i}\epsilon_{aij}\hat{x}_j\left[\frac{1}{|\mathbf{x}|}-\frac{u}{\sinh(u|\mathbf{x}|)}\right]. \tag{29}$$

The field configuration (28) is the desired expression for a single instanton. Under the gauge transformation $A_\mu\to UA_\mu U^\dagger-\partial_\mu UU^\dagger$, we see $C_i\to C_iU^\dagger$ and $S\to SU^\dagger$.

Notice that $\mathcal{M}$ vanishes at only one point

$$x_{\text{singular}}=(\mathbf{x}_{\text{cm}},x_4=0). \tag{30}$$

The gauge field (28) turns out to have a gauge singularity at this point as we will see later.

III. NEAR EACH MONOPOLE

To see the field configuration (28) describe two magnetic monopoles of opposite charge, let us consider the limit $D\gg 1/u,(2\pi/\beta-u)^{-1}$ so that their cores do not overlap. We expect the configuration to approach that of each monopole near $\mathbf{x}_1$ or $\mathbf{x}_2$. If we are near the first monopole so that $|\mathbf{y}_1|\ll D$, we see easily that

$$C_2,S_1,S_2\sim\frac{1}{\sqrt{D}}, \tag{31}$$

and

$$C_1=\frac{\sigma_3\cosh\frac{s_1}{2}-\boldsymbol{\sigma}\cdot\hat{\mathbf{y}}_1\sinh\frac{s_1}{2}}{\sqrt{\cosh s_1-(\hat{\mathbf{y}}_1)_3\sinh s_1}}+\mathcal{O}\left(\frac{1}{D}\right), \tag{32}$$

which is a single-valued unitary matrix. Thus, the whole gauge configuration (28) becomes approximately a gauge transformation of the single monopole configuration $V_\mu(\mathbf{y}_1;u)$.

Similarly, near the second monopole, $|\mathbf{y}_2|\ll D$, we see that

$$C_1,S_1,S_2\sim\frac{1}{\sqrt{D}}, \tag{33}$$

and

$$C_2=\frac{-\sigma_3\cosh\frac{s_2}{2}-\boldsymbol{\sigma}\cdot\hat{\mathbf{y}}_2\sinh\frac{s_2}{2}}{\sqrt{\cosh s_2+(\hat{\mathbf{y}}_2)_3\sinh s_2}}e^{-i(\pi x_4/\beta)\sigma_3}, \tag{34}$$

which is a unitary matrix. Thus, the field configuration (28) becomes a gauge transformation of the second monopole configuration, but the sign of magnetic charge is changed by the large gauge transformation $e^{-i(\pi x_4/\beta)\sigma_3}$. The above discussion shows that one can identify individual magnetic monopoles when their cores are not overlapping.

IV. OUTSIDE MONOPOLE CORE

Outside monopole core $s_1,s_2\gg 1$, we can neglect exponentially small terms. Especially we see

$$\mathcal{M}\approx\frac{1}{2}e^{s_1+s_2}(1+\hat{\mathbf{y}}_1\cdot\hat{\mathbf{y}}_2),$$

$$\mathcal{N}\approx 1+\frac{D}{y_1y_2}\frac{y_1+y_2+D}{1+\hat{\mathbf{y}}_1\cdot\hat{\mathbf{y}}_2}. \tag{35}$$

From this we get

$$C_1\approx\sqrt{\frac{D}{y_1\mathcal{N}}}\frac{2}{1+\hat{\mathbf{y}}_1\cdot\hat{\mathbf{y}}_2}(P_{1-}P_{2-}Q_+ - P_{1+}P_{2+}Q_-), \tag{36}$$

$$C_2\approx\sqrt{\frac{D}{y_2\mathcal{N}}}\frac{2}{1+\hat{\mathbf{y}}_1\cdot\hat{\mathbf{y}}_2}(P_{2-}P_{1-}Q_- - P_{2+}P_{1+}Q_+)e^{-i(\pi/\beta)x_4\sigma_3}, \tag{37}$$

where

$$P_{i\pm}=\frac{1\pm\hat{\mathbf{y}}_i\cdot\boldsymbol{\sigma}}{2}, \quad (i=1,2) \tag{38}$$

are projection operators. Using these approximations, we can obtain the field configuration outside the monopole core region. We expect this to be purely Abelian and so a simple superposition of Abelian fields in a unitary gauge. However seeing this explicitly does not seem to be simple.

