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Light propagation in nontrivial QED vacua

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Within the framework of effective action QED, we derive the light cone condition for homogeneous nontrivial QED vacua in the geometric optics approximation. Our result generalizes the ''unified formula'' suggested by Latorre, Pascual and Tarrach and allows for the calculation of velocity shifts and refractive indices for soft photons travelling through these vacua. Furthermore, we clarify the connection between the light velocity shift and the scale anomaly. This study motivates the introduction of a so-called effective action charge that characterizes the velocity modifying properties of the vacuum. Several applications are given concerning vacuum modifications caused by, e.g., strong fields, Casimir systems and high temperature.
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I. INTRODUCTION

The vacuum considered as a medium has become a popular picture in quantum field theory. With reservations due to the lack of understanding of non-perturbative vacuum phenomena, it is astonishing that analogies between the quantum vacuum and classical media are frequently useful.

One particular example is represented by the propagation of light in a vacuum which is modified by various external environments, e.g., electromagnetic (EM) fields, temperature, geometric boundary configurations, gravitational background and non-trivial topologies. The concept of drawing the analogy is common to all of these cases: vacuum polarization allows the photon to exist as a virtual e^+e^--pair on which the various vacuum modifications can act. Under certain assumptions, this influence on the loop process can effectively be described by an immediate influence of a (generally non-linear) medium on the photon itself, e.g., by refractive indices. This program was carried out among others by Adler [1], Brezin and Itzykson [2] for magnetic fields, by Drummond and Hathrell [3] for gravitation, and by Scharnhorst [4] and Barton [5] for a Casimir configuration. Further important examples are found in Refs. [6–9].

A new physical insight into the phenomenon of photon propagation in non-trivial vacua has been given by Latorre, Pascual and Tarrach [8]. Comparing the known velocity shifts arising from different vacuum modifications, they were able to identify an intriguing general, so-called ''unified'' formula covering all these cases.[1] They concluded that the polarization and direction averaged velocity shift is related to the (renormalized) background energy density u with a ''universal'' numerical coefficient

$$\overline{\delta v} = -\frac{44}{135}\frac{\alpha^2}{m^4}u \tag{1}$$

where m denotes the electron mass and $\alpha \simeq 1/137$. [In the case of gravitation, one α has to be replaced by the combination $(G_N m^2)$ involving Newton's constant.] However, a complete derivation of the ''unified formula'' has not been given up to now.

In the case of gravitation, light was shed on the problem by Shore [10] who proved a polarization sum rule that represents a generalization of Eq. (1). Furthermore, he pointed out that the ''universal'' coefficient in Eq. (1) can be related to the trace anomaly of the energy-momentum tensor in the case of weak EM background fields.

One of the most remarkable features concerning vacuum induced velocity shifts certainly is the fact that $\overline{\delta v} > 0$ is not intrinsically forbidden in quantum field theories. This seems to offer the possibility of superluminal propagation, e.g., in curved spaces and Casimir vacua. Both examples share the property of a possible negative energy density u in Eq. (1).

The two questions, whether the signal (=wave front) velocity indeed exceeds c and whether superluminal propagation is observable in principle, could be resolved by calculating the velocity shift in the infinite frequency limit. But this is presently out of reach, because a resummation of the derivative expansion has to be achieved. However, without being able to answer these questions, let us just say that we find no grounds for violation of (micro-)causality in accordance with [3,8,10,11]. For a causality violation, a space-like signal *and* Lorentz invariance (in the gravitational case, strong principle of equivalence) are necessary conditions. The latter is explicitly violated in the above-mentioned examples. For an excellent discussion, the reader is referred to the work of Shore [10].

In the present work, we confine ourselves to the case of non-trivial vacua modified by QED phenomena. Within the effective action approach [12], we derive a covariant light cone condition in Sec. II which turns out to be a generalization of the ''unified formula.'' The necessary assumptions are analyzed in detail.

In this framework, we are able to clarify the relation between the velocity shift and the trace anomaly in Sec. III. Our findings do not unveil a natural and physically meaningful connection. An alternative physical picture of the ''uni-

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[1]In fact, the results of [9] cannot be embedded in the ''unified formula.'' The solution to this problem is an aim of the present work.

versal'' pre-factor is given instead which is called the effective action charge. Several applications of our light cone condition concerning EM fields, Casimir configurations and temperature are elaborated on in Sec. IV. In the low-energy domain, we can easily recover all of the well-known results described by the ''unified formula.'' However, the ''universal constant'' turns out to be neither constant nor universal when we drop the low-energy restriction. Instead, the concept of an effective action charge provides for an intuitive understanding of the velocity shifts at arbitrary energies.

Conclusions are drawn in Sec. V.

II. LIGHT CONE CONDITION

Consider light propagation in a non-trivial QED vacuum (we will specify this terminology soon) characterized by a certain energy scale. Suppose that there exists an effective action which takes into account any QED quantum phenomena on higher scales and hence provides for an exact description of the propagation. In principle, this effective action will depend on any gauge and Lorentz invariant scalar which we can construct. Throughout the paper, we will stick to the following essential assumptions:

(1) The propagating photons characterized by $f^{\mu\nu}$ are considered to be soft. This is equivalent to calculating the properties of the vacuum in the limit $\omega/m \ll 1$ where the scale is set by the Compton wavelength.

(2) The vacuum modification is homogeneous in space and time (but not necessarily isotropic).

Referring to these assumptions, we can neglect any term in the effective action that involves derivatives of the field, since a derivative either acting on the background field vanishes [assumption (2)] or acting on the photon field $f^{\mu\nu}$ contributes terms of the order $\mathcal{O}(\omega^2/m^2)$ to the equation of motion. In the latter case, it is negligible because of assumption (1).

We furthermore assume the following:

(3) Vacuum modifications caused by the propagating light itself are negligible.

Assumption (3) justifies a linearization of the equations of motion with respect to $f^{\mu\nu}$ but does not stand on the same footing as the former assumptions, since it is not essential for the formalism. Note that we do *not* demand that the deviation from the Maxwell Lagrangian should be small, corresponding to small vacuum modifications.[2]

Since it is unwieldy to establish a general formalism for arbitrary numbers of Lorentz vectors and tensors characterizing the vacuum, we first consider a vacuum only modified by EM fields. Hence, the dynamical building blocks of the effective action which respect Lorentz and gauge invariance are given by the field strength tensor and its dual

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{2a}$$

$$^\star F^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}. \tag{2b}$$

The lowest-order linearly independent scalars are

$$x := \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}(\boldsymbol{B}^2 - \boldsymbol{E}^2) \tag{3a}$$

$$y := \tfrac{1}{4}F_{\mu\nu}{}^\star F^{\mu\nu} = \boldsymbol{E}\cdot\boldsymbol{B}. \tag{3b}$$

The normalization is chosen in such a way that the Maxwell Lagrangian can be written $\mathcal{L}_{\rm M} = -x$.[3] By taking advantage of the antisymmetry of $F^{\mu\nu}$ and by virtue of the relations [13]

$$F^{\mu\alpha}F^\nu{}_\alpha - {}^\star F^{\mu\alpha\star}F^\nu{}_\alpha = 2xg^{\mu\nu}, \tag{4a}$$

$$F^{\mu\alpha\star}F^\nu{}_\alpha = {}^\star F^{\mu\alpha}F^\nu{}_\alpha = y\; g^{\mu\nu}, \tag{4b}$$

using the metric $g = \mathrm{diag}(-,+,+,+)$, it is easy to verify (i) the vanishing of odd-order invariants and (ii) that invariants of arbitrary order can be reduced to expressions only involving $x^n y^m$ where $n,m = 0,1,2 \ldots$. Besides, note that parity invariance demands for m to be even.

Consequently, the complete effective action becomes extremely simplified, turning out to be a function of x and y only. The corresponding Lagrangian reads

$$\mathcal{L} = \mathcal{L}(x,y). \tag{5}$$

We obtain the equations of motion from $\mathcal{L}$ by variation:

$$0 = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial\mathcal{L}}{\partial A_\nu} = \partial_\mu(\partial_x\mathcal{L}F^{\mu\nu} + \partial_y\mathcal{L}^\star F^{\mu\nu}) \tag{6}$$

where ∂_x, ∂_y denote the partial derivatives with respect to the field strength invariants (3) (and should not be confused with space-time derivatives ∂_μ).

If we take advantage of the Bianchi identity while moving ∂_μ to the right, we arrive at

$$0 = (\partial_x\mathcal{L})\partial_\mu F^{\mu\nu} + (\tfrac{1}{2}M^{\mu\nu}_{\alpha\beta})\partial_\mu F^{\alpha\beta}, \tag{7}$$

where $M^{\mu\nu}_{\alpha\beta}$ is given by

$$M^{\mu\nu}_{\alpha\beta} := F^{\mu\nu}F_{\alpha\beta}(\partial_x^2\mathcal{L}) + {}^\star F^{\mu\nu\star}F_{\alpha\beta}(\partial_y^2\mathcal{L}) + \partial_{xy}\mathcal{L}(F^{\mu\nu\star}F_{\alpha\beta} + {}^\star F^{\mu\nu}F_{\alpha\beta}). \tag{8}$$

Note that M is antisymmetric in the upper as well as the lower indices: $M^{\mu\nu}_{\alpha\beta} = -M^{\nu\mu}_{\alpha\beta} = M^{\nu\mu}_{\beta\alpha}$.

