Features of time-independent Wigner functions

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The Wigner phase-space distribution function provides the basis for Moyal's deformation quantization alternative to the more conventional Hilbert space and path integral quantizations. The general features of time-independent Wigner functions are explored here, including the functional (''star'') eigenvalue equations they satisfy; their projective orthogonality spectral properties; their Darboux (''supersymmetric'') isospectral potential recursions; and their canonical transformations. These features are illustrated explicitly through simple solvable potentials: the harmonic oscillator, the linear potential, the Pöschl-Teller potential, and the Liouville potential. [S0556-2821(98)00714-0]

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I. INTRODUCTION

Wigner functions have been receiving increasing attention in quantum optics, dynamical systems, and the algebraic structures of M theory [1]. They were invented by Wigner and Szilard [2], and serve as a phase-space distribution alternative to the density matrix, to whose matrix elements they are related by Fourier transformation. The diagonal, hence, real, time-independent pure-state Wigner function $f(x,p)$ corresponding to the eigenfunction ψ of $\mathbf{H}\psi = E\psi$, is

$$f(x,p)=\frac{1}{2\pi}\int dy\,\psi^*\left(x-\frac{\hbar}{2}y\right)e^{-iyp}\psi\left(x+\frac{\hbar}{2}y\right). \qquad (1)$$

These functions are not quite probability distribution functions, as they are not necessarily positive—this is illustrated below. However, upon integration over p or x, they yield bona fide positive probability distributions, in x or p, respectively.

Wigner functions underlie Moyal's formulation of quantum mechanics [3], through the unique [4,5] one-parameter ($\hbar$) associative deformation of the Poisson-brackets structure of classical mechanics. Expectation values can be computed on the basis of phase-space c-number functions: given an operator $\mathbf{A}(\mathbf{x},\mathbf{p})$, the corresponding phase-space function $A(x,p)$ obtained by $\mathbf{p}\mapsto p$, $\mathbf{x}\mapsto x$ yields that operator's expectation value through

$$\langle \mathbf{A}\rangle = \int dx dp f(x,p)A(x,p), \qquad (2)$$

assuming the usual normalization $\int dx dp f(x,p)=1$ and further assuming Weyl ordering, as addressed by Moyal, who took matrix elements of all such operators:

$$\mathbf{A}(\mathbf{x},\mathbf{p})=\frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp A(x,p) \times \exp[i\tau(\mathbf{p}-p)+i\sigma(\mathbf{x}-x)]. \qquad (3)$$

Wigner functions are c numbers, but they compose with each other nonlocally. The properties of these compositions were explored in, e.g., [6,7], and were codified in an elegant system in [5]: to parallel operator multiplication, the Wigner functions compose with each other through the *associative* star product

$$\star \equiv e^{(i\hbar/2)(\overleftarrow{\partial}_x\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_x)}. \qquad (4)$$

Recalling the action of a translation operator $\exp(a\partial_x)h(x) = h(x+a)$, it is evident that the $\star$ product induces simple ''Bopp'' shifts:

$$f(x,p)\star g(x,p)=f\left(x,p-\frac{i\hbar}{2}\overrightarrow{\partial}_x\right)g\left(x,p+\frac{i\hbar}{2}\overleftarrow{\partial}_x\right) = f\left(x+\frac{i\hbar}{2}\overrightarrow{\partial}_p,p-\frac{i\hbar}{2}\overrightarrow{\partial}_x\right)g(x,p), \qquad (5)$$

etc., where $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ here act on the arguments of f and g, respectively. This intricate convolution samples the Wigner function over the entire phase space, and thus provides an alternative to operator multiplication in Hilbert space.

Antisymmetrizing and symmetrizing the star product, yields the Moyal (sine) brackets [3]

$$\{\{f,g\}\}\equiv\frac{f\star g-g\star f}{2i} \qquad (6)$$

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and Baker's [6] cosine brackets

$$((f,g)) \equiv \frac{f\star g+g\star f}{2}, \tag{7}$$

respectively. Note [7,8] that

$$\int dp\,dx\, f\star g=\int dp\,dx\, fg. \tag{8}$$

Further note the Wigner distribution has a $\star$-factorizable integrand:

$$f(x,-2p)=\frac{1}{2\pi}\int dy[\psi^*(x)e^{iyp}]\star[\psi(x)e^{iyp}]. \tag{9}$$

In general, a systematic specification of time-dependent Wigner functions is predicated on the eigenvalue spectrum of the time-independent problem. For pure-state static distributions, Wigner and, more explicitly, Moyal showed that

$$\{\{H(x,p),f(x,p)\}\}=0; \tag{10}$$

i.e., H and f $\star$ commute. However, there is a more powerful functional equation, the ''star-genvalue'' equation, which holds for the time-independent pure-state Wigner functions (lemma 1), and amounts to a complete characterization of them (lemma 2).

We will explore the features of this $\star$-genvalue equation, and illustrate its utility on a number of solvable potentials, including both the harmonic oscillator and the linear one. The $\star$ multiplications of Wigner functions will be seen to parallel Hilbert-space operations in marked detail. The Pöschl-Teller potential will reveal how the hierarchy of factorizable Hamiltonians familiar from supersymmetric quantum mechanics finds its full analogue in $\star$ space. We determine the Wigner function's transformation properties under (phase-space volume-preserving) canonical transformations, which we finally elaborate in the context of the Liouville potential.

II. $\star$-GENVALUE EQUATION

Lemma 1. Static, pure-state Wigner functions obey the $\star$-genvalue equation

$$H(x,p)\star f(x,p)=Ef(x,p). \tag{11}$$

Without essential loss of generality, consider $H(x,p)=p^2/2m+V(x)$,

$$\begin{aligned}
H(x,p)\star f(x,p)&=\frac{1}{2\pi}\left[\left(p-i\frac{\hbar}{2}\vec{\partial}_x\right)^2\Big/2m+V(x)\right]\int dy\,e^{-iy[p+i(\hbar/2)\overleftarrow{\partial}_x]}\psi^*\left(x-\frac{\hbar}{2}y\right)\psi\left(x+\frac{\hbar}{2}y\right)\\
&=\frac{1}{2\pi}\int dy\left[\left(p-i\frac{\hbar}{2}\vec{\partial}_x\right)^2\Big/2m+V\left(x+\frac{\hbar}{2}y\right)\right]e^{-iyp}\psi^*\left(x-\frac{\hbar}{2}y\right)\psi\left(x+\frac{\hbar}{2}y\right)\\
&=\frac{1}{2\pi}\int dy\,e^{-iyp}\left[\left(i\vec{\partial}_y+i\frac{\hbar}{2}\vec{\partial}_x\right)^2\Big/2m+V\left(x+\frac{\hbar}{2}y\right)\right]\psi^*\left(x-\frac{\hbar}{2}y\right)\psi\left(x+\frac{\hbar}{2}y\right)\\
&=\frac{1}{2\pi}\int dy\,e^{-iyp}\psi^*\left(x-\frac{\hbar}{2}y\right)E\psi\left(x+\frac{\hbar}{2}y\right)=Ef(x,p),
\end{aligned} \tag{12}$$

since the action of the effective differential operators on ψ^* turns out to be null, and, likewise,

$$\begin{aligned}
f\star H&=\frac{1}{2\pi}\int dy\,e^{-iyp}\left[-\left(\vec{\partial}_y-\frac{\hbar}{2}\vec{\partial}_x\right)^2\Big/2m+V\left(x-\frac{\hbar}{2}y\right)\right]\psi^*\left(x-\frac{\hbar}{2}y\right)\psi\left(x+\frac{\hbar}{2}y\right)\\
&=Ef(x,p).
\end{aligned} \tag{13}$$

Thus, both of the above relations (10) and lemma 1 obtain. ■

This time-independent equation was introduced in Ref. [7], such that the expectation of the energy $H(x,p)$ in a pure state time-independent Wigner function $f(x,p)$ is given by

$$\int H(x,p)f(x,p)dx\,dp=E\int f(x,p)dx\,dp. \tag{14}$$

On account of the integration property of the star product, Eq. (8), the left-hand side of this amounts to $\int dx\,dp\,H(x,p)\star f(x,p)$. Implicitly, this equation could have been inferred from the Bloch equation of the temperature- and time-dependent Wigner function, in the early work of [9]. $\star$-genvalue equations are discussed in some depth in the second reference of Ref. [5] and in [10].

