

Classical tests for Weyl gravity: Deflection of light and time delay

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Weyl gravity has been advanced in the recent past as an alternative to general relativity. The theory has had some success in fitting galactic rotation curves without the need for copious amounts of dark matter. To check the viability of Weyl gravity, we propose two additional classical tests of the theory: the deflection of light and time delay in the exterior of a static spherically symmetric source. The result for the deflection of light is remarkably simple: in addition to the usual positive (attractive) Einstein deflection of $4GM/r_0$ we obtain an extra deflection term of $-\gamma r_0$ where γ is a constant and r_0 is the radius of closest approach. With a negative γ , the extra term can increase the deflection on large distance scales (galactic or greater) and therefore imitate the effect of dark matter. Notably, the negative sign required for γ is opposite to the sign of γ used to fit galactic rotation curves. The experimental constraints show explicitly that the magnitude of γ is of the order of the inverse Hubble length something already noted as an interesting numerical coincidence in the fitting of galactic rotation curves. [S0556-2821(98)05514-3]

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I. INTRODUCTION

The higher-derivative conformally invariant Weyl action, the integral of the square of the Weyl tensor, has attracted much interest as a candidate action for quantum gravity. Unlike general relativity (GR), the lack of scale in the theory probably implies that it is perturbatively renormalizable [1,2]. The theory is also asymptotically free [3,4].

Weyl gravity, as a classical theory, has attracted less attention because GR has been so remarkably successful at large distances, i.e., on solar system scales, and therefore there seems no pressing need to study a higher-derivative alternative classical theory. However, GR may not be free of difficulties either theoretical or experimental. At present, it is faced with one long-standing problem: the notorious cosmological constant problem [5] whose solution is not yet in sight. There may however be an experimental problem with GR: the so-called dark matter problem. The clearest evidence for the existence of large amounts of dark matter comes from the flat rotation curves of galaxies, velocities of galaxies in clusters and the deflection of light from galaxies and clusters [6] (for short, we will call these observations “galactic phenomenology”). From this evidence, there is a consensus in the astrophysical community that most of the mass of galaxies (and of our universe) consists of non-luminous matter. However, the nature of this dark matter is still unknown and is one of the great unsolved problems in astrophysics. At first it was thought that it may be faint stars or other forms of baryonic matter i.e. the so-called massive compact halo objects (MACHOs). However, it is safe to say that observations have obtained much fewer events than required for an explanation of the galactic phenomenology with a dark halo dominated by MACHOs [7] (though there is still the possibility that future experiments might show otherwise). One is then left to consider non-baryonic forms of dark matter such as massive neutrinos, axions, and the weakly interacting massive particles, as predicted for example by supersymmetric theories. The direct experimental observation of such non-

baryonic candidates is of date singularly lacking (though many experiments are currently under development) [8]. Hence, to date, the nature of the dark matter that is thought to comprise most of the mass of our universe is still elusive. Is it possible that the copious amounts of dark matter we are searching for is simply not there? We believe it is reasonable at this juncture to consider such a possibility.

As far as we know, the deviation of galactic rotation curves from the Newtonian expectation occurs at distances way beyond the solar-system scale [9]. In other words, it is a galactic scale phenomena. Newton’s gravity theory, which GR recovers in the non-relativistic weak gravity limit, was originally formulated to explain solar-system phenomenology and it may be incorrect to extrapolate this theory to galactic scales. It has therefore been suggested by a handful of authors [10,11,9] that there may not be large amounts of dark matter after all and that the “galactic phenomenology” may be signaling a breakdown of Newtonian gravity (and hence GR) on galactic scales.

Some authors have therefore proposed alternative classical theories of gravity. Most notably there is Milgrom’s modified nonrelativistic dynamics (MOND) program [10], the Kazanas-Mannheim Weyl (conformal) gravity program [11], and the Bekenstein-Sanders scalar-tensor gravity theory [12]. In MOND, Newtonian dynamics are modified at low accelerations typical of orbits on galactic scales. It has had success in fitting galactic rotation curves without the need for dark matter [10,9]. MOND, however, is a non-relativistic theory and therefore cannot make any predictions on relativistic phenomena such as the deflection of light, cosmology, etc. In the scalar-tensor theory, it has been shown that the bending of light cannot exceed that which is predicted by GR [12], in conflict with the observations i.e. the observed bending is actually even greater than that predicted by GR. On aesthetic grounds, conformal gravity is more appealing than other alternative theories because it is based on a local invariance principle i.e. conformal invariance of the metric. Weyl gravity encompasses the largest symmetry group