In general, $F^{\mu\nu}$ contains background fields $F_{\rm B}^{\mu\nu}$ and the propagating photon field $f^{\mu\nu}$. According to assumption (2), the derivative acting on $F_{\rm B}^{\mu\nu}$ vanishes:

$$\partial_\mu F^{\lambda\kappa} = \partial_\mu f^{\lambda\kappa}. \tag{9}$$

Inserting Eq. (9), Eq. (7) yields, in Fourier space,

[2] In principle, the Lagrangian $\mathcal{L}$ can contain imaginary parts indicating the instability of the modified vacuum. In the following, it is understood that we take into account only the real part of $\mathcal{L}$ which is solely responsible for the field equations.

[3] x and y are usually called $\mathcal{F}$ and $\mathcal{G}$. We do not follow this convention for reasons of simplicity.

$$0=(\partial_x\mathcal{L})k_\mu f^{\mu\nu}+(\tfrac{1}{2}M^{\mu\nu}_{\alpha\beta})k_\mu f^{\alpha\beta}. \tag{10}$$

Introducing a gauge potential a^μ for the propagating field $f^{\mu\nu}$, we may write

$$f^{\mu\nu}=k^\mu a^\nu-k^\nu a^\mu=a(k^\mu\epsilon^\nu-k^\nu\epsilon^\mu), \tag{11}$$

where $a:=\sqrt{a^\mu a_\mu}$ and $\epsilon^\mu=a^\mu/a$. Here, the polarization vectors ϵ^μ are normalized to 1.

Establishing the Lorentz gauge $k_\mu\epsilon^\mu=0$, we get

$$0=(\partial_x\mathcal{L})k^2\epsilon^\nu+M^{\mu\nu}_{\alpha\beta}k_\mu k^\alpha\epsilon^\beta, \tag{12}$$

where we used the antisymmetry of $M^{\mu\nu}_{\alpha\beta}$.

The next important step is to multiply Eq. (12) by ϵ_ν and average over polarization states according to the well-known rule

$$\sum_{\text{pol}}\ \epsilon^\beta\epsilon^\nu\to g^{\beta\nu}, \tag{13}$$

where the additional terms on the right-hand side of Eq. (13) vanish with the aid of the antisymmetry of $M^{\mu\nu}_{\alpha\beta}$. We find, for Eq. (12),

$$0=2(\partial_x\mathcal{L})k^2+M^{\mu\nu}_{\alpha\nu}k_\mu k^\alpha. \tag{14}$$

Equation (14) already represents a light cone condition and actually indicates that the familiar $k^2=0$ will in general not hold for arbitrary Lagrangians. Our final task is to put $M^{\mu\nu}_{\alpha\nu}$ in a convenient shape. Using the powerful relations (4), we obtain

$$M^{\mu\nu}_{\alpha\nu}=2[\tfrac{1}{2}F^{\mu\nu}F_{\alpha\nu}(\partial_x^2+\partial_y^2)\mathcal{L}+\delta^\mu_\alpha(y\partial_{xy}\mathcal{L}-x\partial_y^2\mathcal{L})]. \tag{15}$$

Introducing the Maxwell energy-momentum tensor

$$T^\mu_{\ \alpha}=F^{\mu\nu}F_{\alpha\nu}-x\delta^\mu_\alpha, \tag{16}$$

this leads to

$$M^{\mu\nu}_{\alpha\nu}=2[\tfrac{1}{2}T^\mu{}_\alpha(\partial_x^2+\partial_y^2)\mathcal{L}+\delta^\mu_\alpha(\tfrac{1}{2}x(\partial_x^2-\partial_y^2)\mathcal{L}+y\partial_{xy}\mathcal{L})]. \tag{17}$$

However, the Maxwell energy-momentum tensor in general is devoid of any physical meaning, since we are simply not dealing with the Maxwell Lagrangian. The right quantity to deal with is therefore the vacuum expectation value (VEV) of the energy-momentum tensor defined by[4]

$$\langle T^{\mu\nu}\rangle:=\frac{2}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g_{\mu\nu}},\quad \Gamma:=\int d^4x\ \sqrt{-g}\mathcal{L}, \tag{18}$$

where Γ denotes the effective action. Performing the calculation, we arrive at

$$\langle T^{\mu\nu}\rangle_{xy}=-T^{\mu\nu}(\partial_x\mathcal{L})+g^{\mu\nu}(\mathcal{L}-x\partial_x\mathcal{L}-y\partial_y\mathcal{L}). \tag{19}$$

Solving Eq. (19) for $T^{\mu\nu}$ and inserting it into Eq. (17), we can present $M^{\mu\nu}_{\alpha\nu}$ in its final form

$$M^{\mu\nu}_{\alpha\nu}=2\Bigg[-\frac{1}{2}\frac{(\partial_x^2+\partial_y^2)\mathcal{L}}{\partial_x\mathcal{L}}\langle T^\mu{}_\alpha\rangle_{xy}+\delta^\mu_\alpha\Bigg(\tfrac{1}{2}x(\partial_x^2-\partial_y^2)\mathcal{L}+y\partial_{xy}\mathcal{L}+\frac{\tfrac{1}{2}(\partial_x^2+\partial_y^2)\mathcal{L}}{\partial_x\mathcal{L}}(\mathcal{L}-x\partial_x\mathcal{L}-y\partial_y\mathcal{L})\Bigg)\Bigg]. \tag{20}$$

Substituting $M^{\mu\nu}_{\alpha\nu}$ into Eq. (14), we end up with the desired light cone condition for EM field modified vacua fulfilling the above-mentioned assumptions:

$$k^2=Q\langle T^{\mu\nu}\rangle_{xy}k_\mu k_\nu, \tag{21}$$

where

$$Q=\frac{\frac{1}{2}(\partial_x^2+\partial_y^2)\mathcal{L}}{\left[(\partial_x\mathcal{L})^2+(\partial_x\mathcal{L})\left(\frac{x}{2}(\partial_x^2-\partial_y^2)+y\partial_{xy}\right)\mathcal{L}+\frac{1}{2}(\partial_x^2+\partial_y^2)\mathcal{L}(1-x\partial_x-y\partial_y)\mathcal{L}\right]}. \tag{22}$$

To extend the validity of the light cone condition to arbitrary non-trivial vacua, we have to take the vacuum expectation value of Eq. (21) with respect to the additional vacuum modifications parametrized by the (collective) label z:

$$k^2={}_z\langle 0|Q\langle T^{\mu\nu}\rangle_{xy}|0\rangle_z k_\mu k_\nu. \tag{23}$$

Inserting a complete set of intermediate states, we obtain

$$k^2=\sum_i\ {}_z\langle 0|Q|i\rangle_z\,{}_z\langle i|\langle T^{\mu\nu}\rangle_{xy}|0\rangle_z k_\mu k_\nu. \tag{24}$$

In the following, we consider the vacuum to behave as a *passive* medium in which the EM fields and the further vacuum modifications z remain in a state of static equilibrium. Since Q solely depends on x and y [via $\mathcal{L}(x,y)$], this *assumption of passivity* leads to

[4]Note that the variation with respect to the metric tensor is just a trick to calculate the symmetric energy-momentum tensor. With some care, the same result can be obtained by canonical methods.

$$_z\langle 0|Q|i\rangle_z = \langle Q\rangle_z \delta_{0i}. \tag{25}$$

Equation (25) states that the vacuum exhibits no back-reaction caused by the EM fields while switching on z.

Q depends functionally on $\mathcal{L}(x,y)$, which is, as usual, defined via the functional integral over the fluctuating fields. Taking the expectation value of Q hence leads back to integrating over the field configurations which respect the modified vacuum. E.g., if the modification z imposes boundary conditions on the fields, the functional integral has to be taken over the fields which fulfill these boundary conditions. Therefore, taking the VEV of Q defines the new effective Lagrangian characterizing the complete non-trivial vacuum:

$$\langle Q\rangle_z = \langle Q(\mathcal{L}(x,y))\rangle_z = Q(\mathcal{L}(x,y;z)). \tag{26}$$

We finally arrive at the light cone condition for arbitrary homogeneous non-trivial vacua:

$$k^2 = Q(x,y,z)\langle T^{\mu\nu}\rangle_{xyz} k_\mu k_\nu. \tag{27}$$

Remember that the validity of the light cone condition (27) is not restricted to results of perturbation theory or only small modifications of the Maxwell Lagrangian. It is an exact statement in the sense of effective theories.

Now, the terminology ''modified QED vacuum'' should be clarified: from the derivation of the light cone condition, it is obvious that the implicit space-time dependence of $\mathcal{L}$ should only be contained in the field variables. Furthermore, the vacuum has to fulfill the demand for passivity. Otherwise the light cone condition (27) only represents a zeroth order approximation of the infinite sum over intermediate states in Eq. (25).