By virtue of this equation, Fairlie also derived the general $\star$-orthogonality and spectral projection properties of static Wigner functions [7]. His results were later formalized in the spectral theory of the second of Ref. [5] [e.g., Eq. (4.4)]. Consider g corresponding to the (normalized) eigenfunction ψ_g corresponding to energy E_g. By lemma 1 and the associativity of the $\star$ product,

$$f\star H\star g=E_f f\star g=E_g f\star g. \tag{15}$$

Then, if $E_g\neq E_f$, this is only satisfied by

$$f\star g=0. \tag{16}$$

N.B. The integrated version is familiar from Wigner's paper,

$$\int dxdp\, f\star g=\int dxdp\, fg=0, \tag{17}$$

and demonstrates that all overlapping Wigner functions cannot be everywhere positive. The unintegrated relation introduced by Fairlie appears local, but is, of course, highly nonlocal, by virtue of the convolving action of the $\star$ product.

Precluding degeneracy, for $f=g$,

$$f\star H\star f=E_f f\star f=H\star f\star f, \tag{18}$$

which leads, by virtue of associativity, to the normalization relation [6]

$$f\star f\propto f. \tag{19}$$

Both relations (16) and (19) can be checked directly:

$$\begin{aligned}
f(x,p)\star g(x,p)&=f\left(x,p-\frac{i\hbar}{2}\vec{\partial}_x\right)g\left(x,p+\frac{i\hbar}{2}\overleftarrow{\partial}_x\right)\\
&=\frac{1}{(2\pi)^2}\int dy\,\psi_f^*\left(x-\frac{\hbar}{2}y\right)\psi_f\left(x+\frac{\hbar}{2}y\right)e^{-iy[p-(i\hbar/2)\vec{\partial}_x]}\int dY e^{-iY[p+(i\hbar/2)\overleftarrow{\partial}_x]}\psi_g^*\left(x-\frac{\hbar}{2}Y\right)\psi_g\left(x+\frac{\hbar}{2}Y\right)\\
&=\frac{1}{(2\pi)^2}\int dydY e^{-i(y+Y)p}\psi_f^*\left(x-\frac{\hbar}{2}y+\frac{\hbar}{2}Y\right)\psi_f\left(x+\frac{\hbar}{2}y+\frac{\hbar}{2}Y\right)\psi_g^*\left(x-\frac{\hbar}{2}Y-\frac{\hbar}{2}y\right)\psi_g\left(x+\frac{\hbar}{2}Y-\frac{\hbar}{2}y\right)\\
&=\left[\frac{1}{2\pi}\int d(Y+y)e^{-i(y+Y)p}\psi_g^*\left(x-\frac{\hbar}{2}(Y+y)\right)\psi_f\left(x+\frac{\hbar}{2}(y+Y)\right)\right]\\
&\quad\times\left[\frac{1}{h}\int d\left(\frac{\hbar(Y-y)}{2}\right)\psi_f^*\left(\frac{\hbar}{2}(Y-y)\right)\psi_g\left(\frac{\hbar}{2}(Y-y)\right)\right].
\end{aligned} \tag{20}$$

The second integral factor is 0 or $1/h$, depending on $f\neq g$ or $f=g$, respectively, specifying the normalization $f*f=f/h$ in Eq. (19). In conclusion,

Corollary 1. $f_a\star f_b=1/h\,\delta_{a,b}f_a$.

These spectral properties are summoned up by their own necessity; much of their meaning, nevertheless, resides in their margins: For nonnormalizable wave functions, the above second integral factor may diverge, as illustrated below for the linear potential, but the orthogonality properties still hold.

Thus, e.g., for an arbitrary function(al) $F(z)$,

$$F[f\star]f=F(1/h)f, \tag{21}$$

and, for $\star$ genfunctions of lemma 1,

$$F[H\star]f=F(E)f. \tag{22}$$

Baker's converse construction extends to a full converse of lemma 1, namely, the following lemma.

Lemma 2. Real solutions of $H(x,p)\star f(x,p)=Ef(x,p)$ $[=f(x,p)\star H(x,p)]$ *must be of the Wigner form,* $f=\int dy e^{-iyp}\psi^*[x-(\hbar/2)y]\psi[x+(\hbar/2)y]/2\pi$, *such that* $\mathbf{H}\psi=E\psi$.

As seen above, the pair of $\star$-eigenvalue equations dictate, for $f(x,p)=\int dy e^{-iyp}\tilde{f}(x,y)$,

$$\int dy e^{-iyp}\left[-\frac{1}{2m}\left(\vec{\partial}_y\pm\frac{\hbar}{2}\vec{\partial}_x\right)^2+V\left(x\pm\frac{\hbar}{2}y\right)-E\right]\tilde{f}(x,y)=0. \tag{23}$$

This constrains $\tilde{f}(x,y)$ to consist of bilinears $\psi^*[x-(\hbar/2)y]\psi[x+(\hbar/2)y]$ of unnormalized eigenfunctions $\psi(x)$ corresponding to the same eigenvalue E in the Schrödinger equation with potential V. ■

These two lemmata then amount to the statement that, *for real functions* $f(x,p)$, *the Wigner form is equivalent to compliance with the* $\star$*-genvalue equation* (real and imaginary part).

III. EXAMPLE: THE SIMPLE HARMONIC OSCILLATOR

The eigenvalue equation of lemma 1 may be solved directly to produce the Wigner functions for specific potentials, without first solving the corresponding Schrödinger problem (as in, e.g., [11]). Following [7], for the harmonic oscillator, $H=(p^2+x^2)/2$ (with $\hbar=1$, $m=1$), the resulting equation is

$$\left[\left(x+\frac{i}{2}\partial_p\right)^2+\left(p-\frac{i}{2}\partial_x\right)^2-2E\right]f(x,p)=0. \tag{24}$$

By virtue of its imaginary part $(x\partial_p - p\partial_x)f=0$, f is seen to depend on only one variable, $z=4H=2(x^2+p^2)$, and so the equation reduces to a simple ordinary differential equation;

$$\left(\frac{z}{4} - z\partial_z^2 - \partial_z - E\right)f(z)=0. \tag{25}$$

Moreover, setting $f(z)=\exp(-z/2)L(z)$, this yields

$$\left(z\partial_z^2 + (1-z)\partial_z + E - \frac{1}{2}\right)L(z)=0, \tag{26}$$

which is the equation satisfied by Laguerre polynomials $L_n = e^z\partial^n(e^{-z}z^n)$, for $n=E-1/2=0,1,2,\ldots$, so that the unnormalized Wigner eigenfunctions are

$$f_n = e^{-2H}L_n(4H),$$
$$L_0=1, \quad L_1=1-4H, \quad L_2=16H^2-16H+2,\ldots. \tag{27}$$

Note that the eigenfunctions are not positive definite, and are the only ones satisfying the boundary conditions, $f(0)$ finite and $f(z)\to 0$, as $z\to\infty$.

