Scalar, vector, and tensor contributions to CMB anisotropies from cosmic defects

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Recent work has emphasised the importance of vector and tensor contributions to the large scale microwave anisotropy fluctuations produced by cosmic defects. In this paper we provide a general discussion of these contributions, and how their magnitude is constrained by the fundamental assumptions of causality, scaling, and statistical isotropy. We discuss an analytic model which illustrates and explains how the ratios of isotropic and anisotropic scalar, vector and tensor stress-energy sources are determined. This provides a check of the results from large scale numerical simulations, confirming the numerical finding that vector and tensor modes provide substantial contributions to the large angle anisotropies. We show that the qualitative features of the stress-energy tensor carry over to the microwave background anisotropies. This leads to a suppression of the scalar normalization and consequently of the Doppler peaks. [S0556-2821(98)03514-0]

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I. INTRODUCTION

The idea that the breakdown of some fundamental symmetry and the consequent field ordering might be responsible for structure formation in the universe is an attractive one. Recently we have performed the first complete calculations of the power spectra of perturbations in symmetry breaking theories, including global cosmic strings, monopoles and texture [1,2]. These calculations revealed that vector and tensor modes give a larger contribution to the large scale anisotropies than previously suspected, and that their fractional contributions to the total microwave anisotropy power spectrum are comparable for each theory considered (see Fig. 1). Simultaneous work on local strings [3] has produced compatible conclusions.

The main implication of the large vector and tensor contribution on large angular scales is in reducing the normalization of the scalar perturbations, which are responsible for the Doppler peaks. Once the vector and tensor contributions are properly included, the height of the Doppler peaks is low relative to the large angular scale Sachs-Wolfe plateau [1].

The present paper represents an analytical attempt to explain why vector and tensor contributions are substantial on large angular scales, using only the most fundamental properties of the simplest defect theories, namely scaling, causality and statistical isotropy. We illustrate the arguments through comparison with the results of numerical computations [1,4].

II. CAUSALITY AND ANALYTICITY

As discussed in [5], all perturbation power spectra are determined by the unequal time correlator (UETC) of the defect source stress energy tensor $\Theta_{\mu\nu}$:

$$\langle\Theta_{\mu\nu}(\mathbf{k},\tau)\Theta_{\rho\lambda}(-\mathbf{k},\tau')\rangle\equiv\mathcal{C}_{\mu\nu,\rho\lambda}(k,\tau,\tau') \qquad (1)$$

where τ, τ' denote conformal time, and k a comoving wave number. Note that Eq. (1) is real because complex conjugation is equivalent to the replacement $\mathbf{k}\to-\mathbf{k}$. The correlators are invariant under this replacement because the statistical ensemble is rotation invariant.

Causality means that the real space correlators of the fluctuating part of $\Theta_{\mu\nu}$ must be zero for $r>\tau+\tau'$ [5]. Scaling dictates that in the pure matter or radiation eras $\mathcal{C}_{\mu\nu,\rho\lambda}\propto\phi_0^4/(\tau\tau')^{1/2}c_{\mu\nu,\rho\lambda}(k\tau,k\tau')$, where ϕ_0 is the symmetry breaking scale and c is a dimensionless scaling function. Finally, $\Theta_{\mu\nu}$ must obey the equations for stress energy conservation with respect to the background metric (see next section). These provide two linear constraints on the four scalar components of the source. Any pair determines the other two up to possible integration constants. In the matter era the pair Θ and Θ^S [4] provides a convenient choice, allowing an analytical integral solution to the linearized Einstein equations. But for work including the matter-radiation transition [1] the pair Θ_{00} and Θ^S is better, because it results in the correct redshifting away of all components of the source stress energy inside the horizon. In this paper we shall use both pairs Θ and Θ^S in our analytical discussion of an ''incoherent'' model and Θ_{00} and Θ^S for a numericaly solved ''coherent'' model. In the former case, we shall constrain Θ and Θ^S so that on subhorizon scales the source is negligible (see Sec. IV).