As a third remark, we want to point out that the sum over polarization states is not necessary for the derivation of a light cone condition. By summing, we even exclude the study of birefringence from the formalism which is certainly the most important experimental application [14–16]. But for a projection on the polarization eigenstates, the y^n-terms have to be rewritten in terms of the field strength tensor which is practically impossible for arbitrary $\mathcal{L}$.

In the remainder of the section, we calculate further representations of Eq. (27) by choosing a certain reference frame and introducing

$$\overline{k^\mu} = \frac{k^\mu}{|\boldsymbol{k}|} = \left(\frac{k^0}{|\boldsymbol{k}|}, \hat{\boldsymbol{k}}\right) =: (v, \hat{\boldsymbol{k}}), \tag{28}$$

where we defined the phase velocity by $v := k^0/|\boldsymbol{k}|$. For Eq. (27), we obtain

$$v^2 = 1 - Q\langle T^{\mu\nu}\rangle \overline{k_\mu}\,\overline{k_\nu}. \tag{29}$$

Equation (29) clearly demonstrates that the light cone condition is a generalization of the ''unified formula'' of Latorre, Pascual and Tarrach [8].

In general, the Q-factor will depend on all the variables and parameters of $\mathcal{L}$ and hence will naturally be neither universal nor constant. Besides, the daunting structure of the Q-factor will simplify in the case of small corrections to $\mathcal{L}_{\rm M}$. As will be shown in Sec. IV, the denominator then reduces to 1.

Another representation of the light cone condition is found by averaging over propagation directions; i.e., integrating over $\hat{\boldsymbol{k}} \in S^2$,

$$v^2 = \frac{1 - Q\left(\frac{1}{3}\langle T^{00}\rangle + \frac{1}{3}\langle T^\alpha{}_\alpha\rangle\right)}{1 + Q\,\langle T^{00}\rangle}. \tag{30}$$

For $Q\langle T^{00}\rangle \ll 1$ and $\langle T^\alpha{}_\alpha\rangle$ being even of lower order, this reduces to

$$v^2 = 1 - \frac{4}{3} Q\langle T^{00}\rangle = 1 - \frac{4}{3} Qu, \tag{31}$$

where u denotes the (renormalized) energy density of the modified vacuum.

III. VELOCITY SHIFT AND SCALE ANOMALY

In his paper, Shore [10] suggested a deeper connection between the velocity shift and the scale anomaly. For the Heisenberg-Euler Lagrangian, he showed that the coefficients of the x^2 and y^2 terms in the scale anomaly are precisely those appearing in the velocity shift for the different polarization states.

Within the framework developed so far, we will attempt to clarify the relation between the scale anomaly and the velocity shift. Therefore, we have to investigate whether the terms in the Q-factor can be expressed in terms of the anomaly. For reasons of simplicity, we limit this consideration to the case of a purely EM field modified vacuum. From Eq. (19), we can read off the scale anomaly

$$\langle T^\alpha{}_\alpha\rangle = 4(\mathcal{L} - x\partial_x\mathcal{L} - y\partial_y\mathcal{L}). \tag{32}$$

By differentiation, we find

$$\partial_x\langle T^\alpha{}_\alpha\rangle = -4(x\partial_x^2\mathcal{L} + y\partial_{xy}\mathcal{L}), \tag{33a}$$

$$\partial_y\langle T^\alpha{}_\alpha\rangle = -4(y\partial_y^2\mathcal{L} + x\partial_{xy}\mathcal{L}). \tag{33b}$$

From Eqs. (33) immediately follows

$$(\partial_x^2 + \partial_y^2)\mathcal{L} = -\left(\frac{y}{x} + \frac{x}{y}\right)\partial_{xy}\mathcal{L} - \frac{1}{4}\left(\frac{1}{x}\partial_x + \frac{1}{y}\partial_y\right)\langle T^\alpha{}_\alpha\rangle. \tag{34}$$

This expression is proportional to the numerator of the Q-factor, Eq. (22). Using similar techniques, we can also rewrite the denominator, but the result is not very illuminating:

$$\text{denominator}\ \ (Q)=(\partial_x\mathcal{L})^2-(\partial_x\mathcal{L})\left(\frac{x}{2}(\partial_x^2+\partial_y^2)\mathcal{L}+\frac{1}{4}\partial_x\langle T^\alpha{}_\alpha\rangle\right)+\frac{1}{8}[(\partial_x^2+\partial_y^2)\mathcal{L}]\langle T^\alpha{}_\alpha\rangle. \tag{35}$$

Fortunately, approximating Eq. (35) by 1 will be appropriate to the applications of Sec. IV.

It is already obvious from Eq. (34) that there is no immediate connection between $\langle T^\alpha{}_\alpha\rangle$ and the velocity shift, Eq. (29). The findings of Shore arise from the special structure of the Heisenberg-Euler Lagrangian where $\partial_{xy}\mathcal{L}=0$. In general, higher-order mixed terms are not forbidden by gauge, Lorentz or parity invariance. Referring to Eq. (34), the introduction of the scale anomaly appears to be artificial rather than interpretable. Even Shore's conjecture that the sign of the scale anomaly is linked to the sign of the velocity shift cannot be maintained.

Instead, we favor the pure effective action formulation, i.e., the left-hand side of Eq. (34), since it offers a new intuitive picture. Referring to Eq. (29), the value and sign of the velocity shift result from the competition between the VEV of the energy-momentum tensor and the Q-factor. Both are *a priori* neither positive nor bounded by symmetry principles. Let us restrict the following investigation to the case of small corrections to the Maxwell Lagrangian, i.e.,

$$Q\simeq\frac{1}{2}(\partial_x^2+\partial_y^2)\mathcal{L}\ \Rightarrow\ \nabla^2\mathcal{L}=2Q. \tag{36}$$

Because of the similarity to the (2D) Poisson equation, we will call Q from now on the *effective action charge* in field space. The classical vacuum $\mathcal{L}_{\rm M}=-x$ is *uncharged* and hence $v=1$. As we will soon demonstrate, the pure QED vacuum has a small positive charge at the origin in field space $(x=y=0)$. For increasing the field strength, $\langle T^{\mu\nu}\rangle$ certainly also increases without an upper bound, and so we expect Q to decrease in order to produce no unphysical velocity shift >1. It is therefore reasonable to presume localized effective action charge distributions centered upon the origin in field space. The results of Sec. IV will confirm this charge-like picture.

IV. APPLICATIONS OF THE LIGHT CONE CONDITION

Up to now, the light cone condition might be regarded as a nice frame without a picture enclosed, since it is much easier to talk about all-loop or non-perturbative effective actions than to calculate one.

Indeed, the effective actions which we are going to insert will not reach beyond two-loop order. Their general structure can be characterized by

$$\mathcal{L}=\mathcal{L}_{\rm M}+\mathcal{L}_{\rm c},\quad \frac{\mathcal{L}_{\rm c}}{\mathcal{L}_{\rm M}}\ll 1, \tag{37}$$

where $\mathcal{L}_{\rm c}$ contains the correction terms.

Regarding the denominator expression of the effective action charge (35), the scale anomaly $\langle T^\alpha{}_\alpha\rangle$ is of the same order as $\mathcal{L}_{\rm c}$. Hence, Eq. (35) simply reduces to

$$\text{Eq. (35)}=1+\mathcal{O}(\mathcal{L}_{\rm c}), \tag{38}$$

and the approximation $Q=\frac{1}{2}\nabla^2\mathcal{L}=\frac{1}{2}\nabla^2\mathcal{L}_{\rm c}$ is justified.

A. Weak EM fields

According to the authors of Ref. [17], the two-loop corrected Heisenberg-Euler Lagrangian (weak-field limit of the complete one-loop approximated effective QED Lagrangian) reads

$$\mathcal{L}=-x+c_1x^2+c_2y^2, \tag{39}$$

where

$$c_1=\frac{8\alpha^2}{45m^4}\left(1+\frac{40}{9}\frac{\alpha}{\pi}\right), \tag{40a}$$

$$c_2=\frac{14\alpha^2}{45m^4}\left(1+\frac{1315}{252}\frac{\alpha}{\pi}\right). \tag{40b}$$

With the aid of the light cone condition (31), we immediately obtain, for the polarization and propagation direction averaged velocity $(v\equiv\sqrt{\overline{v^2}})$,

$$Q=c_1+c_2, \tag{41}$$

$$\rightsquigarrow v=1-\frac{4\alpha^2}{135m^4}\left(11+\frac{1955}{36}\frac{\alpha}{\pi}\right)\left[\frac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2)\right]. \tag{42}$$

In the well-known one-loop part of Eq. (42), we can identify the factor of $44\alpha^2/135m^4$ as the "universal constant" of the "unified formula" (1). At this stage, it is already understandable that all of the known QED induced velocity shifts share this universal factor in the low-energy limit, since they are all based on the Heisenberg-Euler Lagrangian. Even the results in gravitation involve the same e^+e^--loop calculation (of course, in a curved space-time [3]).[5] It is furthermore obvious that the two-loop correction $\frac{1955}{36}\alpha/\pi$ is as universal as the number 11. [Note that modifications from the denominator of Q, Eq. (35), contribute to the order $\mathcal{O}(\alpha^4)$.]