In fact, Dirac's Hamiltonian factorization method for algebraic solution carries through (cf. [5]) intact in $\star$ space. Indeed,

$$H=\frac{1}{2}(x-ip)\star(x+ip)+\frac{1}{2}, \tag{28}$$

motivating the definition of

$$a \equiv \frac{1}{\sqrt{2}}(x+ip), \quad a^\dagger \equiv \frac{1}{\sqrt{2}}(x-ip). \tag{29}$$

Thus, noting that

$$a\star a^\dagger - a^\dagger\star a = 1 \tag{30}$$

and also that, by the above,

$$a\star f_0 = \frac{1}{\sqrt{2}}(x+ip)\star e^{-(x^2+p^2)}=0 \tag{31}$$

provides a $\star$-Fock vacuum, it is evident that associativity of the $\star$ product permits the entire ladder spectrum generation to go through as usual. The $\star$ genstates of the Hamiltonian, such that $H\star f = f\star H$, are thus

$$f_n \propto (a^\dagger\star)^n f_0 (\star a)^n. \tag{32}$$

These states are real, like the Gaussian ground state, and are thus left-right symmetric $\star$ genstates. They are also transparently $\star$ orthogonal for different eigenvalues, and they project to themselves, as they should, since the Gaussian ground state does, $f_0\star f_0 \propto f_0$. It will be seen below that even the generalization of this factorization method for isospectral potential pairs goes through without difficulty.

IV. FURTHER EXAMPLE: THE LINEAR POTENTIAL

For simplicity, take $m=1/2$, $\hbar=1$. Recall [12] that the problem readily reduces to a free particle: $H(x,p)=p^2+x \mapsto H_{free}=P$ is accomplished by canonically transforming through the generating function $F(x,X)=-\frac{1}{3}X^3 - xX$. The energy eigenfunctions are Airy functions,

$$\psi_E(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} dX e^{iF(x,X)}e^{iEX} = \mathrm{Ai}(x-E). \tag{33}$$

The $\star$-genvalue equation in this case is

$$\left[\left(x+\frac{i}{2}\partial_p\right)+\left(p-\frac{i}{2}\partial_x\right)^2 - E\right]f(x,p)=0, \tag{34}$$

whose imaginary part $(\frac{1}{2}\partial_p - p\partial_x)f(x,p)=0$ gives $f(x,p) = f(x+p^2)=f(H)$. The real part of the equation is then an ordinary second-order equation, just as in the above harmonic oscillator case. Moreover, here the real part of the $\star$-genvalue equation is essentially the same as the usual energy eigenvalue equation:

$$\left(z-\frac{1}{4}\partial_z^2 - E\right)f(z)=0, \tag{35}$$

where $z=x+p^2$. Hence, the Wigner function is again an Airy function, like the above wave functions, except that the argument has a different scale and shift:[1]

$$f(x,p)=\frac{2^{2/3}}{2\pi}\mathrm{Ai}(2^{2/3}(z-E))=\frac{2^{2/3}}{2\pi}\mathrm{Ai}(2^{2/3}(x+p^2-E))$$
$$=\frac{1}{(2\pi)^2}\int dy\, e^{iy(E-x-p^2-y^2/12)}. \tag{36}$$

The Airy functions are not square integrable, so that the conventional normalization $f\star f=(1/2\pi)f$ does not strictly apply. On the other hand, the energy eigenfunctions are nondegenerate, and the general corollary 1 projection relations $f_a\star f_b \propto \delta_{a,b}f_a$ *still* hold for the continuous spectrum:

[1]This case is similar to the Gaussian wave function, i.e., the harmonic oscillator ground state encountered above, whose Wigner function is also a Gaussian, but of different width. S. Habib kindly informed us that this solution is also given in Ref. [13], Eq. (29).

$$f_{E1}\star f_{E2}=f_{E1}\left[\left(x+\frac{i}{2}\vec{\partial}_p\right)+p^2\right]f_{E2}\left[\left(x-\frac{i}{2}\overleftarrow{\partial}_p\right)+p^2\right]=\frac{1}{(2\pi)^4}\int dy\,dY e^{iy[E1-x-(p-Y/2)^2-y^2/12]}e^{iY[E2-x-(p+y/2)^2-Y^2/12]}$$

$$=\frac{1}{(2\pi)^4}\int d(y+Y)e^{i(y+Y)[(E1+E2)/2-x-p^2-(y+Y)^2/12]}\int d\frac{(y-Y)}{2}e^{i[(y-Y)/2](E1-E2)}$$

$$=\frac{1}{(2\pi)}\delta(E1-E2)f_{(E1+E2)/2}(x+p^2), \tag{37}$$

by virtue of the direct definition (36).

V. DARBOUX CONSTRUCTION OF WIGNER FUNCTION RECURSIONS

Analogous ladder operators for eigenstates corresponding to ‘‘essentially isospectral’’ pairs of partner potentials [14] [familiar from supersymmetric quantum mechanics (SSQM)] can also be defined *mutatis mutandis* for Wigner functions and $\star$ products. They faithfully parallel the differential equation structures.

Consider a positive semidefinite Hamiltonian

$$H=p^2/2m+V(x). \tag{38}$$

This can be written as a $\star$ product of two operators,

$$H=Q^*\star Q=\left(\frac{p}{\sqrt{2m}}+iW(x)\right)\star\left(\frac{p}{\sqrt{2m}}-iW(x)\right), \tag{39}$$

provided

$$W^2-\frac{\hbar}{\sqrt{2m}}\partial_x W=V(x). \tag{40}$$

This Riccati equation, familiar from SSQM, can be Darboux transformed by changing variable for the ‘‘superpotential’’ $W(x)$,

$$W=-\frac{\hbar\partial_x\psi_0}{\sqrt{2m}\psi_0}, \tag{41}$$

which reduces the condition to the Schrödinger equation for a zero eigenvalue:

$$-\frac{\hbar^2}{2m}\partial_x^2\psi_0+V(x)\psi_0=0. \tag{42}$$

Also note $Q\star f_0=0$ for the corresponding Wigner function. It is easy to generalize this by adding a constant to H to shift the ground state eigenvalue from zero.

By virtue of associativity, it is evident that the partner Hamiltonian

$$H'=Q\star Q^*=H+\frac{2\hbar}{\sqrt{2m}}\partial_x W, \tag{43}$$

i.e., the one with a partner potential

$$V'=W^2+\frac{\hbar}{\sqrt{2m}}\partial_x W, \tag{44}$$

has Wigner function $\star$ genstates of the same energy as those of H. Specifically,

$$H\star f=Q^*\star Q\star f=f\star Q^*\star Q=Ef \tag{45}$$

implies that the real functions $Q\star f\star Q^*$ are $\star$ genfunctions of H' with the same eigenvalue E,

$$H'\star(Q\star f\star Q^*)=Q\star Q^*\star Q\star f\star Q^*=E(Q\star f\star Q^*), \tag{46}$$

unless f is the Wigner function corresponding to ψ_0, since $Q\star f_0=0$.

In consequence, $E'_n=E_{n+1}$ for $n\geqslant 0$. Conversely, for g $\star$ genfunctions of H', $Q^*\star g\star Q$ are $\star$ genfunctions of H with the same eigenvalues.

Moreover, $\psi'_0\equiv 1/\psi_0$ will be an invalid zero mode eigenfunction of $\mathbf{H}'$, as seen from the sign flip in Eqs. (41) and (44). Consequently, an unnormalized, runaway zero-energy solution of the Schrödinger equation with $V'(x)$ will invert to the legitimate ground state of $\mathbf{H}$ and will permit construction of V given V'.

