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Hairs on the cosmological horizon

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We investigate the possibility of having hair on the cosmological horizon. The cosmological horizon shares similar properties of black hole horizons in the aspect of having hair on the horizon. For those theories admitting haired black hole solutions, the nontrivial matter fields may reach and extend beyond the cosmological horizon. For Q stars and boson stars, the matter fields cannot reach the cosmological horizon. The no short hair conjecture stays valid, despite the asymptotic behavior (de Sitter or anti–de Sitter) of black hole solutions. We prove the no scalar hair theorem for anti–de Sitter black holes. Using Bekenstein's identity method, we also prove the no scalar hair theorem for de Sitter space and de Sitter black holes if the scalar potential is convex. [S0556-2821(98)00514-1]

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I. INTRODUCTION

There are two no hair conjectures in gravitational physics. Although they have not been proven rigorously, they are often referred to as no hair theorems. One is the cosmic no hair theorem [1], which states as follows [2]: Any solution of the Einstein equations with a positive cosmological constant that (i) accepts a synchronous coordinate system, (ii) has a nonpositive three-curvature, (iii) has an energy-momentum tensor satisfying the strong and dominant energy conditions, will become asymptotically de Sitter (at least on patch). This would imply that the inflation of the universe is a natural phenomenon that can explain the isotropy and homogeneity seen today in the universe.

The other is the no hair theorem of black holes [3]. It is generally believed that the collapse of a massive body will finally lead to the formation of a black hole and the external gravitational field of the black hole settles down to the Kerr-Newman solution of the Einstein-Maxwell equations, specified by only three parameters: mass, electric (and/or magnetic) charge, and angular momentum [4]. This theorem indeed excludes scalars [5], massive vectors [6], spinors [7], and Abelian Higgs hair (Maxwell-complex Higgs scalars) [8] from a stationary black hole exterior. However, this situation has been changed dramatically since the discovery of colored black holes in the Einstein-Yang-Mills theory [9] in 1990. Since then, a lot of black holes with different hair have been found (see, e.g., Ref. [10]).

With the discovery of haired black holes, naturally much attention has been drawn to reexamine the no hair theorem of black holes. de Alwis [11] discussed the validity of the old no hair theorem in the stringy black holes. Using a conformal transformation, Saa [12] showed the nonexistence of scalar hair in a large kind of theories. In the Einstein-conformal scalar field theory, as is well known, there exists the so-called Bekenstein black hole solution [13]. But the conformal scalar diverges at the horizon and the solution is dynamically unstable [14]. So Zannias [15] proved that black hole horizon cannot support the conformal scalar hair in the sense of no hair theorem. Furthermore, Sudarsky and Zannias [16] recently showed that the stress-energy tensor is ill defined and the Einstein equations do not hold at the horizon. And hence the Bekenstein solution fails to represent a genuine black hole solution. In Ref. [17] Bekenstein proposed a novel ''no-scalar-hair theorem'' of black holes, which rules out a multicomponent scalar field dressing of any asymptotically flat, static, spherically symmetric black holes. This theorem also holds for scalar-tensor gravity. Further Mayo and Bekenstein [18] investigated this theorem for the charged self-interacting scalar field coupled to an Abelian gauge field, or nonminimally coupled to gravity. In addition, Sudarsky [19] suggested a very simple proof of the no hair theorem in the Einstein-Higgs theory. Núñez, Quevedo, and Sudarsky (NQS) [10] made important progress in understanding the haired black holes by showing that black holes have no short hair: Under some conditions they assumed, the region with nontrivial structure of the nonlinear matter fields must extend beyond 3/2 the horizon radius of black holes. More issues related to the no hair theorem and uniqueness theorem of black holes can be found in Ref. [20].

When a positive (negative) cosmological constant is present, it is widely believed that the Kerr-Newman solution to the Einstein-Maxwell equations will become the Kerr-Newman–(anti–)de Sitter solution, whose spacetime is the asymptotically (anti–)de Sitter one. But the uniqueness theorem for this solution is still lacking, contrary to the Kerr-Newman solution. Further the presence of a positive cosmological constant will usually accompany the occurrence of a cosmological horizon. Thus the cosmological constant changes greatly the asymptotic behavior and structure of spacetimes. However all proofs of the no hair theorem have been carried out on the assumption that the black hole spacetime is asymptotically flat. It is therefore of some interest to investigate the effects of the cosmological constant on the no

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hair theorem and no short hair conjecture of NQS. The cosmological horizon has many similar properties of black hole horizon. For example, like the black hole horizon, the cosmological horizon has the Hawking evaporation and the entropy associated with the horizon [21], and is classically stable [22]. Has the cosmological horizon the similar property in having hairs on the black hole horizon? On the other hand, the regular solutions such as Q stars, boson stars, and gravitational solitons may be surrounded by a cosmological horizon, when a positive cosmological constant is present. Can these nontrivial matter fields reach and extend beyond the cosmological horizon? In the present work we try to make some investigations.