which keep the light cones invariant i.e. the 15-parameter conformal group. It has already been stressed in the past that unlike Weyl gravity and gauge theories, GR is not based on an invariance principle. The principle of general covariance, which follows from the principle of equivalence, is not an invariance principle. It describes how physical systems behave in a given arbitrary gravitational field but it does not tell us much about the gravitational field itself beyond restricting the gravitational action to a scalar. The lack of an invariance principle is partly the reason why guesswork is inevitable in the derivation of Einstein's gravitational field equations (see [13] for details). In contrast, the Weyl action is unique due to its conformal invariance. Besides its aesthetic appeal, Weyl gravity has many other attractive features not the least being that it is renormalizable owing to its lack of length scale. Since the early days of GR, it has been known that the vacuum GR equations $R_{\mu\nu}=0$ are also vacuum solutions of the Weyl theory. One therefore expects the Schwarzschild metric to be one possible solution to the spherically symmetric Weyl vacuum equations. More recently, Weyl gravity has attracted some interest because it has had reasonable success in fitting galactic rotation curves without recourse to any dark matter [14].

The principal reason that Weyl gravity has not received general acceptance is because some solutions of the classical theory are expected to have no lower energy bound and therefore exhibit instabilities [15] i.e. runaway solutions common to higher-derivative theories. For example, there may exist some Weyl vacuum solutions other than $R_{\mu\nu}=0$ which are not desirable. Though it has been shown that the Einstein-Hilbert action plus higher-derivative terms have a well posed initial value problem [16] this has yet to be shown for the pure fourth order Weyl gravity. Fortunately, however, the static spherically symmetric vacuum solutions [11], the analog to the Schwarzschild metric, has been found to be stable and to make important corrections to the Schwarzschild metric at large distances i.e. it contains a linear potential that plays a non-trivial role on galactic scales. It therefore becomes compelling and interesting to compare Weyl gravity to GR in their classical predictions.

II. GEODESIC EQUATIONS

Weyl gravity is a theory that is invariant under the conformal transformation $g_{\mu\nu}(x) \rightarrow \Omega^2(x)g_{\mu\nu}(x)$ where $\Omega^2(x)$ is a finite, non-vanishing, continuous real function. The metric exterior to a static spherically symmetric source (i.e. the analog of the Schwarzschild solution in GR) has already been obtained in Weyl gravity by Mannheim and Kazanas [11]. For a metric in the standard form

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (1)$$

they obtain the vacuum solutions

$$B(r) = A^{-1}(r) = 1 - \frac{2\beta}{r} + \gamma r - kr^2 \quad (2)$$

where β , γ , and k are constants. The authors note that with $\beta = GM$, the Schwarzschild metric can be recovered on a

certain distance scale (say the solar system) provided γ and k are small enough. The linear γ term would then be significant only on larger distance scales (say galactic or greater) and hence would deviate from Schwarzschild only on those scales. The constant k , which should be taken negative, can then be made even smaller so that the kr^2 term becomes significant only on cosmological scales (in fact, it has been shown [11] that k is proportional to the cosmological scalar curvature). It should be noted that solution (2) is not unique. The Weyl gravitational field equations are conformally invariant so that any metric which is related to the standard metric (1) by a conformal factor $\Omega^2(r)$ is also a valid solution. This is in contrast to GR where the Schwarzschild solution is the unique vacuum solution for a spherically symmetric source. Two metrics that differ by a conformal factor of course have different curvatures. Remarkably, however, the geodesic equations for light are conformally invariant. Massive particles, on the other hand, have geodesics that depend on the conformal factor (though it is conceivable to envisage some spontaneous conformal symmetry breaking mechanism which gives rise to conformally covariant massive geodesics. e.g. see [17]. We do not entertain conformal symmetry breaking in this paper).

The geodesic equations along the equatorial plane ($\theta = \pi/2$) for a metric of form (1) are [13]

$$r^2 \frac{d\varphi}{dt} = JB(r), \quad (3)$$

$$\frac{A(r)}{B^2(r)} \left(\frac{dr}{dt} \right)^2 + \frac{J^2}{r^2} - \frac{1}{B(r)} = -E, \quad (4)$$

$$d\tau^2 = EB^2(r)dt^2, \quad (5)$$

where E and J are constants with $E=0$ for null geodesics (photons) and $E>0$ for massive particles. The above geodesic equations are only conformally invariant for photons and therefore two classical tests can be carried out unambiguously: the deflection of light and the time delay of radar echos.