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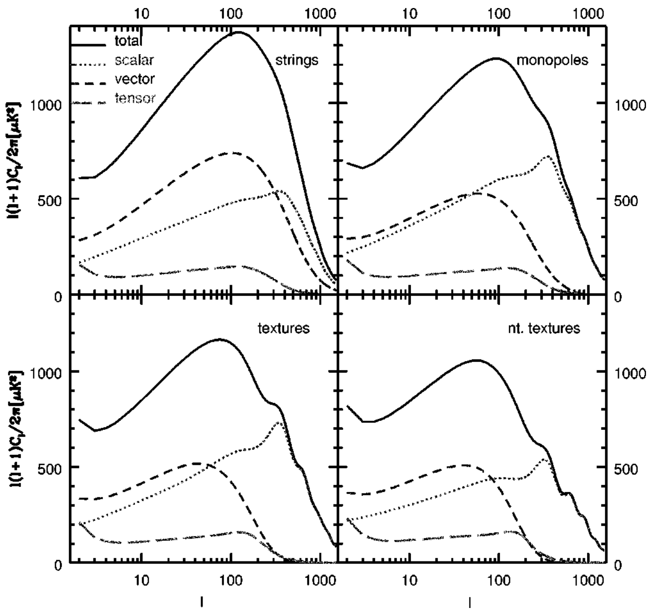

FIG. 1. The contributions to the total anisotropy power spectrum from scalar, vector and tensor components, in the theories of global strings, monopoles, texture and nontopological texture (taken from Ref. [1]).

The unequal time correlator in k space is the Fourier transform of the real space correlator: $\langle\Theta_{ij}(\mathbf{k},\tau)\Theta_{kl}(-\mathbf{k},\tau')\rangle=\int d^3\mathbf{r}e^{-i\mathbf{k}\cdot\mathbf{r}}\langle\Theta_{ij}(r,\tau)\Theta_{kl}(0,\tau')\rangle$. The integral is finite because the real space correlator has compact support, and it follows that the unequal time correlators are analytic in $\mathbf{k}$ for all finite k. They may thus be expanded as a Taylor series in the Cartesian components k_i about $k_i=0$. As k_i tends to zero, isotropy and symmetry impose

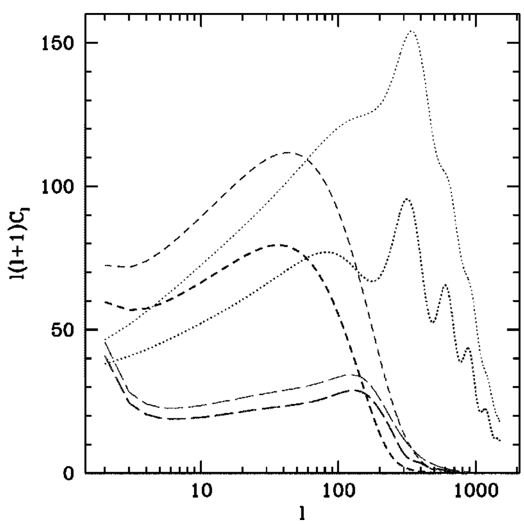

FIG. 2. The importance of the long wavelength modes in the anisotropy power spectra from cosmic textures. The power spectra due to scalar (dotted line), vector (dashed line) and tensor (long-dashed line) components of the sources are compared to those where the source stress energy components Θ_{00} and Θ^S as well as the vector and tensor stresses are set to zero for all $k\tau>5$. The upper curves show the full spectra, the lower ones the results where the cutoff is imposed.

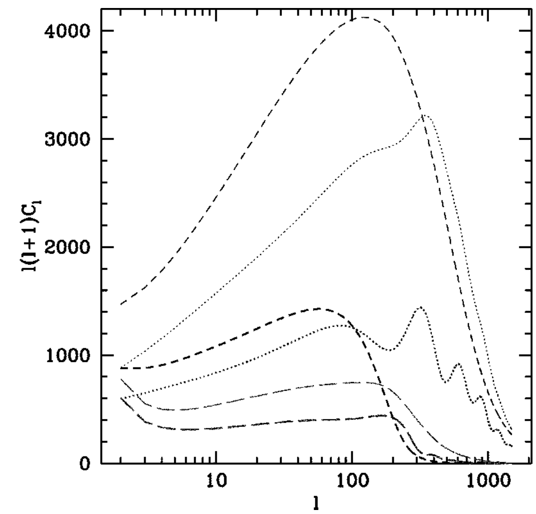

FIG. 3. As in Fig. 2 but for global strings.

$$\lim_{\mathbf{k}\to 0}\langle\Theta_{ij}(\mathbf{k},\tau)\Theta_{kl}(-\mathbf{k},\tau')\rangle=A\delta_{ij}\delta_{kl}+B(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \tag{2}$$

with A and B independent of $\mathbf{k}$.