The organization of this paper is as follows. In Sec. II we consider the Einstein-Yang-Mills theory with a cosmological constant. For the regular Bartnik-McKinnon [23] soliton solution surrounded by a cosmological horizon, we find that the nontrivial Yang-Mills field may reach and extend beyond the cosmological horizon. As the case in the asymptotically flat spacetime, to have the required asymptotic behavior, the nontrivial Yang-Mills field also must extend beyond a critical point satisfying $r=3m(r)$, where $m(r)$ is a mass function in the metric. In Sec. III we extend to discuss the hairs on the cosmological horizon in those theories allowing haired black holes, and investigate the effect of cosmological constant on the no short hair conjecture of black holes. For the Q stars and boson stars, however, we find that the matter fields cannot reach the cosmological horizon. In Sec. IV we discuss the no scalar hair theorem for asymptotically (anti−)de Sitter black holes. The conclusion and discussion are given in Sec. V.

II. EINSTEIN-YANG-MILLS THEORY WITH A COSMOLOGICAL CONSTANT

It is the Einstein-Yang-Mills theory in which Bartnik and McKinnon [23] first found the nontrivial gravitational soliton solution and subsequently some authors [9] discovered numerically the first haired black hole. Here the meaning of ‘‘hair’’ follows Ref. [10]: In a given theory, there is black hole hair, when the spacetime metric and the configuration of the other fields of a stationary black hole solution are not completely specified by the conserved charges defined at asymptotic infinity. In this section we discuss the Einstein-Yang-Mills theory with a cosmological constant. For a positive cosmological constant, more recently, many authors [24,25] have investigated the system. Due to the nontrivial asymptotic behavior, some new phenomena have been revealed.

The action of the Einstein-Yang-Mills theory with a cosmological constant is

$$S=\frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R-2\Lambda-\frac{1}{g^2}\mathrm{Tr}F^2\right), \qquad (2.1)$$

where R is the scalar curvature, Λ is the cosmological constant, F is the SU(2) Yang-Mills field strength, and g is the coupling constant of the field. Throughout this paper the units $G=c=1$ have been used. We now consider the solution whose metric is of the form

$$ds^2=-\mu(r)e^{-2\delta(r)}dt^2+\mu^{-1}(r)dr^2+r^2d\Omega^2, \qquad (2.2)$$

where

$$\mu(r)=1-\frac{2m(r)}{r}-\frac{1}{3}\Lambda r^2, \qquad (2.3)$$

$m(r)$ denotes the mass function and $d\Omega^2$ represents the line element on the unit 2-sphere. Throughout this paper, we require that the solution is asymptotically de Sitter, when $\Lambda>0$,[1] or anti−de Sitter for $\Lambda<0$.[2] For the metric (2.2) we have

$$\lim_{r\to\infty} m(r)=M, \quad \text{and} \quad \lim_{r\to\infty}\delta(r)=0, \qquad (2.4)$$

where M is a constant. In the solution (2.2), when $\delta(r)$ and $m(r)$ vanish, the line element (2.2) describes the de Sitter space ($\Lambda>0$) or anti−de Sitter space ($\Lambda<0$). For the de Sitter space the future infinity is spacelike. This means that for each observer moving on a timelike world line, there is an event horizon separating the region of spacetime which the observer can never see from the region that he can see if he waits long enough. In other words, the event horizon is the boundary of the past of the observer's world line. This event horizon is called a cosmological event horizon. It is located at the coordinate singularity $r_c=\sqrt{3/\Lambda}$ in the solution (2.2). When $\delta(r)=0$, $m(r)=M$ is a constant, and $\Lambda>0$ in Eq. (2.2), the solution is just the Schwarzschild-de Sitter spacetime. If $9\Lambda M^2<1$, the equation $\mu(r)=0$ then has two positive roots. The large one is just the cosmological horizon, beyond which $\mu(r)<0$, while the small one is the black hole horizon. When $\Lambda<0$, the cosmological horizon is absent. For more details about the cosmological and black hole horizons, the reader is referred to Ref. [21]. For the solution we are considering in the action (2.1), the black hole horizons and/or a cosmological horizon of the metric (2.2), if exist, are regular and hence the metric functions $m(r)$, $\delta(r)$ are finite on the horizons. Further we require that the matter fields are also finite on the horizons.

For the Yang-Mills gauge potential, we take the following ansatz:

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[1]The authors in [25] found that in the Einstein-Yang-Mills theory with a positive cosmological constant, there are not only the asymptotically de Sitter solution, but also the so-called the bag of gold solution and the compact regular solution with space topology S^3, depending on the cosmological constant and the node number of the Yang-Mills amplitude. It would be a quite interesting subject to further investigate the latter two asymptotic behaviors from the point of view of the no hair conjecture. Here we restrict ourselves to the case of the first asymptotic behavior.