III. DEFLECTION OF LIGHT

The geodesic equations (3)–(5) enable one to express the angle φ as a function of r :

$$\varphi(r) = \int \frac{A^{1/2}(r)}{r^2 \left(\frac{1}{J^2 B(r)} - \frac{E}{J^2} - \frac{1}{r^2} \right)^{1/2}} dr, \quad (6)$$

where the functions $A(r)$ and $B(r)$ are given by Eq. (2). To do a scattering experiment, the light is taken to approach the source from infinity. Unlike the Schwarzschild solution where the metric is Minkowskian at large distances from the source i.e. $B(r)$ and $A(r) \rightarrow 1$ as $r \rightarrow \infty$, $B(r)$ given by solution (2) diverges as $r \rightarrow \infty$ and we do not recover Minkowski space at large distances. However, this is not a problem. At large r it has been shown that the metric is conformal to a Robertson-Walker metric with three space curvature

$K = -k - \gamma^2/4$ [11]. Hence, at large r the photon is simply moving in a “straight” line in this background geometry [i.e. with $B(r)$ given by Eq. (2) and $\varphi(r)$ given by Eq. (6), it is easy to see that $d\varphi/dr \rightarrow 0$ as $r \rightarrow \infty$]. The photon then deviates from this “straight” line path as it approaches the source.

We now substitute the appropriate quantities in Eq. (6). For the photon we set $E=0$. At the point of closest approach $r=r_0$, we have that $dr/d\varphi=0$ and using Eqs. (5) one obtains $(1/J^2)=B(r_0)/r_0^2$. From the solutions (2) we know that $A^{1/2}(r)=B^{-1/2}(r)$. The deflection of the photon as it moves from infinity to r_0 and off to infinity can be expressed as

$$\Delta\varphi = 2 \int_{r_0}^{\infty} \left(\frac{B(r_0)}{r_0^2} - \frac{B(r)}{r^2} \right)^{-1/2} \frac{dr}{r^2} - \pi, \quad (7)$$

where π is the change in the angle φ for straight line motion and is therefore subtracted out. We now calculate the integral in Eq. (7) using $B(r)=1-2\beta/r+\gamma r-kr^2$. This yields

$$\int_{r_0}^{\infty} \left[\left(1 - \frac{2\beta}{r_0} + \gamma r_0 \right) \frac{r^4}{r_0^2} - \gamma r^3 - r^2 + 2\beta r \right]^{-1/2} dr. \quad (8)$$

The above integral, being the inverse of the square root of a fourth-degree polynomial, can be expressed in terms of elliptic integrals. However, this is not very illuminating. It will prove more instructive to evaluate the integral after expanding the integrand in some small parameters. Note that the constant k , important on cosmological scales, has cancelled out and does not appear in the integral (8). The deflection of light is insensitive to the cosmology of the theory and in general would not be affected by a spherically symmetric Hubble flow. On the other hand, the motion of massive particles on galactic or greater scales is affected by the Hubble flow [14,18]. Hence, the bending of light is highly appropriate for testing Weyl gravity.

We now evaluate the integral (8). It can be rewritten in the form

$$\int_{r_0}^{\infty} \left(\frac{1}{r_0^2} - \frac{1}{r^2} \right)^{-1/2} \left\{ 1 - 2\beta \left(\frac{1}{r_0} + \frac{1}{r} - \frac{1}{r+r_0} \right) + \frac{\gamma r_0}{1+r_0/r} \right\}^{-1/2} \frac{dr}{r^2}. \quad (9)$$

After making the substitution $\sin\theta=r_0/r$ the integral becomes

$$\int_0^{\pi/2} \left[1 - \frac{2\beta}{r_0} \left(1 + \sin\theta - \frac{\sin\theta}{1+\sin\theta} \right) + \frac{\gamma r_0}{(1+\sin\theta)} \right]^{-1/2} d\theta. \quad (10)$$