For example, starting from the trivial potential with a continuous (unnormalizable) spectrum,

$$V'=1, \tag{47}$$

and the solution

$$\psi'_0=\cosh\left(\frac{\sqrt{2m}x}{\hbar}\right), \quad \Rightarrow W=\tanh\left(\frac{\sqrt{2m}x}{\hbar}\right), \tag{48}$$

results via Eq. (40) in the symmetric, reflectionless Pöschl-Teller potential [15], $V=1-2/\cosh^2[(\sqrt{2m}x/\hbar)]$. Conversely, starting from this potential,

$$V(x)=1-\frac{2}{\cosh^2\left(\frac{\sqrt{2m}x}{\hbar}\right)}, \tag{49}$$

there is a single bound state (normalizable to $\int\psi_0^2=2$),

$$\psi_0=\operatorname{sech}\left(\frac{\sqrt{2m}x}{\hbar}\right),\quad \Rightarrow W=\tanh\left(\frac{\sqrt{2m}x}{\hbar}\right),\tag{50}$$

so that

$$V'=1.\tag{51}$$

Thus, the Wigner function ground state (for $m=1/2$) is

$$f_0(x,p)=\frac{1}{2\pi}\int dy\,\frac{e^{-iyp}}{2\cosh(x/\hbar-y/2)\cosh(x/\hbar+y/2)}$$
$$=\frac{1}{\pi}\int_0^\infty dy\,\frac{\cos(yp)}{\cosh(2x/\hbar)+\cosh(y)}$$
$$=\frac{\sin(2xp/\hbar)}{\sinh(2x/\hbar)\sinh(\pi p)}.\tag{52}$$

[N.B. It is not positive definite or a function of just $H(x,p)$.] It may be verified directly that

$$Q\star f_0=\left[p-\frac{i\hbar}{2}\partial_x-i\,\tanh\left(\frac{x}{\hbar}+\frac{i}{2}\partial_p\right)\right]f_0(x,p)=0.\tag{53}$$

This appendage of bound states to a potential generalizes [16] to the hierarchy associated with the Korteweg–de Vries (KdV) equation. Specifically,

$$W(n)=n\,\tanh\left(\frac{\sqrt{2m}x}{\hbar}\right)\tag{54}$$

connects the reflectionless Pöschl-Teller potential

$$V'(x)=n^2-n(n-1)/\cosh^2\left(\frac{\sqrt{2m}x}{\hbar}\right)$$

to its contiguous

$$V(x)=n^2-n(n+1)/\cosh^2\left(\frac{\sqrt{2m}x}{\hbar}\right),\tag{55}$$

which has one more bound state (shape invariance). Recursively, then, one may go in N steps, with the suitable shifts of the potential by $2n-1$ in each step, from the constant potential to

$$V(N;x)=N^2-N(N+1)/\cosh^2\left(\frac{\sqrt{2m}x}{\hbar}\right).\tag{56}$$

Shifting this potential down by N^2 assigns the energy $E=-N^2$ to the corresponding ground state $\psi_0(N)=\operatorname{sech}^N(x)$ (unnormalized), which is the null state of $(\hbar/\sqrt{2m})\partial_x+W(N)$. The corresponding (unnormalized) Wigner function is the $\star$-null state of $Q(N)$,

$$f_0(N;x,p)=\frac{1}{\pi}\int_0^\infty dy\,\frac{\cos(yp)}{[\cosh(2x/\hbar)+\cosh(y)]^N}$$
$$=\frac{1}{(N-1)!}\left(\frac{-\hbar}{2\sinh(2x/\hbar)}\partial_x\right)^{N-1}f_0(1;x,p),\tag{57}$$

where the integral only need be evaluated from the above $f_0(1;x,p)$. Alternatively,

$$f_0(N;x,p)=[\operatorname{sech}(x/\hbar)\star]^{N-1}f_0(1;x,p)[\star\ \operatorname{sech}(x/\hbar)]^{N-1}.\tag{58}$$

The (unnormalized) state above the ground state at $E=-(N-1)^2$ is $[(\hbar/\sqrt{2m})\partial_x-W(N)]\psi_0(N-1)$, and its corresponding Wigner function (setting $m=1/2$) is found recursively from the ground state of $H(N-1)$, through $Q^*(N)\star f_0(N-1)\star Q(N)$,

$$\left[p\star f_0(N-1)+iN\,\tanh\left(\frac{x}{\hbar}\right)\star f_0(N-1)\right]\star Q(N)$$
$$=\left(p\star f_0(N-1)+\frac{N}{N-1}p\star f_0(N-1)\right)\star Q(N)$$
$$=\left(\frac{2N-1}{N-1}\right)^2 p\star f_0(N-1)\star p,\tag{59}$$

by virtue of

$$Q(N-1)\star f_0(N-1)=0=f_0(N-1)\star Q^*(N-1).\tag{60}$$

The state above that, at $E=-(N-2)^2$, is found recursively through

$$Q^*(N)\star Q^*(N-1)\star f_0(N-2)\star Q(N-1)\star Q(N),\tag{61}$$

and so forth. Thus, the entire Wigner $\star$-genfunction spectrum of $H(N)$ is obtained with hardly any reliance on Schrödinger eigenfunctions.

VI. CANONICAL TRANSFORMATION OF THE WIGNER FUNCTION

For notational simplicity, take $\hbar=1$ in this section. The area element in phase space is preserved by canonical transformations

$$(x,p)\mapsto(X(x,p),P(x,p))\tag{62}$$

which yield trivial Jacobians ($dXdP=dxdp\{X,P\}$) by preserving the Poisson brackets

$$\{u,v\}_{xp}\equiv\frac{\partial u}{\partial x}\frac{\partial v}{\partial p}-\frac{\partial u}{\partial p}\frac{\partial v}{\partial x}.\tag{63}$$

They thus preserve the "canonical invariants" of their functions:

$$\{X,P\}_{xp}=1\quad\text{and hence}\quad\{x,p\}_{XP}=1.\tag{64}$$

Equivalently,

$$\{x,p\}=\{X,P\}, \tag{65}$$

in any basis. Motion being a canonical transformation, Hamilton's classical equations of motion are preserved, for $\mathcal{H}(X,P)\equiv H(x,p)$, as well [17]. What happens upon quantization?

Since, in deformation quantization, the Hamiltonian is a c-number function, and so transforms ''classically,'' $\mathcal{H}(X,P)\equiv H(x,p)$, the effects of a canonical transformation on the quantum $\star$-genvalue equation of lemma 1 will be carried by a suitably transformed Wigner function. Predictably, the answer can be deduced from Dirac's quantum transformation theory. Consider the canonical transformations generated by $F(x,X)$:

$$p=\frac{\partial F(x,X)}{\partial x}, \quad P=-\frac{\partial F(x,X)}{\partial X}. \tag{66}$$

Following Dirac's celebrated exponentiation [18] of such a generator, in the implementation of [12,19], the energy eigenfunctions transform canonically through a generalization of the ''representation-changing'' Fourier transform:

$$\psi_E(x)=N_E\int dX e^{iF(x,X)}\Psi_E(X). \tag{67}$$

Thus,

$$f(x,p)=\frac{|N_E|^2}{2\pi}\int dy\int dX_1 e^{-iF^*(x-y/2,X_1)}\Psi_E^*(X_1)e^{-iyp}\int dX_2 e^{iF(x+y/2,X_2)}\Psi_E(X_2). \tag{68}$$

The pair of Wigner functions in the respective canonical variables, $f(x,p)$ and

$$\mathcal{F}(X,P)=\frac{1}{2\pi}\int dY\Psi^*\left(X-\frac{\hbar}{2}Y\right)e^{-iYP}\Psi\left(X+\frac{\hbar}{2}Y\right), \tag{69}$$

are connected by a transformation functional $\mathfrak{T}(x,p;X,P)$,

$$f(x,p)=\int dX\int dP\,\mathfrak{T}(x,p;X,P)\circledast\mathcal{F}(X,P)=\int dX\int dP\,\mathfrak{T}(x,p;X,P)\mathcal{F}(X,P), \tag{70}$$

where $\circledast$ is with respect to the variables X and P.