B. Strong magnetic fields

Since the Heisenberg-Euler Lagrangian (39) represents a weak field limit, Eq. (41) denotes the value of the effective action charge at the origin in field space $(x,y=0)$. In this

[5]In particular, there is nothing mysterious about the factor 11 as it is sometimes found in the literature.

subsection, we analyze the form of Q along the positive x-axis (pure magnetic fields). As our starting point, we use Schwinger's famous formula for the one-loop effective QED Lagrangian [13]:

$$\mathcal{L}_{\rm c}=-\frac{1}{8\pi^2}\int_0^{i\infty}\frac{ds}{s^3}e^{-m^2s}\bigg((es)^2|y|\coth[es(\sqrt{x^2+y^2}+x)^{1/2}]\times\cot[es(\sqrt{x^2+y^2}-x)^{1/2}]-\frac{2}{3}(es)^2x-1\bigg). \tag{43}$$

It is understood that the convergence is implicitly ensured by the prescription $m^2\to m^2-i\epsilon$. (Note that we have not performed a proper time Wick rotation yet.)

It will be useful to reparametrize the field space with new coordinates

$$a:=(\sqrt{x^2+y^2}+x)^{1/2},\quad b:=(\sqrt{x^2+y^2}-x)^{1/2}, \tag{44a}$$

$$\Rightarrow |y|=ab,\quad x=\tfrac{1}{2}(a^2-b^2). \tag{44b}$$

The Laplacian in terms of a and b reads

$$\nabla^2=\frac{1}{a^2+b^2}(\partial_a^2+\partial_b^2). \tag{45}$$

For the term in the square brackets in Eq. (43), we easily find

$$\begin{aligned}\nabla^2[\cdots]&=\frac{(es)^2}{a^2+b^2}(\partial_a^2+\partial_b^2)[ab\,\coth esa\,\cot esb]\\&=\frac{2(es)^2}{a^2+b^2}\bigg[\frac{esb\,\cot esb}{\sinh^2 esa}(esa\,\coth esa-1)\\&\quad+\frac{esa\,\coth esa}{\sinh^2 esb}(esb\,\cot esb-1)\bigg].\end{aligned} \tag{46}$$

Confining ourselves to purely magnetic fields ($x=\frac{1}{2}B^2$, $y=0\Rightarrow b=0$, $a=|\boldsymbol{B}|$), we obtain

$$\text{Eq. (46)}\to\frac{2(es)^2}{a^2}\bigg[\frac{esa\,\coth esa-1}{\sinh^2 esa}-\frac{1}{3}esa\,\coth esa\bigg]. \tag{47}$$

The complete formula for the effective action charge might be written (substitution: $z:=esa$, $h:=m^2/2ea=B^{\rm cr}/2B$)

$$Q(h)=-\frac{1}{2a^2}\frac{\alpha}{\pi}\int_0^{i\infty}\frac{dz}{z}e^{-2hz}\bigg[\frac{z\coth z-1}{\sinh^2 z}-\frac{1}{3}z\coth z\bigg]. \tag{48}$$

With some effort, the evaluation of the integral can be performed analytically by standard means of dimensional regularization. Details are given in Appendix A. The result is

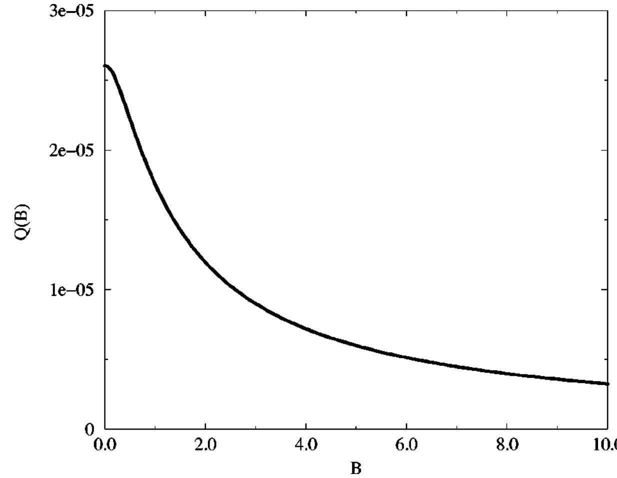

FIG. 1. Effective action charge $Q(B)=$ in units of $1/m^4$ versus magnetic field B in units of the critical field strength $B_{\rm cr}=m^2/e$.

$$Q(h)=\frac{1}{2B^2}\frac{\alpha}{\pi}\bigg[\bigg(2h^2-\frac{1}{3}\bigg)\psi(1+h)-h-3h^2-4h\ln\Gamma(h)+2h\ln 2\pi+\frac{1}{3}+4\zeta'(-1,h)+\frac{1}{6h}\bigg], \tag{49}$$

where ψ denotes the logarithmic derivative of the Γ-function and ζ' is the first derivative of the Hurwitz zeta function with respect to the first argument [23].

For strong fields, the last term of Eq. (49), $\propto 1/6h\propto|\boldsymbol{B}|$, dominates the expression in the square brackets. Hence, the effective action charge decreases with

$$Q(B)\simeq\frac{1}{6}\frac{\alpha}{\pi}\frac{1}{B_{\rm cr}}\frac{1}{B}\quad\text{for}\quad B\to\infty \tag{50}$$

which supports the charge picture (Fig. 1).

The contraction of the energy-momentum tensor VEV may be cast into the form

$$\langle T^{\mu\nu}\rangle\bar{k}_\mu\bar{k}_\nu=\boldsymbol{B}^2-(\boldsymbol{B}\cdot\hat{k})^2+\mathcal{O}(\alpha)=B^2\sin^2\theta+\mathcal{O}(\alpha), \tag{51}$$

where θ measures the angle between the $\boldsymbol{B}$-field and the propagation direction.

Finally, the light cone condition (29) yields, for arbitrary background fields consistent with the one-loop approximation ($h=B_{\rm cr}/2B$),

$$v^2=1-\frac{\alpha}{\pi}\frac{\sin^2\theta}{2}\bigg[\bigg(\frac{B_{\rm cr}^2}{2B^2}-\frac{1}{3}\bigg)\psi\bigg(1+\frac{B_{\rm cr}}{2B}\bigg)-\frac{2B_{\rm cr}}{B}\ln\Gamma\bigg(\frac{B_{\rm cr}}{2B}\bigg)-\frac{3B_{\rm cr}^2}{4B^2}-\frac{B_{\rm cr}}{2B}+\frac{B_{\rm cr}}{B}\ln 2\pi+\frac{1}{3}+4\zeta'\bigg(-1,\frac{B_{\rm cr}}{2B}\bigg)+\frac{B}{3B_{\rm cr}}\bigg]. \tag{52}$$

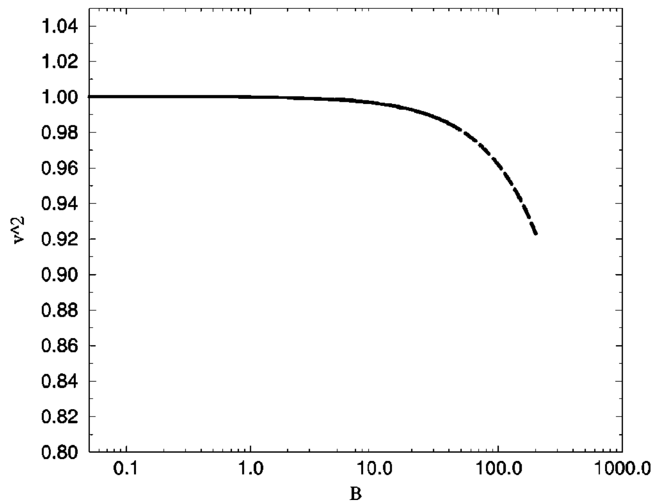

FIG. 2. Square velocity v^2 versus magnetic field B in units of the critical field strength $B_{\rm cr}=m^2/e$. The dashed curve indicates the region where two-loop corrections become important.

The first derivative of the Hurwitz zeta function at -1 can be related to the generalized Γ-function of first order Γ_1 [18],

$$\zeta'(-1,h)=-h\ln h+\ln\Gamma_1(1+h)-L_1+\frac{1}{12}, \qquad (53)$$

where $L_1=0.248\,754\,477\ldots$ is a pure number and can be obtained from the Raabe integral [19].