For any realistic situation, such as the bending of light from the sun, galaxies or cluster of galaxies the deflection is of the order of arc seconds and therefore the parameters β/r_0 and γr_0 , which measure the deviation from straight line motion

in Eq. (10), must be much less than one. We will therefore expand the integrand to first order in the small parameters β/r_0 and γr_0 . One obtains

$$\begin{aligned} & \int_0^{\pi/2} \left[1 + \frac{\beta}{r_0} \left(1 + \sin\theta - \frac{\sin\theta}{1+\sin\theta} \right) - \frac{\gamma r_0}{2(1+\sin\theta)} \right] d\theta \\ &= \frac{\pi}{2} + \frac{2\beta}{r_0} - \frac{\gamma r_0}{2}. \end{aligned} \quad (11)$$

The deflection, given by Eq. (7), is therefore

$$\Delta\varphi = \frac{4\beta}{r_0} - \gamma r_0 \quad (12)$$

a simple modification of the standard “Einstein” result of $4GM/r_0$ (where $\beta=GM$). The constant γ must be small enough such that the extra term $-\gamma r_0$ is negligible compared to $4GM/r_0$ on solar distance scales. The linear γ term, however, can begin to make important contributions on larger distance scales where discrepancies between experiment and theory presently exist i.e. the “Einstein” deflection due to the luminous matter in galaxies or clusters of galaxies is less than the observed deflection. Of course, these discrepancies are usually taken as evidence for the existence of large amounts of dark matter in the halos of galaxies. If the extra term $-\gamma r_0$ is to ever replace or imitate this dark matter on large distance scales it would have to be positive (i.e. attractive), implying that γ must be negative. The sign of γ used to fit galactic rotation curves [14] however, is positive (the reason why the sign of γ is different for null and non-relativistic massive geodesics is discussed in the next section on potentials). Therefore there is an incompatibility between these two analyses. This means that Weyl gravity does not seem to solve the dark matter problem, although this does not signal any inconsistency of Weyl gravity itself. In addition, the mechanism of conformal symmetry breaking is not well understood and it must be addressed in more detail before considering massive geodesics or just mass in general. The analysis of the deflection of light is more reliable since it is completely independent of any such conformal symmetry breaking mechanism.

IV. THE POTENTIAL IN WEYL GRAVITY

In general relativity, the Schwarzschild geodesic equations can be viewed as “Newtonian” equations of motion with a potential (see [19]). In Weyl gravity, a potential can also be extracted from the vacuum equations and for this purpose it is convenient to define a new “time” coordinate p such that $dp=B(r)dt$. The vacuum equations (3)–(5) in these new coordinates are

$$r^2 \frac{d\varphi}{dp} = J, \quad (13)$$

$$\frac{1}{2} \left(\frac{dr}{dp} \right)^2 + \frac{J^2}{2r^2} B(r) - \frac{1}{2} = \frac{-EB(r)}{2}, \quad (14)$$

$$d\tau^2 = Edp^2. \quad (15)$$

Let $B(r) \equiv 1 + 2\phi(r)$ where ϕ is not necessarily a weak field. Equation (14) becomes

$$\frac{1}{2} \left(\frac{dr}{dp} \right)^2 + \frac{J^2}{2r^2} + \phi \left(\frac{J^2}{r^2} + E \right) = \frac{1-E}{2}. \quad (16)$$

The above geodesic equation together with Eq. (13) can be viewed as a particle having energy per unit mass $(1-E)/2$ and angular momentum J moving in ordinary mechanics with a potential

$$V(r) = \phi \left(\frac{J^2}{r^2} + E \right). \quad (17)$$

The derivative of the potential is

$$V'(r) = \frac{\beta}{r^2} \left(\frac{3J^2}{r^2} + E \right) + \frac{\gamma}{2} \left(E - \frac{J^2}{r^2} \right) - krE, \quad (18)$$

where $\phi(r) = -\beta/r + \gamma r/2 - kr^2/2$ was used. There are three terms in Eq. (18): a β , γ , and k term, respectively. The k term vanishes for null geodesics in agreement with our results on the deflection of light. For massive geodesics the k term is non-zero but is negligible unless one is considering cosmological scales. Hence, this term will be ignored. The factor $3J^2/r^2 + E$ in front of the β term is always positive since $E \geq 0$. Therefore, the β term is attractive for both massive and null geodesics (which is the case in GR). On the other hand, the factor $E - J^2/r^2$ in front of the γ term, can be positive or negative depending on the physical situation. For a non-relativistic particle moving in a weak field, which is the case of galactic rotation curves, we obtain $E \approx 1$, $J^2/r^2 \ll 1$, and therefore the factor $E - J^2/r^2$ is positive. For light, E is zero and the factor is negative. The potential (17) is different for non-relativistic particles and light: the γr term in ϕ contributes a linear potential for non-relativistic particles but an inverse r potential for light. Their corresponding derivatives therefore have opposite sign and this explains why γ obtained through galactic rotation curves has the opposite sign to that obtained in the deflection of light.