The trace scalar, anisotropic scalar, vector and tensor components of a tensor T_{ij} are given by

$$T_{ij}(\mathbf{k})=\frac{1}{3}\delta_{ij}T+\left(\hat{k}_i\hat{k}_j-\frac{1}{3}\delta_{ij}\right)T^S+(\hat{k}_iT_j^V+\hat{k}_jT_i^V)+T_{ij}^T$$

$$T_i^Vk_i=k_iT_{ij}^T=T_{ij}^Tk_j=T_{jj}^T=0, \tag{3}$$

where $\hat{k}_i\equiv k_i/k$. Expressing the trace T, T^S, T_i^V and T_{ij}^T in terms of T_{ij} (see e.g. [4]) one finds that the only nonzero correlators consistent with statistical isotropy and homogeneity are $\langle TT\rangle$, $\langle TT^S\rangle$, $\langle T^ST^S\rangle$, $\langle T_i^VT_j^V\rangle$ and $\langle T_{ij}^TT_{kl}^T\rangle$. From Eq. (2) one can compute the small k power spectra of the anisotropic scalar, vector and tensor stresses. One finds that the equal time correlators are in the ratios

$$\langle|\Theta^S|^2\rangle:\langle|\Theta_i^V|^2\rangle:\langle|\Theta_{ij}^T|^2\rangle=3:2:4 \tag{4}$$

where all indices are summed. Thus in a causal theory anisotropic scalar, vector and tensor stresses have white noise components at small k with related amplitudes. A similar argument shows that the correlator $\langle\Theta_{00}\Theta_{ij}\rangle\sim C\delta_{ij}$ at small k, implying that $\langle\Theta_{00}\Theta^S\rangle$ vanishes like k^2 at small k. Likewise $\langle\Theta\Theta^S\rangle$ vanishes like k^2 at small k. So for either of the two choices discussed above, the two scalar source components are uncorrelated outside the horizon.

III. SUPERHORIZON MODES

In cosmic defect theories, perturbations are predominantly produced on the horizon scale. Studies show that the unequal time correlators take the predicted white noise form for $k\tau<5$ or so, and decline strongly at larger $k\tau$. To the extent that the horizon scale modes reflect the causality constraints discussed above, the latter translate into definite relations between the scalar, vector and tensor perturbation power spectra. In Figs. 2 and 3 we show the cosmic microwave background (CMB) anisotropy power spectra calculated in the cosmic global string and texture theories respectively, with and without a cutoff where we switch off the source stress tensor for $k\tau > 5$. The figures show that in the texture theory the effect of suppressing the source for $k\tau > 5$ is relatively minor. For strings, there is a larger effect, but even here the *ratios* of scalar to vector to tensor anisotropies are not much affected. We conclude that the contributions from $k\tau < 5$, which we shall term superhorizon modes, are certainly important in both theories and give at the least a rough measure of the importance of the scalar, vector and tensor contributions to the large angle anisotropies. We will discuss the dependence on the actual cutoff value $k\tau < 5$ in Sec. V.

IV. INTEGRAL CONSTRAINTS

The fact that a cutoff on subhorizon scales does not greatly affect the large angle C_l spectrum has important implications. It means that the short distance structure of the individual defects is not important in determining the qualitative character of the large angle anisotropies, such as the relative scalar, vector and tensor contributions.

Consider the effect of modelling the sources using a ''smoothed'' Θ_{00} tensor, one where we impose a cutoff at $k\tau \sim 5$. We feed in the smoothed Θ_{00} and Θ^S into the stress energy conservation equations:

$$\dot{\Theta}_{00} + \frac{\dot{a}}{a}(\Theta_{00} + \Theta) = \Pi, \quad \dot{\Pi} + 2\frac{\dot{a}}{a}\Pi = -\frac{k^2}{3}(\Theta + 2\Theta^S), \tag{5}$$

where $\Pi = \partial_i \Theta_{0i}$ and Θ and Θ^S are defined in the previous section. It is straightforward to see that the solutions for Π and Θ are well defined.