To find this functional, let $X=\frac{1}{2}(X_1+X_2)$ and $Y=X_2-X_1$, so that $\int dX_1\int dX_2=\int dX\int dY$. Noting that

$$\Psi^*\left(X-\frac{\hbar}{2}Y\right)\Psi\left(X+\frac{\hbar}{2}Y\right)=\int dP e^{iYP}\mathcal{F}(X,P), \tag{71}$$

it follows that Eq. (68) reduces to

$$\begin{aligned} f(x,p)&=\frac{|N|^2}{2\pi}\int dy\int dX_1 e^{-iF^*(x-y/2,X_1)}\Psi^*(X_1)e^{-iyp}\int dX_2 e^{iF(x+y/2,X_2)}\Psi(X_2)\\ &=\frac{|N|^2}{2\pi}\int dX dY dy e^{-iyp}e^{-iF^*(x-y/2,X-Y/2)}\Psi^*(X-Y/2)\Psi(X+Y/2)e^{iF(x+y/2,X+Y/2)}\\ &=\frac{|N|^2}{2\pi}\int dX dP dY dy e^{-iyp+iPY-iF^*(x-y/2,X-Y/2)+iF(x+y/2,X+Y/2)}\mathcal{F}(X,P), \end{aligned} \tag{72}$$

which leads to the following lemma.

Lemma 3. $\mathfrak{T}(x,p;X,P)=(|N|^2/2\pi)\int dY dy \exp[-iyp+iPY-iF^*(x-y/2,X-Y/2)+iF(x+y/2,X+Y/2)]$. ■

Corollary 2. This phase-space transformation functional obeys the ''two-star'' equation

$$H(x,p)\star\mathfrak{T}(x,p;X,P)=\mathfrak{T}(x,p;X,P)\circledast\mathcal{H}(X,P), \quad (73)$$

as follows from $H(x,-i\partial_x)\exp[iF(x,X)]=\mathcal{H}(X,i\partial_X)\exp[iF(x,X)]$. If $\mathcal{F}$ satisfies a $\circledast$-genvalue equation, then f satisfies a $\star$-genvalue equation with the same eigenvalue, and vice versa. ■

Note that, by virtue of the spectral projection feature (16), (19), this equation is also solved by any representation-changing equal-energy bilinear in real Wigner $\star$ genfunctions of H and $\mathcal{H}$,

$$\mathfrak{T}(x,p;X,P)=\sum_E g(E)f_E(x,p)\mathcal{F}_E(X,P), \quad (74)$$

for arbitrary real $g(E)$. Such a bilinear transformation functional is nonsingular (invertible) if and only if $g(E)$ has no zeros on the spectrum of either Hamiltonian.[2]

As an example, consider the linear potential again, which transforms into a free particle ($\mathcal{H}=P$) through

$$F=-\frac{1}{3}X^3-xX\Rightarrow p=-X, \quad x=P-X^2. \quad (75)$$

By direct computation,

$$\mathfrak{T}(x,p;X,P)=2^{2/3}\,\mathrm{Ai}(2^{2/3}(x+X^2-P))\delta(p+X)$$

$$=(2\pi)^2\int dE f_E(x,p)\mathcal{F}_E(X,P)\delta(p+X). \quad (76)$$

Note $N_E=1/\sqrt{2\pi}$ for the free-particle energy eigenfunction normalization choice $\Psi_E(X)=(2\pi)^{-1/2}\exp(iEX)$. Thus, indeed, the free-particle Wigner function $\mathcal{F}_E(X,P)=\delta(E-P)/(2\pi)$ transforms into

$$f(x,p)=\frac{1}{2\pi}\int dP dX \mathfrak{T}\delta(E-P)$$
$$=\frac{2^{2/3}}{2\pi}\mathrm{Ai}(2^{2/3}(x+p^2-E)), \quad (77)$$

as it should, and Eq. (73) is seen to be satisfied directly, by virtue of the linearity of the respective Hamiltonians in the variables P,x, conjugate to those of the arguments of $\delta(p+X)$.

The structure of the result in Eq. (76) underscores that the linear potential is as ''close to classical'' as one can get, in simple quantum mechanics. It has been noted before [12] that the transformation functional for linear potential wave functions is *exactly* the exponential of the classical generating function for the canonical transformation to a free particle, and that this is not the case for any other potential. The present result for the transformation functional for Wigner functions is further evidence for this ''close to classical'' behavior. The delta function $\delta(p+X)$ in Eq. (76) is *half* of the classical story. Were the Airy function also a delta function of its argument, we would have an exact implementation of the $X,P\mapsto x,p$ classical correspondence. As it is, there is some typically quantum mechanical spread around the classical constraint $x+X^2-P=0$, in the form of oscillations of the Airy function, and, in consequence, the Wigner functions of the free particle do not retain their delta-function form under the canonical transformation to the linear potential Wigner functions. Reinstating $\hbar$ into Eq. (36),[3] and taking the limit $\hbar\to 0$ converts the Airy function to a delta function, $\delta(x+X^2-P)$, thereupon producing the complete classical correspondence between the two sets of phase space variables, in that limit.

As already seen, there is substantial nonuniqueness in the choice of transformation functional. For example, for the linear potential again, Eq. (73),

$$(x+p^2)\star\mathfrak{S}(x,p;X,P)=\mathfrak{S}(x,p;X,P)\circledast P \quad (78)$$

is also satisfied by a different (and somewhat simpler) choice:

$$\mathfrak{S}(x,p;X,P)=\exp\{-i[\tfrac{2}{3}X^3+2(x+p^2-P)X]\}. \quad (79)$$

---

[2]In general, if the transformation functional effects a map to a free particle, the P integration is trivial in Eq. (70), and the result for the Wigner function of the x,p theory is just an average over X of the transformation functional. That is, if $\mathcal{F}(X,P)=\delta(P-k(E))$, where $k(E)$ is the momentum-energy relation for the free particle theory in question:

$$f(x,p)=\int dX\int dP\mathfrak{T}(x,p;X,P)\mathcal{F}(X,P)=\int dX\mathfrak{T}(x,p;X,k(E)).$$

One might then be tempted to wonder if just $\mathfrak{T}(x,p;X,P)=\psi_P^*(x-\hbar X/2)e^{-iXp}\psi_P(x+\hbar X/2)/2\pi\equiv\mathfrak{G}(x,p;X,P)$. However, what determines the allowed range for P? It is always possible to embed any real energy spectrum into the real line, but knowing this does not help at all to determine what points are to be embedded. From the point of view of this paper, even when the spectrum is obvious, such a choice for the transformation functional in general does not satisfy the two-$\star$ equation (73). Rather, the equation fails by total derivatives that vary contingent on particularities of the case. E.g., for free-particle plane waves, $\psi_E(x)=\exp(iEx)$, so that $p\star\mathfrak{G}-\mathfrak{G}\circledast P=\partial_X\mathfrak{G}$. This choice for $\mathfrak{T}$, then, does not yield useful information on the Wigner functions.

[3]The exponent of the integrand turns into $iy(E-x-p^2-\hbar^2y^2/12)$.

This transformation functional also converts the free-particle Wigner function $\mathcal{F}_E(X,P)=\delta(E-P)/2\pi$ into an Airy function (as above) after integrating over the free-particle phase space, $\int dX dP$.