Of course, a negative γ term is not reserved to null geodesics only. Any massive particle which is sufficiently relativistic will also have this property. For example consider a particle moving in a weak field ϕ with a negligible ‘‘radial velocity’’ dr/dp . One obtains from Eq. (16) that $J^2/r^2 \approx 1 - E - 2\phi$ and therefore $E - J^2/r^2 \approx 2E + 2\phi - 1$. It follows that if a particle is sufficiently relativistic such that $E < 1/2 - \phi \approx 1/2$ then we obtain a negative γ term.

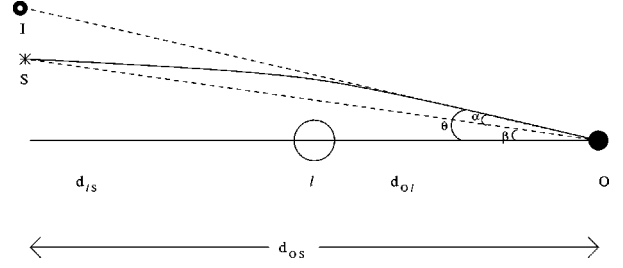


FIG. 1. Light from the source S bends at the lens l and arrives at the observer O who then sees the image I.

We can actually reproduce the deflection of light result Eq. (12) in a most straightforward way using the potential Eq. (17). For null geodesics ($E=0$) the potential is given by

$$V_{\text{null}}(r) = \frac{-\beta J^2}{r^3} + \frac{\gamma J^2}{2r} + \frac{-kJ^2}{2}. \quad (19)$$

The deflection by a potential $V(r)$ is obtained by integrating along the straight line path the gradient of $V(r)$ (in the \perp direction i.e. in the direction of r_0). As long as the deflection is very small, integrating along the straight line path instead of the curved path gives the same results. The deflection is given by

$$\Delta\varphi = \int_{-\infty}^{\infty} \nabla_{\perp} V(r) dZ, \quad (20)$$

where Z is the distance along the straight line path i.e. $r^2 = Z^2 + r_0^2$. In the potential V_{null} , the γ term is an inverse r potential. This is the reason why its contribution to the deflection of light Eq. (20) is finite and comes with a relative negative sign. If V_{null} had contained a linear potential, the integral for the deflection would diverge, implying that no scattering states could exist.

Using $J^2 = r_0^2/B(r_0)$ given in Sec. III and V_{null} as the potential, the deflection Eq. (20) yields

$$\Delta\varphi = \frac{4\beta}{r_0} - \gamma r_0, \quad (21)$$

where only first order terms in β/r_0 and γr_0 were kept. The deflection of light result Eq. (12) is therefore reproduced in a straightforward fashion that allows one to trace clearly the origin of the negative sign in $-\gamma r_0$.

V. THE WEYL RADIUS

The geometry of a typical lens system is shown in Fig. 1, below. A light ray from a source S is deflected by an angle α at the lens and reaches an observer at O. The angle between the optic axis and the true position of the source is β and the angle between the optic axis and the image I is θ . The angular diameter distances between observer and lens, lens and source, and observer and source are d_{ol} , d_{ls} and d_{os} , respec-

tively. For a spherically symmetric lens, image formation is governed by the one-dimensional lens equation

$$\beta = \theta - \alpha(d_{ls}/d_{os}). \quad (22)$$

A source is imaged as a ring if the source, the lens and the observer lie on a ‘‘straight’’ line (i.e. $\beta=0$). For an Einstein deflection angle of $\alpha=4GM/r_0$, the radius of the ring is called the Einstein radius and is given by

$$\theta_E = \left(\frac{4GM}{D} \right)^{1/2}, \quad (23)$$

where $D \equiv d_{ol}d_{os}/d_{ls}$ and M is the mass of the lens enclosed in the Einstein radius. For a Weyl deflection angle given by Eq. (12), the radius of the ring, which we will call the ‘‘Weyl’’ radius, can be readily calculated and yields