[2]In fact, when $\Lambda<0$, the so-called topological black holes, whose topology of event horizons is no longer the 2-sphere S^2, may appear. In this paper the topology of event horizons is restricted to the 2-sphere.

$$A = w(r)\tau_1 d\theta + [w(r)\tau_2 + \cot\theta\tau_3]\sin\theta d\phi, \tag{2.5}$$

where τ_i $(i=1,2,3)$ are three Pauli matrices. In the metric (2.2), we have equations of motion

$$m'(r) = \mu\frac{w'^2}{g^2} + \frac{2V(w)}{g^2 r^2}, \tag{2.6}$$

$$\delta'(r) = -\frac{2w'^2}{g^2 r}, \tag{2.7}$$

$$r^2 e^{\delta}(\mu e^{-\delta} w')' = \frac{\partial V(w)}{\partial w}, \tag{2.8}$$

where

$$V(w) = \tfrac{1}{4}(1-w^2)^2, \tag{2.9}$$

and a prime denotes derivative with respect to r. In Eqs. (2.6)–(2.8) there exist two trivial exact solutions with horizons. One is the Schwarzschild-(anti–)de Sitter solution, when the Yang-Mills potential $w=\pm 1$. The solution includes of course the (anti–)de Sitter space as a special case, that is, $\delta(r)=0$ and $m(r)=0$ in the Eq. (2.2). When $w=0$, the exact solution is the Reissner-Nordström-(anti–)de Sitter spacetime. The authors in [25] also gave a few other exact solutions for a positive cosmological constant. Now we want to discuss the nontrivial solution with horizons. Here the word ‘‘nontrivial’’ means that the Yang-Mills potential w is no longer a trivial constant throughout the whole spacetime.

From Eq. (2.8) and by using Eqs. (2.6) and (2.7), we can obtain

$$r^2\mu w'' + 2w'\left(m - \frac{1}{3}\Lambda r^3 - \frac{2V(w)}{g^2 r}\right) = \frac{\partial V(w)}{\partial w}. \tag{2.10}$$

Multiplying Eq. (2.10) by w', one has

$$\left(\frac{1}{2}\mu r^2 w'^2\right)' + 2w'^2\left(-\frac{1}{4}(\mu r^2)' + m - \frac{1}{3}\Lambda r^3 - \frac{2V(w)}{g^2 r}\right) = \frac{\partial V(w)}{\partial w} w'. \tag{2.11}$$

Defining

$$E(r) = \tfrac{1}{2}\mu r^2 w'^2 - V(w), \tag{2.12}$$

it is easy to show

$$E'(r) = -\left(\frac{2}{g^2 r}E + (3m - r)\right)w'^2. \tag{2.13}$$

Furthermore we can get the following equation

$$\frac{d}{dr}(Ee^{-\delta}) = -[3m(r) - r]e^{-\delta}w'^2. \tag{2.14}$$

We note that this equation is completely identical with Eq. (25) in Ref. [19], although our case is that a cosmological constant is present. Now we discuss two cases, respectively.

(i) Bartnik-McKinnon (BM) soliton surrounded by a cosmological horizon, say r_c. In this case, the cosmological constant is positive and a cosmological horizon appears. The origin is regular, that is, the metric function $m(r)=\delta(r)=0$ and the Yang-Mills potential $w=\pm 1$ at the origin. Can the nontrivial Yang-Mills field reach and extend beyond the cosmological horizon under certain conditions? If the nontrivial Yang-Mills field reaches the cosmological horizon, we then have $\mu(r_c)=0$, $E(r_c)=-V[w(r_c)]<0$ and the function $Ee^{-\delta}$ is negative semidefinite at the cosmological horizon. Obviously, when $r<3m(r)$, the right-hand side (RHS) of Eq. (2.14) is always negative. Thus, to have the asymptotically de Sitter behavior, the nontrivial Yang-Mills field must extend beyond the critical point $r_{\rm crit}$ satisfying

$$r_{\rm crit} = 3m(r_{\rm crit}). \tag{2.15}$$

Otherwise, the Eq. (2.14) cannot be satisfied by a nontrivial Yang-Mills field (that is, w is not a trivial constant). Due to the fact that the solution we are considering has only the cosmological horizon and the black hole horizon is absent, the critical point (2.15) in fact is inside the cosmological horizon. Indeed, the data in Ref. [25] showed this fact. In other words, there is no obstacle for the nontrivial Yang-Mills field reaching and extending beyond the cosmological horizon. This was already showed numerically in [25].

(ii) Colored black hole. In this case, when the cosmological constant is positive, there may exist not only the black hole horizon, but also the cosmological horizon, while only the black hole horizon is present, when the cosmological constant is negative. Inspecting Eq. (2.14), we can see clearly that the cosmological constant does not appear explicitly. Note that the function $Ee^{-\delta}$ is still negative semidefinite at the black hole horizon (and cosmological horizon if $\Lambda>0$). The condition (2.15) remains unchanged in order to have the correct asymptotic behavior of solutions. But it should be pointed out that now the critical point (2.15) is outside the black hole horizon. That is, if the nontrivial Yang-Mills field reaches the black hole horizon, then it must extend beyond the critical point (2.15). The size of the hairosphere will be discussed in the next section. Therefore, for asymptotically (anti–)de Sitter colored black holes, the no short hair conjecture of NQS keeps valid.