There is still immediate information following the above expressions. For $s_1, s_2 \gg D$, $N \approx 1$ and $C_1 \sim C_2 \sim \sqrt{D/|\mathbf{x}|}$, making

$$A_\mu = -i\frac{u}{2}\sigma_3\delta_{\mu 4} + \mathcal{O}\left(\frac{D}{|\mathbf{x}|^2}\right). \tag{39}$$

This implies that when the distance between two monopoles goes to zero, the field configuration becomes trivial, which is exactly what we hope for the zero size instanton. Also the gauge field approaches the vacuum trivially at spatial infinity, implying no boundary contribution from spatial infinity to the topological charge (4). We will see in a moment that the only nontrivial contribution comes from the singularity (30).

V. NEAR SINGULARITY

To consider the singularity at Eq. (30), we put the center-of-mass at the origin, so that $(\mathbf{x}_1)_3 = -k_2 D$ and $(\mathbf{x}_2)_3 = k_1 D$. Then, by expanding the matrices around $x_\mu = 0$, we get

$$C_1 \approx C_2 \approx \frac{i}{\sqrt{2}} U(x)_s, \quad S \approx \mathcal{O}(1), \tag{40}$$

where

$$U(x)_s^\dagger = \frac{x_4 + i\sigma_3 x_3 + i(\sigma_1 x_1 + \sigma_2 x_2) q}{\sqrt{x_4^2 + x_3^2 + (x_1^2 + x_2^2) q^2}} \tag{41}$$

with $q = \sinh(2\pi k_1 k_2 D/\beta)/(2\pi k_1 k_2 D/\beta)$.

Thus the gauge field near the singularity $x_\mu = 0$ becomes

$$A_\mu = U_s \partial_\mu U_s^\dagger + \mathcal{O}(1), \tag{42}$$

showing that it is pure gauge singularity. The nontrivial contribution at this gauge singularity to the topological charge (4) is one, as expected for a single instanton.

VI. MASSLESS MONOPOLE LIMIT

We choose $u = 2\pi/\beta$. In this case the Wilson loop becomes trivial. The gauge symmetry is restored to the original SU(2) [5]. In this limit the isolated second monopole solution disappears as $V_\mu(y_2, 2\pi/\beta - u) = 0$. The size of the second monopole becomes infinite and its topological charge vanishes. It loses its meaning as an isolated object.

We put the massive monopole at the origin so that $\mathbf{y}_1 = \mathbf{x}$ and $\mathbf{y}_2 = D\hat{\mathbf{z}}$. In this limit, $N_2 = C_2 = 0$. After a large gauge transformation, $e^{-i(\pi/\beta)x_4\sigma_3}$, one can see that the solution (28) is

$$A_\mu = \frac{i}{2}\bar{\sigma}_{\mu\nu}\partial_\nu \ln \mathcal{N} \tag{43}$$

with $\bar{\sigma}_{ij} = \epsilon_{ijk}\sigma_k$ and $\bar{\sigma}_{i4} = -\sigma_i$. In this limit the normalization coefficient (23) is

$$\mathcal{N} = 1 + \frac{D}{|\mathbf{x}|} \frac{\sinh\left(\frac{2\pi}{\beta}|\mathbf{x}|\right)}{\cosh\left(\frac{2\pi}{\beta}|\mathbf{x}|\right) - \cos(Tx_4)}. \tag{44}$$

This is exactly the periodic instanton solution [5], once we require a relation

$$D = \frac{\pi\rho^2}{\beta} \tag{45}$$

between the intermonopole distance D and the instanton scale parameter ρ. In the zero temperature limit, $\beta \to \infty$, one can see that the finite size instanton solution can be obtained only if the distance between two magnetic monopoles approaches zero.