Using Eq. (53), one can show that Eq. (52) is identical to the findings of Tsai and Erber [9]. Equation (52) is plotted in Fig. 2. Although the velocity shift increases proportional to the magnetic field for large B, the total amount of the velocity shift remains comparably small,

$$\delta v\simeq 9.58\ldots\times 10^{-5} \quad \text{at} \quad B=B_{\rm cr}=\frac{m^2}{e}, \qquad (54)$$

for strong B-fields, consistent with the one-loop approximation, i.e., $B/B_{\rm cr}<\pi/\alpha\simeq 430$. Taking higher-order loop calculations into account, we expect a stronger decrease of $Q(B)$ for large B in order to let $Q\langle T_{\mu\nu}\rangle\bar{k}_\mu\bar{k}_\nu$ be bounded.

C. Casimir vacua (Scharnhorst effect)

One curious result regarding vacuum induced velocity shifts is the possibility of superluminal phase and group velocities. As mentioned above, e.g., Casimir vacua can create positive velocity shifts, since a negative shift of the zero point energy is permitted. For the configuration of perfectly conducting parallel plates of distance a, $\langle T^{\mu\nu}\rangle$ is found to be [20]

$$\langle T^{\mu\nu}\rangle=\frac{\pi^2}{720a^4}\begin{pmatrix}-1&&&\\&1&&\\&&1&\\&&&-3\end{pmatrix}, \qquad (55)$$

where the symmetry axis points along the 3-direction. The effective action charge has to be evaluated in the zero-field limit. In concordance with experimental facilities, the plate separation a is treated as a macroscopic parameter ($a \propto \mu$m); otherwise, we would violate the soft-photon approximation, since the photon wavelength λ has to obey $\lambda\ll a$ to validate the concept of treating the Casimir region as a (macroscopic) medium.

The magnitude of a implies that we can neglect the a-dependence of Q which is exponentially suppressed by $ma\gg 1$ (this point will become clearer in the following section). Here, Q is simply given by Eq. (41),

$$Q=c_1+c_2=\frac{2\alpha^2}{45m^4}\left(11+\frac{1955}{36}\frac{\alpha}{\pi}\right), \qquad (56)$$

from which directly follows, using Eqs. (55) and (29),

$$v=1+\frac{1}{(90)^2}\frac{\alpha^2}{m^4}\left(11+\frac{1955}{36}\frac{\alpha}{\pi}\right)\frac{\pi^2}{a^4}, \qquad (57)$$

for a propagation perpendicular to the plates (a parallel propagation will, of course, not be modified).

Equation (57) represents the two-loop corrected version of Scharnhorst's formula [4,5]. Note that the two-loop correction enhances the velocity shift. As was recently found by Kong and Ravndal [21], the radiative correction to the Casimir energy is of order α^2/m^4a^8:

$$\langle T^{00}\rangle\equiv u=-\frac{\pi^2}{720a^4}-\frac{11}{(90)^2\times 30\times 16}\frac{\pi^4\alpha^2}{m^4a^8}. \qquad (58)$$

At the two-loop level of Eq. (57), this correction can obviously be neglected. Even three-loop contributions in Q would be more important. But it is interesting to note that this correction also contributes positively to v.

D. Finite temperature

In the remaining sections, we reveal the manifold features of temperature induced velocity shifts. Unlike the Scharnhorst effect, we do not recognize a principal obstacle against measurability here, and the results allow for an immediate physical interpretation. The following calculations are restricted to the one-loop level.

We begin with the one-loop correction to the effective QED Lagrangian at finite temperature which can be decomposed according to

$$\mathcal{L}_{\rm c}(x,y,T)=\mathcal{L}_{\rm c}(x,y,T=0)+\Delta\mathcal{L}(x,y,T), \qquad (59)$$

whereby $\mathcal{L}_{\rm c}(x,y,T=0)$ denotes the usual zero-temperature Lagrangian, Eq. (43).

For purely magnetic fields, $\Delta\mathcal{L}(x,y,T)$ was calculated by Dittrich [22]:

$$\Delta\mathcal{L}(B,T) = -\frac{\sqrt{\pi}}{4\pi^2}\int_0^{i\infty}\frac{ds}{s^{5/2}}e^{-m^2s}esB\cot esB \times T\left[\Theta_2(0,4\pi i s T^2) - \frac{1}{2T\sqrt{\pi s}}\right]. \quad (60)$$

The Jacobi Θ-function is defined by [23]

$$\Theta_2(0,-q) = \sum_{n=-\infty}^{\infty}\exp[-iq(n+\tfrac{1}{2})^2]. \quad (61)$$

The effective action charge can be decomposed similarly to Eq. (59) into

$$Q(x,y,T) = Q(x,y,T=0) + \Delta Q(x,y,T) = \tfrac{1}{2}\nabla^2\mathcal{L}_c(x,y,T=0) + \tfrac{1}{2}\nabla^2\Delta\mathcal{L}(x,y,T). \quad (62)$$

$Q(x,y,T=0)$ clearly corresponds to the zero-temperature case as treated above.

Since we have to differentiate with respect to x *and* y, it is not sufficient for the calculation of ΔQ to consider magnetic fields only in Eq. (60). Not until we have carried out the Laplacian are we allowed to set $\boldsymbol{E}=0$. Indeed, we have to take this limit $\boldsymbol{E}\to 0$ in the end, because the principle of equilibrium thermodynamics would otherwise be violated. In addition, the above-mentioned assumption of passivity of the vacuum is only fulfilled for $\boldsymbol{E}=0$. The appropriate expression is simply obtained by replacing

$$esB\cot esB \quad (63a)$$

by the gauge and Lorentz invariant terms

$$(esa)(esb)\coth(esa)\cot(esb) \quad (63b)$$

in analogy to Eq. (43). Again, we make use of the advantageous coordinates a,b in field space defined in Eqs. (44). The result of the differentiation was already found in Eq. (46); hence we obtain, for the temperature induced effective action charge for purely magnetic fields $(a=B, b=0)$,

$$\begin{aligned}\Delta Q(B,T) &= -\frac{\sqrt{\pi}}{a^2}\frac{\alpha}{\pi}\int_0^{i\infty}\frac{ds}{\sqrt{s}}e^{-m^2s}\left[\frac{esa\coth esa - 1}{\sinh^2 esa} - \frac{esa}{3}\coth esa\right]T\left(\Theta_2(0,4\pi i s T^2) - \frac{1}{\sqrt{\pi s}2T}\right)\\ &= -\frac{\alpha}{\pi}\frac{1}{a^2}\int_0^{i\infty}\frac{ds}{s}e^{-m^2s}\left[\frac{esa\coth esa - 1}{\sinh^2 esa} - \frac{esa}{3}\coth esa\right]\sum_{n=1}^{\infty}e^{-i\pi n}e^{-n^2/4T^2s}. \quad (64)\end{aligned}$$

In the last line, we made use of the identity [22]

$$\Theta_2(0,4\pi i s T^2) = \frac{1}{\sqrt{\pi s}2T}\left(1 + 2\sum_{n=1}^{\infty}e^{-i\pi n}e^{-n^2/4T^2s}\right). \quad (65)$$

Our task is to evaluate Eq. (64) in the various limits. First, we consider pure temperature phenomena with vanishing field strength. The temperature dependent part of the effective action charge reduces to

$$\begin{aligned}\Delta Q(B=0,T) &= 2\frac{22}{45}\alpha^2\int_0^{i\infty}ds\; s e^{-m^2s}\sum_{n=1}^{\infty}e^{-i\pi n}e^{-n^2/4T^2s}\\ &= \frac{22}{45}\frac{\alpha^2}{m^4}\sum_{n=1}^{\infty}(-1)^n\left(\frac{m}{T}n\right)^2 K_2\left(\frac{m}{T}n\right), \quad (66)\end{aligned}$$

whereby we have taken advantage of the representation

$$2\left(\frac{\mu}{2}\right)^{\nu}K_{\nu}(\mu) = \int_0^{\infty}du\; u^{\nu-1}\exp\left(-u - \frac{\mu^2}{4u}\right) \quad (67)$$

for the modified Bessel function and have rotated the contour.

For low temperature, we may use the asymptotic expansion of $K_2(x)$ for $x \gg 1$:

$$K_2(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\left[1 + \mathcal{O}\left(\frac{1}{x}\right)\right]. \quad (68)$$

In this limit, we find

$$\Delta Q(B=0,T\to 0) \simeq -\frac{22}{45}\frac{\alpha^2}{m^4}\sqrt{\frac{\pi}{2}}\left(\frac{m}{T}\right)^{3/2}e^{-m/T}\to 0^-. \quad (69)$$

Hence, the effective action charge is perfectly described by $Q(B=0,T=0)=c_1+c_2$, Eq. (41), in this limit, while the influence of temperature on Q vanishes as it should. [Note that in the case of Scharnhorst's effect a similar term $\Delta Q(B=0, ma\gg 1)$ also vanishes by drawing the analogy $T \propto 1/a$.]

Next, we investigate the high-temperature limit $T/m \gg 1$ of Eq. (66). The calculation is, however, much more involved, and so we simply state the result:

$$\Delta Q(T\gg m) = -\frac{22}{45}\frac{\alpha^2}{m^4}\left[1 - \frac{k_1}{4}\frac{m^4}{T^4} + \mathcal{O}\left(\frac{m^6}{T^6}\right)\right], \quad (70)$$

where $k_1 = 0.123\,749\,077\,470\ldots = \text{const}$. The interested reader is referred to Appendix B.