So far we have seen that for the asymptotically (anti–)de Sitter colored black holes, the nontrivial Yang-Mills field at the black hole horizon must extend beyond a critical point satisfying Eq. (2.15). In fact, Eq. (2.15) is still a universal condition for those theories allowing the haired black holes when a cosmological constant is introduced. For Q stars and boson stars, however, the matter fields cannot reach the cosmological horizon. In the next section, we will discuss the general case.

III. NO SHORT HAIR CONJECTURE AND COSMOLOGICAL CONSTANT

On the basis of the investigation of all black holes with hairs discovered in the different theories in recent years, Núñez, Quevedo, and Sudarsky [10] have found that the region with nontrivial structure of the nonlinear matter fields must extend beyond 3/2 the horizon radius, being independent of all other parameters in this theory. Further they have argued that this is a universal lower bound for asymptotically flat black holes and the matter satisfying the following conditions: (i) The weak energy condition holds; (ii) The energy density ρ falls to zero faster than r^{-4}; (iii) The trace of the stress-energy tensor is negative. Based on this observation, they have put forward the no short hair conjecture to replace the original no hair theorem.

In this section we would like to show that the presence of the cosmological constant does not essentially affect the no short hair conjecture. We will still work in the spherically symmetric metric (2.2). The Einstein equations with a cosmological constant are

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=8\pi T_{\mu\nu}, \tag{3.1}$$

where $T_{\mu\nu}$ represents the stress-energy tensor of the matter fields. In the metric (2.2), Eqs. (3.1) give

$$\delta'(r)=\frac{4\pi r}{\mu}(T^t{}_t-T^r{}_r), \tag{3.2}$$

$$\mu'(r)=r(8\pi T^t{}_t-\Lambda)+\frac{1-\mu}{r}. \tag{3.3}$$

Equation (3.3) can be rewritten as

$$m'(r)=-4\pi r^2 T^t{}_t. \tag{3.4}$$

With the help of the conservation equations of matter fields $T^{\mu}{}_{\nu;\mu}=0$ and Eqs. (3.2) and (3.3), we obtain

$$\begin{aligned} e^{\delta}(e^{-\delta}r^4T^r{}_r)'&=\frac{r^3}{2\mu}[(3\mu+\Lambda r^2-1)(T^r{}_r-T^t{}_t)+2\mu T]\\ &=\frac{r^3}{\mu}\left[\left(1-\frac{3m}{r}\right)(T^r{}_r-T^t{}_t)+\mu T\right], \end{aligned} \tag{3.5}$$

where T denotes the trace of the stress-energy tensor. Now we discuss the Eq. (3.5) under the same assumption as in Ref. [10], that is, matter fields satisfy the weak energy condition, the energy density, $\rho\equiv -T^t{}_t$, goes to zero faster than r^{-4} and the trace of the stress-energy tensor, T, is always negative. Here the weak energy condition implies that ρ is positive semidefinite and $|T^r{}_r|\leqslant -T^t{}_t$. From the regularity of horizons and the finiteness of matter fields on the horizons, inspecting (3.5) we must have

$$T^r{}_r-T^t{}_t=0 \quad \text{at} \ \ r=R_h, \tag{3.6}$$

where R_h stands for a cosmological or black hole horizon location.

First we consider the case, where the only cosmological horizon is present at r_c, that is, a soliton solution surrounded by a cosmological horizon. For all the theories allowing haired black holes such as the Einstein-Yang-Mills theory, Einstein-Skyrme theory, Einstein-Yang-Mills-dilaton theory with or without an additional potential term, Einstein-Yang-Mills-Higgs theory, Einstein-non-Abelian-Procca theory (the quantity $T^r{}_r-T^t{}_t$ of these theories is given in Ref. [10]), self-gravitating global monopoles [26] and gauge monopoles [27], one has

$$T^r{}_r-T^t{}_t=\mu P, \tag{3.7}$$

where P is a positive semidefinite function of r. Hence all these theories satisfy the condition (3.6) and then Eq. (3.5) becomes

$$e^{\delta}(e^{-\delta}r^4T^r{}_r)'=r^3\left[\left(1-\frac{3m}{r}\right)P+T\right]. \tag{3.8}$$

Note that the matter fields satisfy the weak energy condition and Eq. (3.6). The function $e^{-\delta}r^4T^r{}_r$ is negative semidefinite at the cosmological horizon. On the other hand, note that the RHS of Eq. (3.8) is negative semidefinite if $r<r_{\rm crit}$, where

$$r_{\rm crit}=3m(r_{\rm crit}), \tag{3.9}$$

and the function $e^{-\delta}r^4T^r{}_r$ must asymptotically approach zero as $r\to\infty$ so that the solution is asymptotically de Sitter. Therefore if the nontrivial matter fields reach the cosmological horizon, they must satisfy the critical point relation (3.9). Thus we obtain the condition (2.15) for the general case. Because the nontrivial Yang-Mills field can reach and extend beyond the cosmological horizon, we have no reasons to doubt that other nonlinear matters mentioned above cannot reach the cosmological horizon. Thus we conclude that the cosmological horizon can support the nontrivial nonlinear matter fields.