The interpretation of this solution can be done consistently with the previous pictures about massless monopole [12,1]. First of all, when we remove the massless monopole, $D \to \infty$, the configuration becomes pure magnetic monopole [6]. When the massless monopole is at finite distance, the field configuration near the massive monopole is purely magnetic and then the massless monopole or the non-Abelian cloud shields the magnetic charge of the massive monopole at distance scale D and the field configuration at scale $r \gg D$ falls off quickly like a dipole field configuration [7].

VII. ZERO TEMPERATURE LIMIT

Let us now investigate our solution at the zero temperature limit $\beta \to \infty$, which implies $u \to 0$ by Eq. (3). After putting the center-of-mass position (17) at the origin, we see that for finite $x = (\mathbf{x}, x_4)$, $N_1 \approx u$, $N_2 \approx 2\pi/\beta - u$, and

$$\mathcal{M} \approx (2\pi/\beta^2)^2 x^2,$$

$$\mathcal{N} \approx 1 + \frac{\beta D}{\pi x^2}. \tag{46}$$

Thus the zero temperature limit of S in Eq. (22) is nontrivial only if βD remains finite. This is consistent with the argument after Eq. (45). After removing the singularity at the origin by a singular gauge transformation, $U^\dagger = (x_4 + i\boldsymbol{\sigma} \cdot \mathbf{x})/\sqrt{x^2}$, a 2×2 matrix S of Eq. (22) becomes

$$S = \frac{x_4 + i\boldsymbol{\sigma}\cdot\mathbf{x}}{\sqrt{x^2 + \rho^2}} \tag{47}$$

with $\rho^2 = \beta D/\pi$ as shown in Eq. (45). The two matrices v_1, v_2 of Eq. (19) are simply

$$v_1 \approx v_2 \approx -\frac{\beta}{2\pi}\sqrt{\frac{2D}{x^2}}. \tag{48}$$

The gauge field (14) becomes

$$A_\mu = \frac{-i\sigma_{\mu\nu}x_\nu}{x^2+\rho^2}, \tag{49}$$

where $\sigma_{ij}=\epsilon_{ijk}\sigma_k$ and $\sigma_{i4}=\sigma_i$. This is the standard regular expression for a single instanton on R^4 [13].

VIII. MODULI SPACE METRIC

The relative moduli space of two constituent monopoles for a single instanton is known to be the Taub-NUT space with Z_2 division [1]. Here we fix the normalization and provide the global picture of the moduli space, which also sheds light on the zero temperature limit and the trivial Wilson loop limit.

To fix the normalization, we consider the additional real time direction x^0, which makes our theory five dimensional. Instantons and magnetic monopoles appear as self-dual solitons. The number of zero modes of a single instanton is eight and is the sum of the zero modes for constituent monopoles. Each monopole carries four zero modes for its position and internal U(1) phase. We can divide eight instanton zero modes into four for the center of mass motion of monopoles and four for the relative motions of magnetic monopoles. Defining the moduli space is quite similar to the monopole case [14]. For the infinitesimal change of the moduli parameters z_A, $A=1,..,8$, the corresponding infinitesimal change $\delta_A A_\mu$ would satisfy the background gauge and the linearized self-dual equations. Then the moduli space metric is given by

$$\mathcal{G}_{AB} = \int d^4x \;\; \delta_A A_\mu \delta_B A_\mu . \tag{50}$$

One can easily see that this space should be hyper-Kähler by generalizing the argument in Ref. [14].

The detailed derivation of the moduli space of these monopoles is given before [8,9]. [Since their magnetic charges belong to the same U(1) group with opposite sign, the value of the parameter λ in Ref. [8] is two.] Each monopole is imagined to carry corresponding integer quantized electric charge, q_1, q_2. The only modification for the case in hand is that we have to integrate over x_4. This leads to an overall multiplicative factor β on the effective low energy Lagrangian. The center-of-mass moduli space is just $R^3 \times S^1$. Since $\beta(m_1+m_2)=8\pi^2$, the metric for the center-of-mass moduli space becomes

$$ds^2_{\rm c.m.} = 8\pi^2\left(d\mathbf{R}^2 + \frac{\beta^2}{4\pi^2}d\chi^2\right), \tag{51}$$

where $\mathbf{R}$ is the center-of-mass position and χ is the conjugate variable for the total electric charge. The total electric charge is $q_\chi = k_1q_1+k_2q_2$ [8], which turns out to be the x_4 momentum [1]. This charge need not be quantized [1] and χ lies along the real line R. Thus, we cannot identify $\beta\chi/(2\pi)$ with x_4, unless $q_1=q_2$. The overall coefficient $8\pi^2$ is the mass of instanton.