Actually, it is not necessary to integrate over the phase space. In general, $\star$ multiplying a delta function spreads it out, and yields a Fourier transform with respect to the conjugate variable. Thus, for the example considered,

$$e^{i[(-2/3)X^3-2(x+p^2-P)X]}\star\delta(P-E)=e^{2iX(P-E)}\frac{1}{\pi}\int dZ e^{-2iZ(P-E)}e^{i[(-2/3)Z^3-2(x+p^2-P)Z]}$$
$$=e^{2iX(P-E)}\frac{1}{\pi}\int dZ e^{i[(-2/3)Z^3-2(x+p^2-E)Z]}=e^{2iX(P-E)}2^{2/3}\ \mathrm{Ai}(2^{2/3}(x+p^2-E)). \tag{80}$$

Hence,

$$\int dX\int dP e^{i[(-2/3)X^3-2(x+p^2-P)X]}\star\delta(P-E)=2^{2/3}\pi\ \mathrm{Ai}(2^{2/3}(x+p^2-E)). \tag{81}$$

Compare this to the action of the above $\mathfrak{T}(x,p;X,P)$,

$$[\mathrm{Ai}(2^{2/3}(x+X^2-P))\delta(p+X)]\star\delta(P-E)=e^{2iX(P-E)}\frac{1}{\pi}\int dZ e^{-2iZ(P-E)}\ \mathrm{Ai}(2^{2/3}(x+Z^2-P))\delta(p+Z)$$
$$=e^{2i(p+X)(P-E)}\frac{1}{\pi}\mathrm{Ai}(2^{2/3}(x+p^2-P)). \tag{82}$$

Aside from innocuous normalizations, the difference in the two transformation functionals acting on the free-particle Wigner function is just the phase factor $e^{2ip(P-E)}$ and the argument of the Airy function, where E has been replaced by P. Indeed, the phase factor precisely compensates for the different energy eigenvalue occurring in the argument of Ai, when acted upon by $(x+p^2)\star$. Such simple phase factors may be used to shift a $\star$ genvalue whenever the Hamiltonian is linear in any variable.

VII. ILLUSTRATIONS USING LIOUVILLE QUANTUM MECHANICS

A summary illustration of all the above, in particular the canonical transformation effects on Wigner functions, is provided by the Liouville model [20]. Our conventions for the model [which are essentially those of [21], with their $m\equiv 1/(4\pi)$ and their $g\equiv 1$] are given by

$$H_{Liouville}=p^2+e^{2x}. \tag{83}$$

The energy eigenfunctions are then solutions of

$$\left(-\frac{d^2}{dx^2}+e^{2x}\right)\psi_E(x)=E\psi_E(x). \tag{84}$$

The solutions are Kelvin (modified Bessel) K functions, for $0<E<\infty$,

$$\psi_E(x)=\frac{1}{\pi}\sqrt{\sinh(\pi\sqrt{E})}K_{i\sqrt{E}}(e^x), \tag{85}$$

which are normalized such that $\int_{-\infty}^{+\infty}dx\,\psi_{E_1}^*(x)\psi_{E_2}(x)=\delta(E_1-E_2)$. There is no solution [20] for $E=0$.

For completeness, consider the Fourier transform (including a convergence factor, necessary for $x\rightarrow-\infty$ to control plane wave behavior, but not for $x\rightarrow\infty$)

$$\Phi_E(p+i\epsilon)=\int_{-\infty}^{+\infty}dx e^{-ix(p+i\epsilon)}\psi_E(x)$$
$$=\frac{1}{4\pi}\sqrt{\sinh(\pi\sqrt{E})}2^{-i(p+i\epsilon)}$$
$$\times\Gamma\left(\frac{-i(p+i\epsilon)+i\sqrt{E}}{2}\right)\Gamma\left(\frac{-i(p+i\epsilon)-i\sqrt{E}}{2}\right). \tag{86}$$

This follows, e.g., from a result in [22], Vol. II, p 51, Eq. (27):

$$\int_0^{+\infty}dz z^{\mu}K_{\nu}(z)=2^{\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right), \tag{87}$$

valid for $\Re(1+\mu\pm\nu)>0$ (i.e., the previous transform is valid for $\epsilon>0$). The right-hand side of this last relation clearly displays the symmetry $\nu\rightarrow-\nu$, which just amounts to the physical statement that the energy eigenfunctions are nondegenerate for the transmissionless exponential potential of the Liouville model.

Further note the effect on $\Phi_E(p+i\epsilon)$ of shifting $p\to p+2i$, using $\Gamma(1+z)=z\Gamma(z)$,

$$\Phi_E(p+2i+i\epsilon)=4\left(\frac{-i(p+i\epsilon)+i\sqrt{E}}{2}\right)\times\left(\frac{-i(p+i\epsilon)-i\sqrt{E}}{2}\right)\Phi_E(p+i\epsilon)=[E-(p+i\epsilon)^2]\Phi_E(p+i\epsilon). \tag{88}$$

So, as $\epsilon\to0$, $\Phi_E(p+2i)=(E-p^2)\Phi_E(p)$. But this simple difference equation is just the Liouville energy eigenvalue equation in the momentum basis,

$$(p^2-E)\Phi_E(p)+e^{2i\partial_p}\Phi_E(p)=0. \tag{89}$$

Such first-order difference equations invariably lead to gamma functions [23]. Below, it turns out that the Wigner functions also satisfy momentum difference equations, but of second order.

Many, if not all, properties of the Liouville wave functions may be understood from the following integral representation [[24], Chap. VI, Sec. 6.22, Eq. (10)]. Explicitly emphasizing the abovementioned nondegeneracy,

$$K_{ik}(e^x)=K_{-ik}(e^x)=\frac{1}{2}e^{\pi k/2}\int_{-\infty}^{+\infty}dX e^{ie^x\sinh X}e^{ikX}. \tag{90}$$

(Also see [25], Eq. 9.6.22.) This integral representation may be effectively regarded as the canonical transformation of a free-particle energy eigenfunction e^{ikX} through use of the generating function $F(x,X)=e^x\sinh X$. Classically, $p=\partial F/\partial x=e^x\sinh X$ and $P=-\partial F/\partial X=-e^x\cosh X$, and so $P^2-p^2=e^{2x}$. That is, $H_{Liouville}=\mathcal{H}_{free}\equiv P^2$ under the classical effects of the canonical transformation. The quantum effects are detailed below, by $\star$ acting with the Liouville and free Hamiltonians on the suitable transformation functional.

The Liouville Wigner function may be obtained from the definition (1) in terms of known higher transcendental functions:

$$f(x,p)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dy\ \frac{1}{\pi^2}\sinh(\pi\sqrt{E})K_{i\sqrt{E}}(e^{x-y/2})e^{-iyp}K_{i\sqrt{E}}(e^{x+y/2})$$
$$=\frac{1}{4\pi^3}\sinh(\pi\sqrt{E})2^{2ip}e^{(-1-2ip)x}G_{04}^{40}\left(\frac{e^{4x}}{16}\middle|\frac{1+2i\sqrt{E}}{4},\frac{1-2i\sqrt{E}}{4},\frac{1+2i\sqrt{E}+4ip}{4},\frac{1-2i\sqrt{E}+4ip}{4}\right). \tag{91}$$

The following K transform was utilized to express this result in closed form:

$$\int_0^\infty dw(wz)^{1/2}w^{\sigma-1}K_\mu(a/w)K_\nu(wz)=2^{-\sigma-5/2}a^\sigma G_{04}^{40}\left(\frac{a^2z^2}{16}\middle|\frac{\mu-\sigma}{2},\frac{-\mu-\sigma}{2},\frac{1}{4}+\frac{\nu}{2},\frac{1}{4}-\frac{\nu}{2}\right). \tag{92}$$

The right-hand side involves a special case of Meijer's G function,

$$G_{pq}^{mn}\left(z\middle|\begin{matrix}a_i, & i=1,\dots,p\\ b_j, & j=1,\dots,q\end{matrix}\right) \tag{93}$$

(cf. [22], Sec. 5.3), which is fully symmetric in the parameter subsets $\{a_1,\dots,a_n\}$, $\{a_{n+1},\dots,a_p\}$, $\{b_1,\dots,b_m\}$, and $\{b_{m+1},\dots,b_q\}$. It is possible to reexpress the result as a linear combination of generalized hypergeometric functions of type ${}_0F_3$, but there is little reason to do so here. This transform is valid for $\Re a>0$, and is taken from [26], p. 711, Eq. (55).[4] The transform is complementary to [27], Sec. 10.3, Eq. (49), in an obvious way, a K transform which appears in perturbative computations of certain Liouville correlation functions [21].