Equations (3.5) and (3.7) and the property of the cosmological horizon lead to the following theorem.

Theorem 1: In the spherically symmetric, asymptotically (anti–)de Sitter black hole spacetime with matter fields satisfying the weak energy condition, the energy density going to zero faster than r^{-4}, and the trace of stress-energy tensor being nonpositive, if the nontrivial matter configuration reaches the black hole horizon, it must extend beyond a universal critical point satisfying $r_{\rm crit}=3m(r_{\rm crit})$, where $m(r)$ is the mass function in the metric. Or the no short hair conjecture keeps valid for asymptotically (anti–)de Sitter black holes.

Proof: For the asymptotically anti–de Sitter black hole solution, we have

$$\mu(r_b)=0, \quad \text{and} \ \ \mu(r)>0 \quad \text{for} \ \ r>r_b, \tag{3.10}$$

and $\delta(r)$ is finite at the horizon and approaches zero as $r\to\infty$, where r_b denotes the black hole horizon. For the asymptotically de Sitter solution, we have

$$\mu(r_b)=\mu(r_c)=0, \tag{3.11}$$

and

$$\mu(r)>0 \quad \text{for } r_b<r<r_c, \quad \text{and } \mu(r)<0 \quad \text{for } r>r_c, \tag{3.12}$$

where r_c represents the cosmological horizon. Note that the function $e^{-\delta}r^4T^r{}_r$ is always negative semidefinite at the black hole horizon (and cosmological horizon if $\Lambda>0$) and the cosmological constant does not appear explicitly in Eq. (3.8). To have the required asymptotic behavior, the function $e^{-\delta}r^4T^r{}_r$ must go to zero as $r\to\infty$. Therefore the nontrivial matter fields must extend beyond the critical point (3.9), if they reach the black hole horizon.

Here it is worth stressing that from mathematical expressions, our result (3.9) is completely the same as the one of NQS. Note that $m'(r)\geqslant 0$ because of the positive semidefinite energy density and requiring the asymptotic flatness of spacetime, they could further write down:

$$r>r_{\rm crit}=3m(r_{\rm crit})>3m(r_b)=\tfrac{3}{2}r_b. \tag{3.13}$$

Therefore they can assert that the nontrivial matter fields must extend beyond 3/2 the horizon radius of black holes. For our case, because the horizons are determined by the equation $\mu(r)=1-2m/r-\Lambda r^2/3=0$, we cannot obtain the last equality in Eq. (3.13). But in general, we can say that the positive cosmological constant widens the hairosphere, while the hairosphere thins for a negative cosmological constant, compared to the one in the asymptotically flat black hole solutions [10]. From our result, furthermore, we can see that the essence of the no short hair conjecture of black holes keeps valid for the asymptotically (anti–)de Sitter black holes. In addition, we would like to point out that although the cosmological constant does not come explicitly into the critical point relation (3.9), the cosmological constant has effects on the black hole hairs. Adding a positive (negative) cosmological constant corresponds to adding a repulsive (attractive) force. If the cosmological constant is too large, one would have no equilibrium configurations [25].

In recent years, there has been considerable interest in two kinds of stars: Q stars and boson stars [28–30]. When a positive cosmological constant is added to these theories, it is expected that these regular stars would be surrounded by a cosmological horizon. Can the matter fields constructing the star reach the cosmological horizon? Note that a conserved particle number associated with the Noether current appears always in these theories. A system of self-gravitating real massive scalar field does not admit regular static solutions, since there is no conserved current leading to particle number conservation. Consider a general complex scalar Φ with a self-interacting potential $U(\Phi^*\Phi)$. Its Lagrangian is

$$L_{\rm matter}=-g^{\mu\nu}\partial_\mu\Phi^*\partial_\nu\Phi-U(\Phi^*\Phi), \tag{3.14}$$

which has a conserved current

$$j^\mu=-i\sqrt{-g}g^{\mu\nu}(\Phi^*\partial_\nu\Phi-\Phi\partial_\nu\Phi^*), \tag{3.15}$$

and a conserved particle number

$$N=\int d^3x j^0. \tag{3.16}$$

The Noether charge prevents the star from diffusion. From Eq. (3.14) the stress-energy tensor is

$$T_{\mu\nu}=\partial_\mu\Phi^*\partial_\nu\Phi+\partial_\mu\Phi\partial_\nu\Phi^* -g_{\mu\nu}[g^{\alpha\beta}\partial_\alpha\Phi^*\partial_\beta\Phi+U(\Phi^*\Phi)]. \tag{3.17}$$