The relative mass $m_1m_2/(m_1+m_2)$ between two monopoles is $8\pi^2k_1k_2/\beta$. We introduce the relative position between two monopoles $\mathbf{r}=\mathbf{x}_1-\mathbf{x}_2$ and note that $|\mathbf{r}|=D$. The metric for the relative moduli space [obtained after multiplying β to Eq. (5.8) in Ref. [8]] is

$$ds^2_{\rm rel} = 8\pi^2k_1k_2[(1+r_0/r)d\mathbf{r}^2 + r_0^2(1+r_0/r)^{-1}(d\psi+\mathbf{w}(\mathbf{r})\cdot d\mathbf{r})^2], \tag{52}$$

where $r_0=\beta/(2\pi k_1k_2)$ and $\mathbf{w}(\mathbf{r})$ is the Dirac potential such that $\nabla\times\mathbf{w}=\nabla(1/r)$. This is the Taub-NUT space with length paramter $r_0/2$. Since both monopoles can carry only integer electric charge, their relative charge $q_\psi=q_1=q_2$ is integer quantized instead of half-integer quantized as in the SU(3) case [9]. Thus their relative phase ψ should have the interval $[0,2\pi]$ instead of $[0,4\pi]$. This is the origin of Z_2 orbifold singularity of the relative moduli space $\mathcal{M}_0$. The total moduli space can be found by a similar discussion as for monopoles [8,9] and is given as

$$\mathcal{M} = R^3 \times \frac{R^1\times\mathcal{M}_0}{Z}, \tag{53}$$

where the generator of the identity map Z is $(\chi,\psi)=(\chi+2\pi,\psi+2\pi k_2)$.

In the zero temperature limit $\beta\to\infty$, or in the limit where symmetry is restored, say, $k_2\to 0$, the relative metric becomes flat. This is similar to the massless limit of the relative moduli space metric in SO(5) [12]. After using the instanton scale parameter ρ in Eq. (45), the metric (52) becomes

$$ds^2 = 16\pi^2(d\rho^2+\rho^2 d\Omega_3^2), \tag{54}$$

where $d\Omega_3^2$ is the metric of a unit three sphere. The overall coefficient can be checked directly by calculating $\int d^4x(\delta_\rho A^a_\mu)^2$, which is straightforward because $\partial A^a_\mu/\partial\rho$ of Eq. (49) satisfies the background gauge, $D_\mu \delta A_\mu = 0$. Since the adjoint matters belong to SO(3), so the gauge orbit of a single instanton is S^3/Z_2, implying the Z_2 orbifold singularity at origin.

IX. CONCLUDING REMARKS

By using the Nahm construction, we have found the field configuration for a single instanton in the SU(2) gauge theory on $R^3\times S^1$. When the gauge group is spontaneously broken by the Wilson loop, a single instanton is shown to be composed of two fundamental monopoles of opposite magnetic charge. By taking various limits, our solution is shown to be consistent with the previously known ideas about periodic instantons, massless monopoles and zero temperature instantons.

There are several interesting implications from our work as mentioned in Refs. [1,2]. Here we also see that the zero temperature limit may be interesting. At the zero temperature limit of a single caloron, the positions of two monopoles should come together to the center in order to get a finite size instanton, which makes the monopole picture somewhat trivial. However the story cannot be all there is for the two caloron case. Even at the zero temperature limit of two close-by calorons, there are no identifiable instanton positions [15]. Thus, it is not clear where the four constituent monopoles for two calorons will end up at the zero temperature limit. Thus, we hope that the picture of composite instantons and their constituent monopoles still survives even at zero temperature in some sense, say, after Abelian projection [16], and leads to new insight on understanding the chiral symmetry and confinement in zero temperature QCD.

Note added. While writing this paper, we became aware of Ref. [17] which has a considerable overlap with our work.

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