$$\theta_w = \left(\frac{4GM}{D + \gamma(d_{ol})^2} \right)^{1/2}. \quad (24)$$

The above result for the Weyl radius will be used later to obtain an estimate for the constant γ . If the source, lens and observer are not aligned in a ‘‘straight’’ line (i.e. $\beta \neq 0$) then instead of a ring one obtains two images, one inside and one outside the Weyl ring. Using the Weyl deflection angle Eq. (12) and the definitions for the Einstein and Weyl radius, the lens equation (22) gives

$$\beta = (1 + n_\gamma)\theta - \frac{4GM}{D\theta}, \quad (25)$$

where $n_\gamma \equiv \gamma d_{ol}^2/D$. The two solutions to the above equation are

$$\theta_\pm = \frac{1}{2(1+n_\gamma)} [\beta \pm \sqrt{\beta^2 + 4\theta_w^2(1+n_\gamma)^2}]. \quad (26)$$

VI. CIRCULAR ORBITS IN EQUILIBRIUM

In the Schwarzschild metric, it is known that photons do not have circular orbits with stable equilibrium but have one unstable equilibrium at the radius $r=3GM$. We now determine the radii of equilibrium for photons in the Weyl vacuum solution (2). The geodesic equation of interest is Eq. (4) where we substitute $E=0$ for photons and set dr/dt to zero at the radius of orbit $r=R$. Equation (4) becomes

$$\frac{J^2}{R^2} - \frac{1}{B(R)} = 0. \quad (27)$$

For equilibrium, the derivative of the left-hand side (LHS) of Eq. (27) at $r=R$ must vanish and we obtain

$$-\frac{2J^2}{R^3} + \frac{B'(R)}{B^2(R)} = 0. \quad (28)$$

With J^2 given by Eq. (27) and $B(r)$ given by Eq. (2), Eq. (28) becomes

$$\gamma R^2 + 2R - 6GM = 0, \quad (29)$$

where $\beta=GM$ was used. Note that the constant k has again cancelled out. The two solutions to Eq. (29) are

$$R \approx 3GM \quad \text{and} \quad R \approx -2/\gamma, \quad (30)$$

where it has been assumed that $|\beta\gamma| \ll 1$. We see that besides the $R=3GM$ solution a second equilibrium exists at $R=-2/\gamma$ if γ is negative. By differentiating Eq. (29) we see that this second equilibrium is a stable one while the first is an unstable one as in the Schwarzschild case. This stable equilibrium provides us with a natural length scale i.e. a scale which determines the ‘‘region of influence’’ of a particular localized source in contrast to the background or global aspects. A length scale of this sort is probably necessary if we ever want to develop a concept of ‘‘energy of an isolated system’’ in Weyl gravity. In the Schwarzschild case, the metric tends towards Minkowski space in the limit $r \rightarrow \infty$ and a Gauss’s law formulation of total energy of an ‘‘isolated’’ system is possible. In the Weyl case we obtain a metric conformal to a Robertson-Walker spacetime in the limit $r \rightarrow \infty$. We therefore need a natural cutoff radius at which the influence of the specific source in question ceases and the global aspects take over. Indeed, we have shown that the constant k , which is proportional to the cosmological curvature, plays no role in determining the radius of stable equilibrium and lends support to the idea that the stable radius is determined by the localized source. Hence, from the arguments above, a negative γ is desirable.

VII. TIME DELAY

We now calculate the time taken by a photon for a trip between any two points in a gravitational field produced by a central mass. We expect modifications to the standard GR result when the radius of closest approach to the central mass is on the order of galactic scales. The equation governing the time evolution of orbits is Eq. (4), with $E=0$ for light. At the point of closest approach $r=r_0$, $dr/dt=0$ so that Eq. (4) gives $J^2=r_0^2/B(r_0)$. The time for light to travel from r_0 to r_1 , given by Eq. (4), is

$$t = \int_{r_0}^{r_1} \left(\frac{A(r)/B(r)}{1 - \frac{B(r)r_0^2}{B(r_0)r^2}} \right)^{1/2} dr. \quad (31)$$

We evaluate the above integral with $A(r)$ and $B(r)$ given by Eq. (2). This yields

$$t = \int_{r_0}^{r_1} \frac{r(1-2\beta/r + \gamma r - kr^2)^{-1}(1-2\beta/r_0 + \gamma r_0 - kr_0^2)^{1/2}}{\sqrt{r^2 - r_0^2} \left[1 - \frac{2\beta}{r_0} \left(1 + \frac{r_0^2}{r(r+r_0)} \right) + \gamma r_0 \left(\frac{r}{r+r_0} \right) \right]^{1/2}} dr. \quad (32)$$