Therefore, we arrive at the remarkable result that the complete effective action charge

$$Q(T\gg m) = Q(T=0) + \Delta Q(T\gg m) = \frac{11}{90}k_1\frac{\alpha^2}{T^4} + \mathcal{O}\left(\frac{m^2}{T^6}\right) \quad (71)$$

decreases rapidly, $\propto 1/T^4$. $Q(B=0,T)$ is plotted in Fig. 3. The influence of temperature causes the effective action

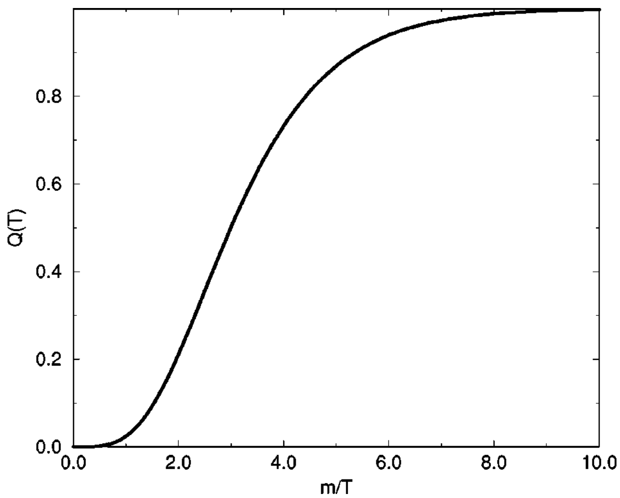

FIG. 3. Effective action charge $Q(T)=Q(B=0,T=0)+\Delta Q(T)$ in units of $\frac{22}{45}\alpha^2/m^4$; for high temperature, $Q(T)$ decreases proportional to $1/T^4$.

charge to evaporate. Numerical results astonishingly indicate that Eq. (71) is already a reasonable approximation for $m/T\simeq 1.4$ (error $\leqslant 5\%$) where real e^+e^--pair creation is energetically impossible and the vacuum is essentially modified by a photon gas. This excludes the interpretation that Eq. (71) is a pure threshold effect of pair production.

To complete the high temperature or zero field analysis of the light cone condition we need the VEV of the energy-momentum tensor which is given by (see, e.g., [24])

$$\langle T_{\mu\nu}\rangle_T=\frac{\pi^2}{90}\left(N_B+\frac{7}{8}N_F\right)T^4\ \mathrm{diag}(3,1,1,1). \tag{72}$$

The integer variables N_B and N_F denote the number of bosonic and fermionic degrees of freedom at a given temperature. For QED, we obtain

$$N_B=2,\quad N_F=0\quad \text{for}\quad T\ll m\quad (\text{photon gas}) \tag{73a}$$

$$N_B=2,\quad N_F=4\quad \text{for}\quad T\gg m\quad (\text{photon} + \text{ultrarelativistic } e^+ \text{and } e^- \text{ fermion gas}). \tag{73b}$$

It is appropriate to employ Eq. (31) for the light cone condition. Using our findings in Eqs. (69), (41), (72), (73a), we recover the well-known result [8] for low temperature,

$$v=1-\frac{44\pi^2}{2025}\alpha^2\frac{T^4}{m^4}, \tag{74}$$

which according to Fig. 3 is valid for $T/m<0.16$ (error $\leqslant 5\%$) ($T<10^9$ K). Substituting Eq. (71) into Eq. (31) and using Eqs. (72), (73b) for $T\gg m$, we finally arrive at the velocity of soft photons moving in a photon and ultrarelativistic e^+e^- gas:

$$v=1-\frac{121}{8100}k_1\pi^2\alpha^2+\mathcal{O}\left(\frac{m^2}{T^2}\right)$$
$$=1-9.72\ldots\times 10^{-7}+\mathcal{O}\left(\frac{m^2}{T^2}\right)=\text{const}+\mathcal{O}\left(\frac{m^2}{T^2}\right). \tag{75}$$

In Eqs. (74), (75), we found that the velocity shift increases proportional to T^4 for low temperature but approaches a constant value in the high-temperature limit. This can be understood in terms of the effective action charge which evaporates sufficiently fast compared to the increase of the energy-momentum tensor VEV. Obviously, the shift described by Eq. (75) remains small; therefore the deviation from the vacuum velocity does not become seriously important (e.g., for the construction of cosmological models). However, one should keep in mind that, if the temperature exceeds the masses of further charged particles, each particle will contribute additively to Q and will increase the respective number of degrees of freedom N_B or N_F.

E. Casimir vacua at finite temperature

The combination of thermal and Casimir phenomena is in itself worthwhile studying, because both effects enter the formalism via boundary conditions but lead to opposite results. In the following, we want to investigate where and why the respective effect dominates the velocity shift. The determining order parameter is the dimensionless combination Ta. Nevertheless, the plate separation a has to be considered as a macroscopic quantity ($a\simeq\mu$m).

First, we consider the low-temperature region. According to Brown and Maclay [20], the VEV of $T^{\mu\nu}$ depending on a and T is given by

$$\langle T^{00}\rangle_T^a=-\frac{\pi^2}{720}\frac{1}{a^4}+\frac{\zeta(3)}{\pi^2}\frac{T^3}{a},$$
$$\langle T^{33}\rangle_T^a=-\frac{\pi^2}{240}\frac{1}{a^4},\quad \text{for}\quad Ta\to 0, \tag{76}$$

for the parallel plate configuration [$\zeta(3)=1.202\,056\ldots$]. The light cone condition (29) for a propagation perpendicular to the plates [$\bar{k}^\mu=(v,0,0,1)$] yields

$$v=1+\frac{1}{(90)^2}\frac{\alpha^2}{m^4}\left(11+\frac{1955}{36}\frac{\alpha}{\pi}\right)\frac{\pi^2}{a^4}\left(1-\frac{180\zeta(3)}{\pi^4}(Ta)^3\right). \tag{77}$$

In the low-T limit, the $(Ta)^3$-term can be neglected and we only rediscover Scharnhorst's result. But we do not find an additional velocity shift proportional to T^4 which could have been expected from Eq. (74). This clearly arises from the fact that none of the (quantized) perpendicular modes can be excited at low temperature. The $(Ta)^3$-term in Eq. (77) will become important for $Ta=\mathcal{O}(1)$, i.e., $T>2000$ K for

$a \simeq \mu$m. This shows that the Scharnhorst effect is not sensitive to temperature perturbations.

For increasing temperature, we encounter an intermediate temperature region characterized by the condition $1 \ll Ta \ll ma$ which corresponds to 0.2 eV$\ll T \ll$0.5 MeV. This implies that $Q = Q(T=0)$ is a justified approximation and the thermal contribution of an e^+e^- gas does not have to be taken into account.

Using further results of Brown and Maclay [20],

$$\langle T^{00}\rangle_T^a = \frac{\pi^2}{15} T^4 \tag{78}$$

$$\langle T^{33}\rangle_T^a = \frac{\pi^2}{45} T^4 + \frac{\zeta(3)}{4\pi} \frac{T}{a^3}, \quad \text{for} \quad Ta \gg 1,$$

we find

$$v = 1 - \frac{4\pi^2}{(45)^2} \frac{\alpha^2}{m^4} \left(11 + \frac{1955}{36} \frac{\alpha}{\pi} \right) T^4 \left(1 - \frac{45\zeta(3)}{16\pi^3} \frac{1}{(Ta)^3} \right). \tag{79}$$

In this limit, only the modifications caused by the blackbody radiation become important. A term proportional to $1/a^4$ as a consequence of certain missing zero-point fluctuations does not occur, since higher (perpendicular) modes have been thermally excited.

For $T \gg m$, we will certainly recover Eq. (75) with negligible $1/(Ta)^3$ Casimir corrections. Anyway, the concept of solid plates is (at least experimentally) meaningless in this domain.