In this scheme, we deduce important relations between Θ and Θ^S. Equations (5) are easily integrated to obtain Θ_{00} in terms of Θ and Θ^S: exchanging the order of the double integral we get

$$\Theta_{00} = \tau^{-2}\left[\int_0^\tau d\tau' 2\tau'\Theta + \frac{1}{3}k^2\tau'^4\left(\frac{1}{\tau'} - \frac{1}{\tau}\right)(\Theta + 2\Theta^S)(\tau')\right], \tag{6}$$

where we used $a(\tau) \propto \tau^2$ in the matter era. But as argued, the smoothed Θ_{00} is identically zero inside the horizon. It follows that both the τ^{-2} and τ^{-3} coefficients integrate to zero. The former gives

$$\int_0^\infty d\tau'\left[2\tau'\Theta + \frac{1}{3}k^2\tau'^3(\Theta + 2\Theta^S)(k,\tau')\right] = 0, \tag{7}$$

and the latter gives

$$\int_0^\infty d\tau' \tau'^4 (\Theta + 2\Theta^S)(k,\tau') = 0. \tag{8}$$

This imposes a negative correlation between Θ and Θ^S, and guarantees that Π vanishes faster than a^{-2} inside the horizon. The constraints (7) and (8) will turn out to be remarkably powerful when building models for the superhorizon components of Θ and Θ^S.

V. PERTURBATIONS IN THE MATTER ERA

We wish to compute the large angular scale anisotropies produced in the matter era. For this purpose we use the following integral solution to the linearized Einstein equations in a matter dominated universe [4]:

$$\frac{\delta T}{T}(\mathbf{n})|_{SW} = -\frac{1}{2}\int_i^f d\tau h_{ij,0}(\tau, \mathbf{n}(\tau_0 - \tau))n^i n^j, \quad h_{ij,0} = h_{ij,0}^{\text{scalar}} + h_{ij,0}^{\text{vector}} + h_{ij,0}^{\text{tensor}}$$

$$h_{ij,0}^{\text{scalar}} = -16\pi G \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \int_0^\tau d\tau'\left[\frac{1}{3}\delta_{ij}\left(\frac{\tau'}{\tau}\right)^6 (\Theta + 2\Theta^S)(\tau', \mathbf{k}) - \hat{k}^i\hat{k}^j\left(\frac{\tau'}{\tau}\right)^4 \Theta^S(\tau', \mathbf{k})\right]$$

$$h_{ij,0}^{\text{vector}} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}(h_{i,0}^V \hat{k}_j + h_{j,0}^V \hat{k}_i), \quad h_{i,0}^V = 16\pi G\int_0^\tau d\tau'\left(\frac{\tau'}{\tau}\right)^4 \Theta_i^V(\eta')$$

$$h_{ij,0}^{\text{tensor}}(\tau, \mathbf{x}) = 16\pi G\int_0^\tau d\tau' k^3 \tau'^4 [G_1(\tau')\dot{G}_2(\tau) - G_2(\tau')\dot{G}_1(\tau)]\Theta_{ij}^T(\tau'\mathbf{x})$$

$$G_1(\tau) = \frac{\cos(k\tau)}{(k\tau)^2} - \frac{\sin(k\tau)}{(k\tau)^3}, \quad G_2(\tau) = \frac{\cos(k\tau)}{(k\tau)^3} + \frac{\sin(k\tau)}{(k\tau)^2}, \tag{9}$$

where G_1 and G_2 are the two homogenous solutions to the tensor (gravity wave) equation.

The model we shall consider is one in which the components of Θ_{ij} have the following unequal time autocorrelators:

$$(16\pi G)^2\langle\Theta(\mathbf{k},\tau)\Theta(-\mathbf{k},\tau')\rangle=\theta(\epsilon-k\tau)\delta(\tau-\tau')\mathcal{A}$$

$$(16\pi G)^2\langle\Theta^{S,V,T}(\mathbf{k},\tau)\Theta(-\mathbf{k},\tau')^{S,V,T}\rangle$$
$$=\theta(\epsilon-k\tau)\delta(\tau-\tau')\mathcal{A}^{S,V,T} \qquad (10)$$

where θ is the Heaviside function, and we define $\langle\Theta^T(\tau)\Theta^T(\tau')\rangle\equiv\frac{1}{4}\langle\Theta^T_{ij}(\tau)\Theta^T_{ij}(\tau')\rangle$ and $\langle\Theta^V(\tau)\Theta^V(\tau')\rangle\equiv\frac{1}{2}\langle\Theta^V_i(\tau)\Theta^V_i(\tau')\rangle$, with all indices summed.