The result (91) may be written in slightly different alternate forms

$$f(x,p)=\frac{\sinh(\pi\sqrt{E})e^{-x}}{4\pi^3}G_{04}^{40}\left(\frac{e^{4x}}{16}\middle|\frac{1+2i\sqrt{E}-2ip}{4},\frac{1-2i\sqrt{E}-2ip}{4},\frac{1+2i\sqrt{E}+2ip}{4},\frac{1-2i\sqrt{E}+2ip}{4}\right)$$
$$=\frac{\sinh(\pi\sqrt{E})}{8\pi^3}G_{04}^{40}\left(\frac{e^{4x}}{16}\middle|\frac{i\sqrt{E}-ip}{2},\frac{-i\sqrt{E}-ip}{2},\frac{i\sqrt{E}+ip}{2},\frac{-i\sqrt{E}+ip}{2}\right), \tag{94}$$

[4]There is an error in this result as it appears in [27], Vol. II, Sec. 10.3, Eq. (58), where the formula has $a^2z^2/4$ instead of $a^2z^2/16$ as the argument of the G function. The latter argument is correct, and appears in Meijer's original paper cited here.

by making use of the parameter translation identity for the G function [[22], Sec. 5.3.1, Eq. (9)]:

$$z^{\lambda} G^{mn}_{pq}\left(z \middle| \begin{matrix} a_r \\ b_s \end{matrix} \right) = G^{mn}_{pq}\left(z \middle| \begin{matrix} a_r+\lambda \\ b_s+\lambda \end{matrix} \right). \tag{95}$$

Yet another way to express the result utilizes the Fourier transform of the wave function, Eq. (86), in terms of which the Wigner function reads, in general,

$$f(x,p) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty} dk \Phi_E^*\left(p - \frac{1}{2}k\right) e^{ixk} \Phi_E\left(p + \frac{1}{2}k\right). \tag{96}$$

The specific result (86) then gives, as $\epsilon \to 0$,

$$f(x,p) = \left(\frac{1}{8\pi^2}\right)^2 \sinh(\pi\sqrt{E}) \int_{-\infty}^{+\infty} dk e^{ixk} 4^{-i(k/2+i\epsilon)} \Gamma\left(\frac{i(p-k/2-i\epsilon)-i\sqrt{E}}{2}\right) \times \Gamma\left(\frac{i(p-k/2-i\epsilon)+i\sqrt{E}}{2}\right) \Gamma\left(\frac{-i(p+k/2+i\epsilon)+i\sqrt{E}}{2}\right) \Gamma\left(\frac{-i(p+k/2+i\epsilon)-i\sqrt{E}}{2}\right). \tag{97}$$

However, this is a contour integral representation of the particular G function given above. Because of the ϵ prescription, the contour in the variable $z = k/2 + i\epsilon$ runs parallel to the real axis, but slightly above the poles of the Γ functions located on the real axis at $z = p - \sqrt{E}$, $z = p + \sqrt{E}$, $z = -p + \sqrt{E}$, and $z = -p - \sqrt{E}$. Changing variables to $s = \frac{1}{2} iz$ yields

$$f(x,p) = \frac{1}{8\pi^3} \sinh(\pi\sqrt{E}) \frac{1}{2\pi i} \int_C ds \left(\frac{e^{4x}}{16}\right)^s \Gamma\left(\frac{ip-i\sqrt{E}}{2} - s\right) \Gamma\left(\frac{ip+i\sqrt{E}}{2} - s\right) \Gamma\left(\frac{-ip+i\sqrt{E}}{2} - s\right) \Gamma\left(\frac{-ip-i\sqrt{E}}{2} - s\right), \tag{98}$$

where the contour C in the s plane runs from $-i\infty$ to $+i\infty$, just to the left of the four poles on the imaginary s axis at $i(p+\sqrt{E})/2$, $i(p-\sqrt{E})/2$, $i(-p+\sqrt{E})/2$, and $i(-p-\sqrt{E})/2$. This is recognized as the Mellin-Barnes-type integral definition of the G^{40}_{04} function [cf. [22], Sec. 5.3, Eq. (1)] in agreement with the second result above, Eq. (94).

The translation identity (95) is seen to hold by virtue of Eq. (98), through simply shifting the variable of integration, s. Moreover, deforming the contour in Eq. (98) to enclose the four sequences of poles $s_n = n + i(\pm p \pm \sqrt{E})/2$ reveals the equivalence of this particular G function to a linear combination of four ${}_0F_3$ functions, one for each of the sequences of poles. Evaluating the integral by the method of residues for all these poles produces the standard ${}_0F_3$ hypergeometric series.

It should now be straightforward to directly check that the explicit result for $f(x,p)$ is indeed a solution to the Liouville $\star$-genvalue equation,

$$H_{Liouville} \star f(x,p) = \left[\left(p - \frac{i}{2}\partial_x\right)^2 + e^{2[x+(i/2)\partial_p]}\right] f(x,p) = E f(x,p). \tag{99}$$

For real E and real $f(x,p)$, the imaginary part of this $\star$-genvalue equation is

$$(-p\partial_x + e^{2x} \sin \partial_p) f(x,p) = 0, \tag{100}$$

while the real part is

$$\left(p^2 - E - \frac{1}{4}\partial_x^2 + e^{2x} \cos \partial_p\right) f(x,p) = 0. \tag{101}$$

The first of these is a first-order differential-difference equation relating the x and p dependence:

$$e^{-2x}\partial_x f(x,p) = \frac{1}{2ip}[f(x,p+i) - f(x,p-i)]. \tag{102}$$

Similarly, the real part of the $\star$-genvalue equation is a second-order differential-difference equation:

$$e^{-2x}\left(p^2 - E - \frac{1}{4}\partial_x^2\right) f(x,p) + \frac{1}{2}[f(x,p+i) + f(x,p-i)] = 0. \tag{103}$$

The previous first-order equation may now be substituted (twice) into this last second-order equation, to convert it from a differential-difference equation into a second-order difference-only equation in the momentum variable, with nonconstant coefficients:

$$0 = (p^2 - E) f(x,p) + \left(\frac{e^{2x}}{4p}\right)^2 [f(x,p+2i) - 2f(x,p) + f(x,p-2i)] + i\frac{e^{2x}}{4p}[f(x,p+i) - f(x,p-i)] + \frac{e^{2x}}{2}[f(x,p+i) + f(x,p-i)]. \tag{104}$$

We leave it as an exercise for the reader to exploit the recursive properties of the Meijer G function and show that this difference equation is indeed obeyed by the result (91). Rather than pursue this in detail, we turn our attention to the transformation functional which connects the above result for f to a free-particle Wigner function.