F. Finite temperature and magnetic fields

For low temperature as well as for weak fields, thermal phenomena decouple from magnetic vacuum modifications, because the effective action charge is not sensitive to weak influences. The velocity shifts can simply be described by an addition of the respective above-calculated ones. The only non-trivial interplay can be found in the domain of strong fields in hot surroundings (e.g., hot neutron stars). Our intention is to evaluate the thermal effective action charge contribution given in Eq. (64) in this limit. Therefore, we substitute $z = esa$ $(h = m^2/2ea = B_{\rm cr}/2B)$,

$$\Delta Q(h,T) = -\frac{\alpha}{\pi} \frac{1}{a^2} \int_0^{i\infty} \frac{dz}{z} e^{-2hz} \left[\frac{z\coth z - 1}{\sinh^2 z} - \frac{1}{3} z\coth z \right] \times \sum_{n=1}^{\infty} (-1)^n e^{-\frac{ea}{4T^2}\frac{n^2}{z}}. \tag{80}$$

In this representation, it is obvious that the integral is dominated by small values of z for weak fields $(h \gg 1)$ and vice versa, i.e., large z for strong fields. We are interested in the latter, and so we expand the term in the square brackets for $z \gg 1$: $[\cdots] \to -\frac{1}{3} z$. Following the manipulations of Eqs. (66), (67), we arrive at

$$\Delta Q(T, B \gg B_{\rm cr}) = \frac{1}{3} \frac{\alpha}{\pi} \frac{1}{B^2} \frac{B}{B_{\rm cr}} \sum_{n=1}^{\infty} (-1)^n \frac{m}{T} n K_1\left(\frac{m}{T} n \right). \tag{81}$$

Note that it was not necessary to impose any conditions on the magnitude of T to arrive at Eq. (81). But, as mentioned above, the field-temperature phenomena decouple for $T \ll m$ due to the asymptotic behavior of $K_1[(m/T)n] \propto \exp(-m/T)$ in Eq. (81); hence, $\Delta Q(m/T \gg 1, B \gg B_{\rm cr}) \to 0$. Using similar techniques as applied in Appendix B, the high-temperature limit of Eq. (81) can be determined. The result for $T \gg m$ and $B \gg B_{\rm cr}$ is

$$\Delta Q(T,B) = \frac{1}{3} \frac{\alpha}{\pi} \frac{1}{B^2} \frac{B}{B_{\rm cr}} \left[-\frac{1}{2} + \frac{1}{2} k_2 \frac{m^2}{T^2} + \mathcal{O}\left(\frac{m^4}{T^4} \right) \right] = -\frac{1}{6} \frac{\alpha}{\pi} \frac{1}{B^2} \frac{B}{B_{\rm cr}} + \frac{1}{6} \frac{\alpha}{\pi} k_2 \frac{e}{BT^2} + \mathcal{O}\left(\frac{m^2}{BT^4} \right), \tag{82}$$

where $k_2 = 0.213\,139\,199\,408\ldots = \text{const}$ [see Eq. (B14)]. To obtain the complete effective action charge Q, we add the strong-field contribution $Q(T=0)$ which was found in Eq. (50):

$$Q(T \gg m, B \gg B_{\rm cr}) = \frac{1}{6} \frac{\alpha}{\pi} k_2 \frac{e}{BT^2} + \mathcal{O}\left(\frac{m^2}{BT^4} \right) = \frac{2}{3} k_2 \frac{\alpha^2}{m^4} \frac{1}{\tilde{B}\tilde{T}^2} + \mathcal{O}\left(\frac{m^2}{BT^4} \right), \tag{83}$$

where we have introduced the convenient dimensionless variables $\tilde{B} = B/B_{\rm cr} = eB/m^2$ and $\tilde{T} = T/m$ which satisfy $\tilde{B}, \tilde{T} \gg 1$. Equation (83) describes the same features of the effective action charge which we have encountered in previous examples: Q is centered upon the origin in field space, decreases proportionally to $1/\tilde{B}$ and evaporates with increasing temperature.

To calculate the velocity shift, we need the energy density which consists of three parts:

$$\langle T^{00}\rangle = \underbrace{\langle T^{00}\rangle_T^{B=0}}_{\text{eq.(75)}} + \underbrace{\langle T^{00}\rangle_{T=0}^{B}}_{\frac{1}{2}B^2} + \Delta\langle T^{00}\rangle_T^B \tag{84}$$

The last term of Eq. (84) is connected with the Lagrangian via the free energy (density) according to

$$\Delta\langle T^{00}\rangle_T^B = F + TS = F - T\frac{\partial F}{\partial T} = -\mathcal{L}_T^B + T\frac{\partial \mathcal{L}_T^B}{\partial T}. \tag{85}$$

The leading mixed contribution $\mathcal{L}_T^B$ to $\mathcal{L}$ is found in Ref. [22]:

$$\mathcal{L}_T^B = \frac{eB}{12} T^2 \rightsquigarrow \Delta\langle T^{00}\rangle_T^B = \frac{eB}{12} T^2. \tag{86}$$

We finally arrive at the polarization and propagation direction averaged velocity shift for strong fields and high temperature:

$$v = 1 - \frac{11\pi^2}{135} k_2 \alpha^2 \frac{\tilde{T}^2}{\tilde{B}} - \frac{k_2}{18\pi}\alpha \frac{\tilde{B}}{\tilde{T}^2} - \frac{k_2}{27}\alpha^2$$
$$= 1 - 9.13\ldots\times 10^{-6}\frac{\tilde{T}^2}{\tilde{B}} - 2.75\ldots\times 10^{-5}\frac{\tilde{B}}{\tilde{T}^2} - 4.21\ldots \times 10^{-7}. \tag{87}$$

At $\tilde{T}^2/\tilde{B} = 1.74\ldots$, we find a minimal velocity shift:

$$|\delta v| \simeq 3.20\times 10^{-5}. \tag{88}$$

At the same time, this number approximately sets the scale of a typical velocity shift for strong fields consistent with the one-loop approximation. This is also confirmed by the result of Eq. (54).

V. CONCLUSIONS

In this work, we studied light propagation in non-trivial QED vacua in the geometric optics approximation. For any given QED effective action describing soft photons, we derived the light cone condition averaged over polarization states. This result generalizes the ''unified formula'' found by Latorre, Pascual and Tarrach [8] which turned out to be the low-energy limit of our light cone condition.

We furthermore clarified the connection between light velocity shifts and the scale anomaly suggested by Shore [10]. Unfortunately, our findings do not indicate an immediate connection hinting at deeper physical grounds.

Instead, the structure of the light cone condition suggests introducing the intuitive physical picture of an effective action charge Q showing a localized profile in field space centered upon the origin. This charge directly characterizes the properties of the modified vacuum which are responsible for velocity shifts.

Within this conceptual framework, we analyzed several modified QED vacua and calculated the respective modified velocities. The inverse velocities are equal to the refractive indices of the modified vacua in the low-frequency limit. Hence, these velocities are phase as well as group velocities—the latter due to their independence of frequency. In the low-energy limit, we recovered all known results which were already perfectly described by the ''unified formula.''

For arbitrary magnetic fields, we reproduced the findings of Tsai and Erber [9] using our comparably simple formalism.

In the sequel, we calculated the next-to-leading order corrections to the Scharnhorst effect.

Finally, we investigated the influence of temperature on the velocity shifts. The evaporation of the effective action charge turns out to be the dominating effect in the high-temperature domain. It causes the velocity shift to approach a constant value. Only when a strong magnetic field is involved does the light cone condition fail to provide for a bound of the velocity shift in our examples. But we expect higher-order loop corrections to stop an unbounded growth of the velocity shift by inducing a faster decrease of the effective action charge far from the origin in field space.

Referring to the light cone condition, the sign of the velocity shift is in general determined by the sign of the effective action charge *and* the vacuum energy density. However, up to now, we have not been able to construct an example which exhibits a negative effective action charge in QED. This might be a general characteristic of the Abelian theory. Indeed, the one-loop effective action of a covariant constant chromomagnetic background field [25] (naively) possesses a negative effective action charge.

We would like to conclude with the remarkable observation that parity violating terms in the effective action proportional to $y^{2n+1}, n=0,1,2\ldots$, will not contribute to the effective action charge in the zero field limit, since the equation for Q is of Poisson type. Thus, e.g., the existence of dyons [26] will not cause a velocity shift in the weak field limit.

APPENDIX A

Our aim is to evaluate the integral of Eq. (48):

$$I(h) = \int_0^{i\infty} \frac{dz}{z}\, e^{-2hz}\left[\frac{z\coth z - 1}{\sinh^2 z} - \frac{1}{3} z\coth z\right]. \tag{A1}$$

For this, we have to decompose it into simple parts which one can handle by standard methods of dimensional regularization. Note that the integral is convergent, since the prescription $h \to h - i\epsilon$ is implicitly understood.

We begin with an integration by parts of the first term in square brackets with respect to the $\sinh^2$ in the denominator. This leads to

$$I(h) = \int_0^{i\infty} dz\, e^{-2hz}\left[\frac{h}{z}\coth z - h\coth^2 z + \frac{1}{2z^2}\coth z - \frac{1}{2z}\frac{1}{\sinh^2 z} - \frac{1}{3}\coth z\right] - \frac{1}{6}. \tag{A2}$$

The last three terms of the expression in square brackets are already in a convenient shape. In the following, we thus consider only the remaining first two terms. The strategy is similar: we extract a term proportional to $1/\sinh^2 z$ and integrate by parts:

$$I_1(h):=h\int_0^{i\infty}dz\,e^{-2hz}\left[\coth a\left(\frac{1}{z}-\coth z\right)\right]$$
$$=h\int_0^{i\infty}dz\,e^{-2hz}\left[\frac{\cosh z\sinh z}{z}-\cosh^2 z\right]\frac{1}{\sinh^2 z}$$
$$=h\int_0^{i\infty}dz\,e^{-2hz}\left[\left(2h+\frac{1}{z}\right)\coth z+\left(h+\frac{1}{z}\right)\sinh 2z\right.$$
$$\left.-\left(\frac{h}{z}+\frac{1}{2z^2}+1\right)\cosh 2z-\left(\frac{h}{z}+\frac{1}{2z^2}+1\right)\right]. \tag{A3}$$

Inserting I_1 into Eq. (A2), we obtain the wanted types of integrals. Each of these can be integrated separately by introducing an extra factor of z^ϵ and rotating the contour onto the positive real axis. At the end, the $1/\epsilon$-poles cancel and we arrive at the result given in Eq. (53) in the limit $\epsilon\to 0$.