The sources are nonzero only on ''superhorizon'' scales ($k\tau<\epsilon$) and they are uncorrelated except at equal times. This latter property means that the model is ''totally incoherent,'' in the terminology of Ref. [6]. These correlators are not strictly causal—in real space they take the form $r^{-3}[\sin x - x\cos x]$ where $x=\epsilon r/\tau$—but they are small and oscillatory beyond $r\sim\tau$ for $\epsilon=5$. So the violations of causality are small. Rotational invariance forbids any cross correlation between scalar, vector or tensor modes. There is, however, one more allowed cross correlator, namely that between the isotropic and anisotropic stresses. The argument given in Sec. III implies that $\langle\Theta\Theta^S\rangle$ vanishes as k^2 for small k, but it cannot be zero because of the constraint (8). We choose to model it as

$$(16\pi G)^2\langle\Theta(\mathbf{k},\tau)\Theta^S(-\mathbf{k},\tau')\rangle$$
$$=\theta(\epsilon-k\tau)\delta(\tau-\tau')\left(\frac{k\tau}{\epsilon}\right)^2\mathcal{A}^{\Theta S}. \qquad (11)$$

If we now compute the equal time correlator of the constraint (8), we determine

$$\mathcal{A}^{\Theta S}=-\frac{11}{36}(\mathcal{A}+4\mathcal{A}^S). \qquad (12)$$

Similarly we compute the equal time correlator of Eq. (7) and obtain

$$\left(\frac{4}{3}+\frac{4}{15}\epsilon^2+\frac{1}{63}\epsilon^4\right)\mathcal{A}+\left(\frac{8}{21}\epsilon^2+\frac{4}{81}\epsilon^4\right)\mathcal{A}^{\Theta S}+\frac{4}{63}\epsilon^4\mathcal{A}^S=0. \qquad (13)$$

These equations yield $\mathcal{A}^{\Theta S}=-2.47\mathcal{A}^S$ and $\mathcal{A}=4.07\mathcal{A}^S$ for $\epsilon=5$. At $k\tau=5$, there is a mild inconsistency with the bound $\mathcal{A}^{\Theta S}<\sqrt{\mathcal{A}\mathcal{A}^S}$; so we shall adopt $\mathcal{A}^{\Theta S}=-2\mathcal{A}^S$ and $\mathcal{A}=4\mathcal{A}^S$.

VI. DELTA FUNCTION APPROXIMATION

The procedure is simple in principle: the correlators (10) translate into correlators of the metric perturbations and thus into correlators of the temperature perturbations, equivalent to the anisotropy power spectrum C_l. But in order to compute the relevant integrals analytically, we shall make two approximations. The first is that we shall replace metric perturbation unequal time correlators under τ integrals using the following formula:

$$\langle\dot{A}(\tau)\dot{B}(\tau')\rangle\rightarrow\delta(\tau-\tau')\frac{d}{d\tau}\langle A(\tau)B(\tau)\rangle. \qquad (14)$$

The weighting function is chosen so that the integrals $\int_0^{\tau_f}d\tau\int_0^{\tau_f}d\tau'$ of both sides are guaranteed to be equal for all τ_f. The formula is also invariant under changing variables from τ to any other function $f(\tau)$. The second approximation is to use the fact that the Green's functions in Eqs. (9) fall off strongly with τ. This means that the metric perturbations fall off rapidly beyond $k\tau=\epsilon$, which justifies us simply setting them zero beyond that point.

VII. CMB ANISOTROPIES

In the usual way we expand the microwave sky temperature in spherical harmonics $\delta T/T=\Sigma a_{lm}Y_{lm}(\theta,\phi)$, and compute $C_l=\langle|a_{lm}|^2\rangle$. The formula for the contribution to the integrated Sachs-Wolfe effect from trace scalar and anisotropic scalar contributions is [7]

$$C_l^{scalar}=\frac{1}{2\pi}\int_0^\infty k^2dk\left\langle\left[\int_0^{\tau_0}d\tau\left(\frac{1}{3}\dot{h}_1(\tau)+\dot{h}_2(\tau)\frac{d^2}{d(k\Delta\tau)^2}\right)j_l(k\Delta\tau)\right]^2\right\rangle \qquad (15)$$

where $\Delta\tau=\tau_0-\tau$, τ_0 is the conformal time today and, as above, $\langle\cdots\rangle$ denotes ensemble averaging. The scalar metric perturbation components are given from Eqs. (9):

$$\dot{h}_1=-16\pi G\int d\tau'(\tau'/\tau)^6(\Theta+2\Theta^S)(\tau')$$

$$\dot{h}_2=-16\pi G\int d\tau'(\tau'/\tau)^4\Theta^S(\tau') \qquad (16)$$

with $\mathbf{k}$ dependence implicit.