Given Eq. (90), it follows that

$$\psi_E(x)=\frac{1}{\pi}\sqrt{\sinh(\pi\sqrt{E})}K_{i\sqrt{E}}(e^x)=\frac{1}{2\pi}\sqrt{\sinh(\pi\sqrt{E})}e^{\pi\sqrt{E}/2}\int_{-\infty}^{+\infty}dXe^{ie^x\sinh X}e^{i\sqrt{E}X}, \tag{105}$$

and hence $N_E=[4\pi\sqrt{E}e^{\pi\sqrt{E}}\sinh(\pi\sqrt{E})]^{1/2}/2\pi$, if we choose a $\delta(E_1-E_2)$ normalization for the free-particle plane waves as well as for the Liouville eigenfunctions. Therefore, lemma 3 yields

$$\begin{aligned}\mathfrak{T}(x,p;X,P)&=\frac{|N|^2}{2\pi}\int dYdy\,\exp[-iyp+iPY-iF^*(x-y/2,X-Y/2)+iF(x+y/2,X+Y/2)]\\&=\frac{1}{(2\pi)^3}[4\pi\sqrt{E}e^{\pi\sqrt{E}}\sinh(\pi\sqrt{E})]\int dYdy\,\exp\left[-iyp+iPY-ie^{x-y/2}\sinh\left(X-\frac{Y}{2}\right)+ie^{x+y/2}\sinh\left(X+\frac{Y}{2}\right)\right]\\&=\frac{1}{4\pi^3}[4\pi\sqrt{E}e^{\pi\sqrt{E}}\sinh(\pi\sqrt{E})]\int d\left(\frac{y+Y}{2}\right)\exp\left[i(P-p)\frac{y+Y}{2}+ie^{x+X}\sinh\left(\frac{y+Y}{2}\right)\right]\\&\quad\times\int d\left(\frac{Y-y}{2}\right)\exp\left[i(P+p)\frac{Y-y}{2}+ie^{x-X}\sinh\left(\frac{Y-y}{2}\right)\right].\end{aligned} \tag{106}$$

We thus conclude that

$$\mathfrak{T}(x,p;X,P)=\frac{4}{\pi^2}\sqrt{E}e^{\pi\sqrt{E}}\sinh(\pi\sqrt{E})e^{-\pi P}K_{i(P-p)}(e^{x+X})K_{i(P+p)}(e^{x-X}). \tag{107}$$

We now check that this result obeys Eq. (73) and, in so doing, carry out the nontrivial steps needed to show the Liouville Wigner functions satisfy the Liouville ⋆-genvalue equation (99). That is to say, we shall show

$$\left(\left(p-\frac{i}{2}\vec{\partial}_x\right)^2+e^{2[x+(i/2)\vec{\partial}_p]}\right)\mathfrak{T}(x,p;X,P)=\mathfrak{T}(x,p;X,P)\left[\left(P+\frac{i}{2}\overleftarrow{\partial}_X\right)^2\right] \tag{108}$$

or, equivalently,

$$\left[\left(p-\frac{i}{2}\vec{\partial}_x\right)^2+e^{2[x+(i/2)\vec{\partial}_p]}-\left(P+\frac{i}{2}\vec{\partial}_X\right)^2\right]K_{i(P-p)}(e^{x+X})K_{i(P+p)}(e^{x-X})=0. \tag{109}$$

Specifically,

$$\frac{-1}{4}(\vec{\partial}_x^2-\vec{\partial}_X^2)K_{i(P-p)}(e^{x+X})K_{i(P+p)}(e^{x-X})=-e^{2x}K'_{i(P-p)}(e^{x+X})K'_{i(P+p)}(e^{x-X}), \tag{110}$$

$$\begin{aligned}(-ip\vec{\partial}_x-iP\vec{\partial}_X)K_{i(P-p)}(e^{x+X})K_{i(P+p)}(e^{x-X})=&-i(p+P)e^{x+X}K'_{i(P-p)}(e^{x+X})K_{i(P+p)}(e^{x-X})-i(p-P)e^{x-X}K_{i(P-p)}\\&\times(e^{x+X})K'_{i(P+p)}(e^{x-X})\end{aligned} \tag{111}$$

and

$$e^{2[x+(i/2)\vec{\partial}_p]}K_{i(P-p)}(e^{x+X})K_{i(P+p)}(e^{x-X})=e^{2x}K_{1+i(P-p)}(e^{x+X})K_{-1+i(P+p)}(e^{x-X}). \tag{112}$$

Now, recall the recurrence relations ([25], Eq. 9.6.26)

$$K_{1+i(P-p)}(e^{x+X})=-K'_{i(P-p)}(e^{x+X})+i(P-p)e^{-x-X}K_{i(P-p)}(e^{x+X}), \tag{113}$$

$$K_{-1+i(P+p)}(e^{x-X})=-K'_{i(P+p)}(e^{x-X})-i(P+p)e^{-x+X}K_{i(P+p)}(e^{x-X}). \tag{114}$$

So the previous relation (112) becomes

$$e^{2(x+i/2\,\vec{\partial}_p)}K_{i(P-p)}(e^{x+X})K_{i(P+p)}(e^{x-X})=e^{2x}K'_{i(P-p)}(e^{x+X})K'_{i(P+p)}(e^{x-X})+i(P+p)e^{x+X}K'_{i(P-p)}(e^{x+X})K_{i(P+p)}(e^{x-X})$$
$$-i(P-p)e^{x-X}K_{i(P-p)}(e^{x+X})K'_{i(P+p)}(e^{x-X})+(P^2-p^2)K_{i(P-p)}(e^{x+X})$$
$$\times K_{i(P+p)}(e^{x-X}). \tag{115}$$

The sum of Eqs. (110), (111), and (115) shows that Eq. (109) is, indeed, satisfied.

Integrating over X and P the product of $\mathfrak{T}(x,p;X,P)$ and the free-particle Wigner function, as given here by $(4\pi\sqrt{E})^{-1}\delta(P-\sqrt{E})$, yields another expression for the Liouville Wigner function which checks against the previous result, Eq. (91). Using Eq. (92) and the parameter translation identity for the G function, this other expression is just Eq. (94).

Supersymmetric Liouville quantum mechanics is obtained by carrying through the Darboux construction detailed above (with $\hbar=1=2m$), for the choice

$$W(x)=e^x. \tag{116}$$

The conventions used essentially follow [28].

The first Hamiltonian of the essentially isospectral pair is then

$$H=p^2+e^{2x}-e^x, \tag{117}$$

and the allowed spectrum is $0\leq E<\infty$, including zero energy, for which there is a bounded wave function normalized as part of the continuum,

$$\psi_0(x)=\frac{1}{\sqrt{\pi}}e^{-e^x}. \tag{118}$$

The other, $E>0$, eigenfunctions are

$$\psi_E(x)=\left[\frac{1}{4\pi^2\sqrt{E}}e^x\cosh(\pi\sqrt{E})\right]^{1/2}$$
$$\times[K_{1/2-i\sqrt{E}}(e^x)+K_{1/2+i\sqrt{E}}(e^x)], \tag{119}$$

again normalized so that $\int_{-\infty}^{+\infty}dx\,\psi^*_{E_1}(x)\psi_{E_2}(x)=\delta(E_1-E_2)$.

The second Hamiltonian of the pair is

$$H'=p^2+e^{2x}+e^x, \tag{120}$$

and the allowed spectrum is $0<E<\infty$, excluding zero energy.[5] The $E>0$ eigenfunctions are then

$$\psi'_E(x)=\left[\frac{1}{4\pi^2\sqrt{E}}e^x\cosh(\pi\sqrt{E})\right]^{1/2}$$
$$\times[iK_{1/2-i\sqrt{E}}(e^x)-iK_{1/2+i\sqrt{E}}(e^x)], \tag{121}$$

and may be obtained from the previous $E>0$ eigenfunctions, as $\psi'_E(x)=(1/\sqrt{E})(\partial_x+W)\psi_E(x)$.

For both Hamiltonians, the Wigner functions are straightforward to construct directly, once again leading to the K transform (92) and particular Meijer G functions. We find it sufficient here to consider only the ground state for H,

$$f_0(x,p)=\frac{1}{2\pi^2}\int_{-\infty}^{+\infty}dy\,e^{-2e^x\cosh(y/2)-iyp}=\frac{2}{\pi^2}K_{2ip}(2e^x), \tag{122}$$

a single modified Bessel function. It smoothly [29] satisfies $[p-iW(x)]\star f_0=0$ and, hence, the $\star$-genvalue equation $H\star f_0=0$.

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