For static, spherically symmetric configurations, the complex scalar field has the form $\Phi=\phi(r)e^{-i\omega t}$, where ω is a nonzero constant. In the metric (2.2), from Eq. (3.17) we have

$$T^r{}_r-T^t{}_t=2\mu(r)\phi'(r)^2+2\omega^2\mu(r)^{-1}e^{2\delta}\phi(r)^2, \tag{3.18}$$

from which we see clearly that $T^r{}_r-T^t{}_t\neq 0$ unless $\phi(r)=0$ at the cosmological horizon. Considering the requirement (3.6), we conclude that the scalar field in the boson stars cannot reach the cosmological horizon. Boson soliton stars [29] and Q stars [31] are two kinds of nontopological soliton stars. The difference between soliton stars and general boson stars is that in the absence of gravitational field, the soliton stars reduce to nontopological solitons. For the general boson stars the theory has no soliton solution. That is, the choice $U(\Phi^*\Phi)$ is very different in the different kinds of stars. But there are generally the complex scalar fields like Φ in the boson soliton stars and Q stars in order to have a conserved charge. These matter configurations in Q starts and boson soliton stars therefore cannot reach the cosmological horizon, either, because the potential $U(\Phi^*\Phi)$ does not appear in Eq. (3.18). In addition, adding other matter fields such as, fermion field, Maxwell field, etc., to these theories cannot change this result. This point can be seen from Eq. (3.18). In Ref. [26] Kastor and Traschen have argued the possibility of having black hole horizons inside various classical field configurations by using the condition (3.6) resulting from the Oppenheimer-Volkoff equation of hydrostatic equilibrium. The Q stars and boson stars do not allow for black hole horizon inside them. Therefore it further shows that the cosmological horizon and black hole horizon share similar properties from the viewpoint of having hairs on them.

IV. NO SCALAR HAIR THEOREM AND COSMOLOGICAL CONSTANT

Nowadays there are some methods to show that for the Einstein-minimally coupled real scalar field system with a positive semidefinite potential, the static, spherically symmetric, and asymptotically flat solution is only the Schwarzschild solution with a constant scalar field corresponding to a zero point of the potential. This just is the no scalar hair theorem of black holes [17,19,20]. In these proofs the condition of asymptotic flatness plays an important role. When a cosmological constant is present, this condition is lost. Can one still prove the no scalar hair theorem? In this section we discuss this problem.

Consider the following action:

$$S=\int d^4x\sqrt{-g}\left(\frac{1}{16\pi}(R-2\Lambda)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)\right), \tag{4.1}$$

where $V(\phi)$ is a positive semidefinite potential of the scalar field ϕ. Varying this action, we have the equations of motion

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}$$
$$=8\pi\{\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}[\tfrac{1}{2}(\nabla\phi)^2+V(\phi)]\}, \tag{4.2}$$

$$\nabla^2\phi=\frac{\partial V(\phi)}{\partial\phi}. \tag{4.3}$$

In the metric (2.2) the Einstein equations (4.2) reduce to

$$\delta'(r)=-4\pi r\phi'^2, \tag{4.4}$$

$$m'(r)=4\pi r^2[\tfrac{1}{2}\mu(r)\phi'^2+V(\phi)]. \tag{4.5}$$

The asymptotic condition requires that

$$\lim_{r\to\infty}\phi'^2\sim O(1/r^{6+\varepsilon}),\quad \text{and}\quad \lim_{r\to\infty}V(\phi)\sim O(1/r^{4+\varepsilon}), \tag{4.6}$$

where ε is a positive small quantity. Comparing with the case of asymptotic flatness [19], we find that ϕ'^2 is required to fall off faster than the one in asymptotically flat spacetime. The equation of motion for the scalar field (4.3) becomes

$$(\mu\phi')'+\left(\frac{2}{r}-\delta'\right)\mu\phi'=\frac{\partial V}{\partial\phi}. \tag{4.7}$$

Following Sudarsky [19], we multiply Eq. (4.7) by ϕ' to obtain

$$\left[\frac{1}{2}\mu\phi'^2\right]'+\left[\frac{1}{2}\mu'+\left(\frac{2}{r}-\delta'\right)\mu\right]\phi'^2=\frac{\partial V(\phi)}{\partial\phi}\phi'. \tag{4.8}$$

Let us study the behavior of solutions for the cases of the asymptotically anti–de Sitter $(\Lambda<0)$ and the asymptotically de Sitter $(\Lambda>0)$, respectively.

(i) $\Lambda<0$. In this case, the solution is required to be asymptotically anti–de Sitter. We have the following theorem.

Theorem 2: In the Einstein-minimally coupled scalar field system with a positive semidefinite scalar potential and a negative cosmological constant, the static, spherically symmetric black hole solution with a regular horizon and possessing asymptotically anti–de Sitter behavior is the Schwarzschild-anti–de Sitter spacetime and the scalar field is a constant corresponding to a local extremum of this potential.