APPENDIX B

In this appendix, we want to expand the infinite sum in Eq. (66) for small values of $\lambda:=m/T$ which corresponds to a high temperature limit:

$$S(\lambda):=\sum_{n=1}^{\infty}(-1)^n(\lambda n)^2K_2(\lambda n). \tag{B1}$$

Since the appearance of Bessel functions reflects the $R^3\times S^1$ topology which is the finite-temperature field theory space, the techniques described in the following are certainly useful for further finite-temperature applications.

The first step is to choose a representation of the modified Bessel function that shows a simple dependence on the summation index [23]:

$$K_2(\lambda n)=\int_0^\infty e^{-\lambda n\cosh t}\cosh 2t\,dt. \tag{B2}$$

Inserting Eq. (B2) into Eq. (B1) leads us to

$$S(\lambda)=\lambda^2\int_0^\infty dt\,\cosh 2t\sum_{n=1}^{\infty}n^2\,e^{-(i\pi+\lambda\cosh t)n}. \tag{B3}$$

By differentiating the geometric series $\Sigma_{n=0}^\infty q^n=1/(1-q)$ twice with respect to q, we find the result for the sum in Eq. (B3):

$$\sum_{n=0}^{\infty}n^2q^n=\frac{q(1+q)}{(1-q)^3}. \tag{B4}$$

Inserting Eq. (B4) into Eq. (B3) and decomposing the $\cosh 2t$ into $2\cosh^2 t-1$, we get

$$S(\lambda)=\lambda^2\int_\lambda^\infty\frac{dp}{\sqrt{p^2-\lambda^2}}\,e^{-p}\frac{(1-e^{-p})}{(1+e^{-p})^3}$$
$$-2\int_\lambda^\infty\frac{p^2\,dp}{\sqrt{p^2-\lambda^2}}\,e^{-p}\frac{(1-e^{-p})}{(1+e^{-p})^3}$$
$$=-2\int_\lambda^\infty dp\,\sqrt{p^2-\lambda^2}\,e^{-p}\frac{(1-e^{-p})}{(1+e^{-p})^3}$$
$$-\lambda^2\int_\lambda^\infty\frac{dp}{\sqrt{p^2-\lambda^2}}\,e^{-p}\frac{(1-e^{-p})}{(1+e^{-p})^3}$$
$$=:J_1(\lambda)+\lambda^2J_2(\lambda), \tag{B5}$$

where we have substituted $p:=\lambda\cosh t$. With some care, the parameter integrals J_1 and J_2 can now be expanded. We have to pay special attention to the process of taking the limit $\lambda\to 0$ for the J's and their derivatives. We can circumvent possible convergence problems at the lower bound by a repeated integration by parts of the square root terms. Using the short form

$$(\%):=\frac{1}{p}\,e^{-p}\frac{(1-e^{-p})}{(1+e^{-p})^3}, \tag{B6}$$

the non-vanishing coefficients of the expansion up to order $\mathcal{O}(\lambda^5)$ can be expressed as

$$J_1(0)=2\int_0^\infty dp\;p^2\;(\%)=1, \tag{B7}$$

$$J_1''(0)=2\int_0^\infty dp\;p\frac{d}{dp}(\%)=-2\int_0^\infty dp\;(\%), \tag{B8}$$

$$J_1''''(0)=6\int_0^\infty dp\;p\frac{d}{dp}\left(\frac{1}{p}\frac{d}{dp}(\%)\right), \tag{B9}$$

$$J_2(0)=\int_0^\infty dp\;(\%)=-2J_1''(0), \tag{B10}$$

$$J_2''(0)=-\int_0^\infty dp\;p\frac{d}{dp}\left(\frac{1}{p}\frac{d}{dp}(\%)\right)$$
$$=-\frac{1}{6}J_1''''(0). \tag{B11}$$

Finally, the Taylor expansion of Eq. (B1) reads

$$S(\lambda)=J_1(0)+\left(\frac{1}{2}J_1''(0)+J_2(0)\right)\lambda^2+\left(\frac{1}{24}J_1''''(0)+\frac{1}{2}J_2''(0)\right)\lambda^4+\mathcal{O}(\lambda^6)=1-\frac{k_1}{4}\lambda^4+\mathcal{O}(\lambda^6), \tag{B12}$$

where the constant k_1 is defined by

$$k_1:=-J_2''(0)\equiv\frac{1}{6}J_1''''(0)=\int_0^\infty dp\;p\,\frac{d}{dp}\left(\frac{1}{p}\,\frac{d}{dp}(\%)\right)=0.123\,749\,077\,479\dots\;. \tag{B13}$$

The constant k_2 that appears in the calculation of the effective action charge for high temperature and strong fields is obtained by similar techniques. Its integral representation is (accidentally) equal to $J_2(0)$:

$$k_2:=J_2(0)=\int_0^\infty\frac{dp}{p}\mathrm{e}^{-p}\,\frac{(1-\mathrm{e}^{-p})}{(1+\mathrm{e}^{-p})^3}=0.213\,139\,199\,408\dots\;. \tag{B14}$$

[1] S.L. Adler, Ann. Phys. (N.Y.) **67**, 599 (1971).
[2] E. Brezin and C. Itzykson, Phys. Rev. D **3**, 618 (1971).
[3] I.T. Drummond and S.J. Hathrell, Phys. Rev. D **22**, 343 (1980).
[4] K. Scharnhorst, Phys. Lett. B **236**, 354 (1990).
[5] G. Barton, Phys. Lett. B **237**, 559 (1990).
[6] R.D. Daniels and G.M. Shore, Nucl. Phys. **B425**, 634 (1994).
[7] R.D. Daniels and G.M. Shore, Phys. Lett. B **367**, 75 (1996).
[8] J.L. Latorre, P. Pascual, and R. Tarrach, Nucl. Phys. **B437**, 60 (1995).
[9] Wu-yang Tsai and T. Erber, Phys. Rev. D **12**, 1132 (1975).
[10] G.M. Shore, Nucl. Phys. **B460**, 379 (1996).
[11] G. Barton and K. Scharnhorst, J. Phys. A **26**, 2037 (1993).
[12] W. Dittrich and M. Reuter, *Effective Lagrangians in Quantum Electrodynamics*, Lecture Notes in Physics Vol. 220 (Springer-Verlag, Berlin, 1985).
[13] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[14] D. Bakalov *et al.*, in *TAUP93*, Proceedings of the Third International Workshop on Theoretical and Phenomenological Aspects of Underground Physics, Assergi, Italy, edited by C. Arpesella *et al.* [Nucl. Phys. B (Proc. Suppl.) **35**, 180 (1994)].
[15] D. Bakalov *et al.*, INFN/AE-94/27, 1994 (unpublished).
[16] E. Iacopini and E. Zavattini, Phys. Lett. **85B**, 151 (1979).
[17] V.I. Ritus, Sov. Phys. JETP **42**, 774 (1975); in *Issues in Intense-Field Quantum Electrodynamics*, Proceedings of the Lebedev Physical Institute Vol. 168, edited by V. L. Ginzburg (Nova Science Publishers, New York, 1987); for a modern treatment employing the string-inspired world line formalism, cf. M. Reuter, M.G. Schmidt and C. Schubert, Ann. Phys. (N.Y.) **259**, 313 (1997); B. Körs and M.G. Schmidt, hep-th/9803144, 1988 (unpublished).
[18] W. Dittrich and H. Gies (unpublished); W. Dittrich, *QED in Constant External Magnetic Fields*, Lecture Notes (University of Tübingen, Tübingen, 1997).
[19] L. Bendersky, Acta Math. **61**, 263 (1933).
[20] L.S. Brown and G.J. Maclay, Phys. Rev. **184**, 1272 (1969).
[21] X. Kong and F. Ravndal, Phys. Rev. Lett. **79**, 545 (1997); hep-ph/9803216, 1998 (unpublished).
[22] W. Dittrich, Phys. Rev. D **19**, 2385 (1979).
[23] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).
[24] D. Bailin and A. Love, *Introduction to Gauge Field Theory* (IOP Publishing, London, 1993).
[25] G.K. Savvidy, Phys. Lett. **71B**, 133 (1977); S.G. Matinyan and G.K. Savvidy, Nucl. Phys. **B134**, 539 (1978).
[26] S.G. Kovalevich, P. Osland, Ya.M. Shnir and E.A. Tolkachev, Phys. Rev. D **55**, 5807 (1997).