We can expand the above integral to first order in the parameters β/r , γr and kr^2 which are much less than 1 within the usual limits of integration. To first order in the parameters, the integral (32) yields

$$t \approx \int_{r_0}^{r_1} r \left[\left(1 - \frac{1}{2}kr_0^2 \right) + \frac{2\beta}{r} + \frac{\beta r_0}{r(r+r_0)} - \gamma r + \frac{\gamma r_0^2}{2(r+r_0)} + kr^2 \right] \frac{dr}{\sqrt{r^2 - r_0^2}}. \quad (33)$$

There are six elementary integrals to evaluate above. The result is

$$t \approx \sqrt{r_1^2 - r_0^2} + 2\beta \ln \left(\frac{r_1 + \sqrt{r_1^2 - r_0^2}}{r_0} \right) + \beta \sqrt{\frac{r_1 - r_0}{r_1 + r_0}} - \frac{\gamma}{2} \left(\frac{r_1^3 - r_0^3}{\sqrt{r_1^2 - r_0^2}} \right) + \frac{k}{6} (2r_1^2 + r_0^2) \sqrt{r_1^2 - r_0^2}. \quad (34)$$

The leading term is identified as the time for light to travel in a straight line in Minkowski space (where $\beta = \gamma = k = 0$) and we recognize the β terms as the standard ‘‘Shapiro’’ time delay. The γ and k terms evidently produce a modification of the time delay. We see that the effect of the γ term is to increase the time delay if γ is negative and to decrease it if γ is positive.

VIII. CONSTRAINTS ON γ FROM EXPERIMENTS

A. Solar gravitational deflection

In solar experiments, the sun can be treated as a point mass and no lens model is required. To date, the best measurements on the deflection of light from the sun were obtained using radio-interferometric methods and verified Einstein’s prediction to within 1%. The measured deflection at the solar limb was 1.761 ± 0.016 arc sec [20] compared to Einstein’s prediction of $4GM_\odot/R_\odot = 1.75$ arc sec. Using the Weyl deflection angle Eq. (12) these measurements constrain the constant γ to the range $3.45 \times 10^{-19} \text{ cm}^{-1} \geq \gamma \geq -1.87 \times 10^{-18} \text{ cm}^{-1}$. Clearly, the solar gravitational deflection experiments constrain strongly the order of magnitude of γ but leave open the possibility for a positive or negative γ .

B. Signal retardation by solar gravity

The results of the Viking relativity experiment published in 1979 [21] confirmed the ‘‘Shapiro’’ time delay on solar system scales to an accuracy of 0.1%. For example, a ray that

leaves the earth, grazes the sun, reaches Mars and comes back would have a time delay of $248 \pm 0.25 \mu\text{s}$ where the $248 \mu\text{s}$ is the exact prediction of the ‘‘Shapiro’’ time delay and the uncertainty $\pm 0.25 \mu\text{s}$ can be used to constrain γ . At superior conjunction, the radius of the sun to the Earth r_e , and to Mars r_m , are much greater than the radius of the sun R_\odot so that r_0 can be neglected in the factor in front of γ in Eq. (34). We therefore have $-\gamma(r_e^2 + r_m^2) = \pm 0.25 \times 10^{-6} \text{ s}$. This constrains γ to the range $|\gamma| \leq 1.02 \times 10^{-23} \text{ cm}^{-1}$. This is roughly five orders of magnitude better than the constraint on γ from solar deflection experiments but does not allow us to draw any conclusions on the sign of γ .

C. Deflection of light by galaxies and clusters

One should expect measurements on the deflection of light by galaxies and clusters to determine the most accurate value for γ because it is on those scales where the γ term plays a significant role. However, the interpretation of the experimental data on those scales is more difficult than in the solar system because galaxies and clusters have unknown matter distributions and in general cannot be assumed to be either point masses or spherically symmetric. A parametrized lens model is therefore required for each case of gravitational lensing. For example, to understand the time delay in the gravitational lens 0957+561, one has to describe not only the distribution of the lensing galaxy G1 but also the effect of the surrounding cluster. Nonetheless, it has been pointed out [22,23] that though a spherically symmetric lens model is an idealization, it is a good first approximation in understanding the large arcs that are observed in clusters and is extremely useful in obtaining the same order of magnitude results as the more realistic case. We will therefore use data on the large arcs found in clusters to obtain a value for γ with confidence that the order of magnitude is correct. To constrain γ beyond ‘‘order of magnitude’’ accuracy, a detailed lens model for each cluster is required and would take us beyond the scope of the present paper.