Proof: To prove this theorem, following [19] we define

$$E=\tfrac{1}{2}\mu\phi'^2-V(\phi). \tag{4.9}$$

it is easy to show

$$E'(r)=-b\phi'^2, \tag{4.10}$$

where

$$b=4\pi rE-\Lambda r+\frac{3}{2r}\left(\frac{4}{3}-\frac{2m}{r}\right). \tag{4.11}$$

With the help of Eqs. (4.9), (4.10), and (4.4), we reach

$$\frac{d}{dr}(Ee^{-\delta})=\left[\Lambda r-\frac{3}{2r}\left(\frac{4}{3}-\frac{2m}{r}\right)\right]e^{-\delta}\phi'^2. \tag{4.12}$$

Noting that $\Lambda<0$, it is obvious that the RHS of Eq. (4.12) is always negative for $r>r_b$, where r_b represents the location of the black hole horizon. Therefore the function $Ee^{-\delta}$ should be a decreasing function for $r>r_b$. From Eq. (4.9) we also note that $E(r_b)=-V[\phi(r_b)]<0$. Therefore $Ee^{-\delta}(r>r_b)$ is always more negative than $Ee^{-\delta}(r_b)$. However the asymptotically anti–de Sitter solution satisfying the asymptotic behavior (4.6) requires that $\lim_{r\to\infty}Ee^{-\delta}=0$. Hence the only solution is $\phi'=0$ and $V(\phi)=0$ throughout the spacetime. In fact, the constant scalar can correspond to a local extremum of the potential satisfying $\partial V(\phi)/\partial\phi=0$, because one can absorb the extremum of the potential to the cosmological constant so that the cosmological constant is an effective one and the potential becomes an effective potential. Thus the above proof continues to hold. We will discuss this point later.

(ii) $\Lambda>0$. This is another story due to the different asymptotic behavior. On the one hand, the cosmological horizon may appear. On the other hand, obviously, the RHS of Eq. (4.12) is positive semidefinite as r is larger than a critical value. Thus we have no way to rule out the possibility of having scalar hairs in this method. If defining

$$E(r)=\frac{1}{2}\mu\phi'^2-V(\phi)-\frac{1}{4\pi}\Lambda, \tag{4.13}$$

we have

$$\frac{d}{dr}(Ee^{-\delta})=-\frac{3}{2r}\left(\frac{4}{3}-\frac{2m}{r}\right)e^{-\delta}\phi'^2. \tag{4.14}$$

The RHS of Eq. (4.14) now is always negative for $r>r_b$. This indicates that the function $Ee^{-\delta}$ should be a decreasing function for $r>r_b$. From Eq. (4.13) we note that $E(r_b)=-V[\phi(r_b)]-\Lambda/4\pi<0$ and $E(r_c)=-V[\phi(r_c)]-\Lambda/4\pi<0$. Therefore $Ee^{-\delta}(r>r_b)$ is always negative semidefinite for $r\geqslant r_b$. Note that $\lim_{r\to\infty}Ee^{-\delta}=-\Lambda/4\pi$ in contrast to the cases of the asymptotically flat and asymptotically anti–de Sitter spacetimes, where $\lim_{r\to\infty}Ee^{-\delta}=0$. Thus, for asymptotically de Sitter black holes, we still have no reason to rule out the scalar hair and hence the Sudarsky's method does not work. In fact, both the Bekenstein's method proving his novel no scalar hair theorem [17] and the scaling techniques [20] do not work in this case, either. But by using Bekenstein's identity method [6], we can show the following theorem.

Theorem 3: In the Einstein-minimally coupled scalar field system with a positive semidefinite, convex scalar potential and a positive cosmological constant, the spherically symmetric, asymptotically de Sitter solution (with a cosmological horizon and no black hole horizon) and the spherically symmetric, asymptotically de Sitter black hole solution (with a regular black hole horizon and a cosmological horizon) are the de Sitter space and the Schwarzschild-de Sitter black hole, respectively. And the scalar field is a constant corresponding to the point of the minimum of this potential.

Proof: Multiplying Eq. (4.3) by $(\phi-\phi_0)$ and integrating from r_0 to r_c, we have

$$r^2 e^{-\delta}\mu(r)\phi'(\phi-\phi_0)|_{r_0}^{r_c}=\int_{r_0}^{r_c} dr\ r^2 e^{-\delta} \times\left(\mu(r)\phi'^2+(\phi-\phi_0)\frac{\partial V}{\partial\phi}\right). \tag{4.15}$$

Here ϕ_0 is a constant which gives the minimum value of the potential V, r_c denotes the cosmological horizon and r_0 represents the origin for the asymptotically de Sitter solution and black hole horizon r_b for de Sitter black holes. The left-hand side (LHS) of Eq. (4.15) always vanishes: For the black hole solution, the metric function $\mu=0$ both at the black hole and cosmological horizons; for the asymptotically de Sitter solution, $\mu=0$ at the cosmological horizon and $\phi'=0$ at the origin, which comes from the regularity requirement. If the scalar potential $V(\phi)$ is convex, both the two terms of integrand of Eq. (4.15) are then positive semidefinite in the region between $r_0\leqslant r\leqslant r_c$. Therefore the only solution is $\phi=\phi_0$, and the metrics are the Schwarzschild-de Sitter and the de Sitter space (as a special case of the Schwarzschild-de Sitter metric), respectively.

When the solution is an asymptotically anti–de Sitter one, if we set that r_0 is the black hole horizon and r_c is asymptotic infinity, then the LHS of Eq. (4.15) still vanishes: $\mu(r)=0$ at the horizon and $\phi'=0$ at infinity. And the integrand of Eq. (4.15) is still positive semidefinite. Thus the only solution is the Schwarzschild-anti–de Sitter spacetime with a constant scalar field. In a word, the Bekenstein's identity method can exclude the scalar hair of spherically symmetric (anti–)de Sitter black holes and the de Sitter space if the scalar potential is convex. In fact, this method can also exclude the scalar hair for the Kerr-(anti–)de Sitter black holes.

V. CONCLUSION AND DISCUSSION

In this work we have investigated the possibility of having hairs on the cosmological horizon and the effects of a cosmological constant on the no short hair conjecture and no scalar hair theorem of black holes. From the viewpoint of having hairs on horizons, the cosmological horizon shares similar properties of the black hole horizon. For the theories admitting haired black hole solutions, the nontrivial matter configurations may reach and extend beyond the cosmological horizon. An explicit example is that in the Bartnik-McKinnon soliton surrounded by a cosmological horizon the Yang-Mills field can extend beyond the cosmological horizon. This has been found numerically in Refs. [24,25]. For Q stars and boson stars, the matter fields cannot reach the cosmological horizon. The no short hair conjecture of black holes remains valid, in spite of the different asymptotic behaviors (de Sitter or anti–de Sitter) of black hole solutions in the sense that the presence of the cosmological constant does not change the expression of the critical point (3.9). But we would like to point out that, although the NQS's no short hair conjecture remains valid for asymptotically (anti–)de Sitter black holes, we cannot say that the hair must extend beyond 3/2 the horizon radius of black holes, since in our case the horizons are determined by the equation, $1-2m(r)/r-\Lambda r^2/3=0$. The positive cosmological constant widens the hairosphere, while the hairosphere thins for a negative cosmological constant, compared to the one in the asymptotically flat black hole solutions [10].

For a negative cosmological constant, we have shown: For the Einstein-minimally coupled scalar field system with a positive semidefinite scalar potential, if the spacetime is static and spherically symmetric, has regular black hole horizon, and is of the asymptotically anti–de Sitter behavior, the only solution is the Schwarzschild-anti–de Sitter spacetime and the scalar field is a constant corresponding to a local extremum of the potential. For a positive cosmological constant, if the scalar potential is convex, both the cosmological horizon and black hole horizon cannot support the scalar hair, that is, the only solution is the de Sitter space or Schwarzschild-de Sitter spacetime. Therefore the no scalar hair theorem hold not only for the asymptotically flat black holes, but also for the asymptotically (anti–)de Sitter black holes.

It is a difficult task to prove no scalar hair theorem of black holes which are not necessarily spherically symmetric. By now one can prove the no scalar hair theorem for asymptotically flat black holes only when the scalar potential is convex. When the scalar potential is non-negative, in order to prove the no scalar hair theorem one must have more conditions: The spacetime is static, spherically symmetric. When the spacetime is asymptotically anti–de Sitter or de Sitter, certainly, it will become more difficult to show the no scalar hair theorem. In Sec. IV we have proved the no scalar hair theorem of anti–de Sitter black holes in the case for a minimally coupled scalar field with a positive semidefinite scalar potential and the metric being of form (2.2). For the asymptotically de Sitter solution and de Sitter black hole, however, it should be emphasized that when the scalar potential is convex, the result is always valid, in spite of whether or not the cosmological constant is present and whether or not the spacetime is spherically symmetric.

Here it should also be stressed that a *general* scalar potential which is *bounded from below* can be redefined to be positive semidefinite by absorbing the negative value into the cosmological constant. Then our cosmological constant is an effective one and it determines the asymptotic behavior of the spacetime. So in the sense of no scalar hair theorem we have shown: In the Einstein-minimally coupled scalar field system with an arbitrary scalar potential, the only spherically symmetric black hole solution is the Schwarzschild-anti–de Sitter spacetime if the effective cosmological constant is negative, or the Schwarzschild-de Sitter spacetime if the effective cosmological constant is positive and the effective potential is convex. The scalar field is a trivial constant corresponding to a local extremum of the potential [$\partial V(\phi)/\partial\phi=0$]. Since the hair is a characteristic of black holes, it may be closely related to the thermodynamics of black holes [32]. Therefore it is significant to study under what conditions black holes have hairs, and under what conditions black holes have no hairs for asymptotically flat black holes, asymptotically (anti−)de Sitter black holes, and even for black holes with unusually asymptotic behavior and nonspherically topological black holes. In addition, it also should be of interest to verify numerically that the cosmological horizon can support the nontrivial matter hairs in those theories admitting haired black hole solutions.

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