In a spherically symmetric lens, the radius of the tangentially oriented large arcs, θ_{arc} , is a good estimate of the radius which occurs at $\beta=0$ in the lens equation (22). In GR, the radius of the arc is therefore interpreted as the Einstein radius where M is the total mass (M_{total}) i.e. the sum of the luminous and presumed dark matter. In the context of Weyl gravity, the *same* arc is to be interpreted as the Weyl radius, with M representing only the luminous matter (M_L) and γ a constant to be determined. Using Eqs. (23) and (24) for the Einstein and Weyl radius, respectively and equating them for the same observed arc, one obtains

$$\gamma = \left(\frac{d_{os}}{d_{ls}d_{ol}} \right) \left(\frac{M_L}{M_{\text{total}}} - 1 \right). \quad (35)$$

In experiments one measures the redshifts Z_l and Z_s in the spectrum of the light reaching us from the lens (i.e., the cluster) and source, respectively. We define a dimensionless quantity (often called the angular size distance) $y \equiv (1 + Z)d/L_{H_0}$ where $L_{H_0} \equiv c/H_0$ is the Hubble length and d are the angular diameter distances that appear in Eq. (35) (we have reinserted the speed of light c for clarity). The values of y are related to redshifts by [24]

$$y_{ox} = \begin{cases} \frac{Z_x(1 + Z_x/2)}{1 + Z_x} & \text{for } \Omega = 0, \\ 2 - \frac{2}{\sqrt{1 + Z_x}} & \text{for } \Omega = 1, \end{cases} \quad (36)$$

$$y_{ls} = \begin{cases} y_{os}(1 + y_{ol}^2)^{1/2} - y_{ol}(1 + y_{os}^2)^{1/2} & \text{for } \Omega = 0, \\ y_{os} - y_{ol} & \text{for } \Omega = 1, \end{cases} \quad (37)$$

where x represents either the lens l or the source s and Ω is

the cosmological density parameter (we have taken $\Omega = 0$ and $\Omega = 1$ as examples though we will see that γ is insensitive to Ω). Equation (35) can be rewritten in terms of y and yields

$$\gamma = \frac{1}{L_{H_0}} \left(\frac{y_{os}}{y_{ls}y_{ol}} \right) \left(\frac{M_L}{M_{\text{total}}} - 1 \right). \quad (38)$$

To obtain a value for γ reliable data on the redshifts Z_l and Z_s is required as well as values for the mass-to-light ratios of clusters derived from gravitational lensing. Fortunately, such data exists for many large arcs in clusters. Before looking at the data it is important to note that the mass-to-light ratio is large for a typical cluster (> 100) and therefore $M_L/M_{\text{total}} \ll 1$. It follows that the factor $(M_L/M_{\text{total}} - 1)$ will not differ from cluster to cluster. Data is shown below (taken from [25,26,27]) for different clusters with the value of γ calculated in each case.

Cluster	Z_l	Z_s	M_L/M_{total}	$\frac{y_{os}}{y_{ls}y_{ol}} \Big _{\Omega=1,0}$	$\gamma _{\Omega=1,0}$
A370	0.375	0.724	$\approx 1/200$	6.877, 7.765	-6.83, -7.72
A2390	0.231	0.913	$\approx 1/120$	7.885, 7.308	-7.82, -7.25
Cl2244-02	0.331	0.83	$< 1/100$	7.68, 6.87	-7.60, -6.80

where γ is in units of the inverse Hubble length, $1/L_{H_0}$ [which is equal to $(H_0/100) \times 1.08 \times 10^{-28} \text{ cm}^{-1}$.] As can be seen from the data, γ is negative, reasonably constant from one cluster to the next and its order of magnitude is clearly the inverse Hubble length. Interestingly enough, the value for $|\gamma|$ obtained by Mannheim and Kazanas [11] in the context of galactic rotation curves is of the same order of magnitude. Though no use of Hubble's constant was made in their calculation, the authors recognized that γ was close numerically to the inverse Hubble length. Lensing data in clusters confirms via Eq. (38) that γ is not independent of Hubble's constant. In conclusion, we obtain the same order of magnitude for γ as in galactic rotation curves but with opposite sign. This discrepancy merits further investigation.

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