The vector and tensor contributions to C_l are [7]

$$C_l^V=\frac{2}{\pi}\int_0^\infty k^2dk\,l(l+1)\times\left\langle\left[\int_0^{\tau_0}d\tau\dot{h}^V(\tau)\frac{d}{d(k\Delta\tau)}[j_l(k\Delta\tau)/k\Delta\tau]\right]^2\right\rangle$$

$$C_l^T=\frac{1}{2\pi}\int_0^\infty k^2dk\frac{(l+2)!}{(l-2)!}\left\langle\left[\int_0^{\tau_0}\frac{d\tau}{k^2\Delta\tau^2}\dot{h}^T(\tau)j_l(k\Delta\tau)\right]^2\right\rangle \qquad (17)$$

where $\langle\dot{h}^T(\tau)\dot{h}^T(\tau')\rangle\equiv\frac{1}{4}\langle\dot{h}^T_{ij}(\tau)\dot{h}^T_{ij}(\tau')\rangle$ and $\langle\dot{h}^V(\tau)\dot{h}^V(\tau')\rangle\equiv\frac{1}{2}\langle\dot{h}^V_i(\tau)\dot{h}^V_i(\tau')\rangle$, with all indices summed.

We now compute the relevant metric perturbation correlators: from Eqs. (10), (12) and (16), using $\mathcal{A}^{\Theta S}=-2\mathcal{A}^S$ and $\mathcal{A}=4\mathcal{A}^S$ as discussed above, we obtain

$$\langle h_1(\tau)^2\rangle \approx \frac{1}{40}\tau^3\mathcal{A}^S$$

$$\langle h_2(\tau)^2\rangle = \frac{1}{81}\tau^3\mathcal{A}^S$$

$$\langle h_1(\tau)h_2(\tau)\rangle \approx \frac{1}{217}\tau^3\mathcal{A}^S$$

$$\langle h^V(\tau)^2\rangle = \frac{1}{81}\tau^3\mathcal{A}^V$$

$$\langle h^T(\tau)^2\rangle = \int_0^\tau d\tau' G(\tau,\tau')^2\mathcal{A}^T$$

$$G(\tau,\tau') = k^3\tau'^4[G_1(\tau')G_2(\tau) - G_2(\tau')G_1(\tau)] \qquad (18)$$

where we have evaluated the scalar correlators at $k\tau=5$, and the tensor modes G_1 and G_2 are given in Eqs. (9). The tensor integral is straightforwardly performed, yielding

$$\langle h^T(\tau)^2\rangle = \frac{\mathcal{A}^T}{60k^3z^6}\left(\frac{105}{2}(1-z^2)\sin(2z) - 105z\cos(2z) + 6z^7 - 14z^5 - 35z^3\right) \qquad (19)$$

where $z=k\tau$. This function is $\sim\mathcal{A}^T\tau^3/81$ at small $k\tau$, identical to the vector expression. But for larger $k\tau$ it is suppressed, with the suppression factor being ≈ 0.29 at $k\tau=5$. The suppression is due to the oscillatory nature of the tensor compared to the vector response.

We now compute the integrals in Eqs. (17), starting with the tensor contribution C_l^T. The delta function allows one of the τ integrations to be performed. Then we change variables from k to $x=k\Delta\tau$. The Heaviside function gives the upper limit $x<\epsilon(\tau_0/\tau-1)$ or $\tau<\tau_0/(1+x/\epsilon)$. Exchanging orders of the integrals we find

$$C_l^T = \frac{1}{2\pi}\frac{(l+2)!}{(l-2)!} \times \int_0^\infty \frac{dx}{x^2} j_l^2(x) \int_0^{\tau_0/(1+x/\epsilon)} \frac{d\tau}{(\tau_0-\tau)^3}\frac{d}{d\tau}\langle h^T(\tau)^2\rangle. \qquad (20)$$

For large l the integral is dominated by large x, since $j_l(x) \sim x^l$ at small x. But at large x, $\tau \ll \tau_0$ and the τ integral is trivial. Thus one finds, at large l,

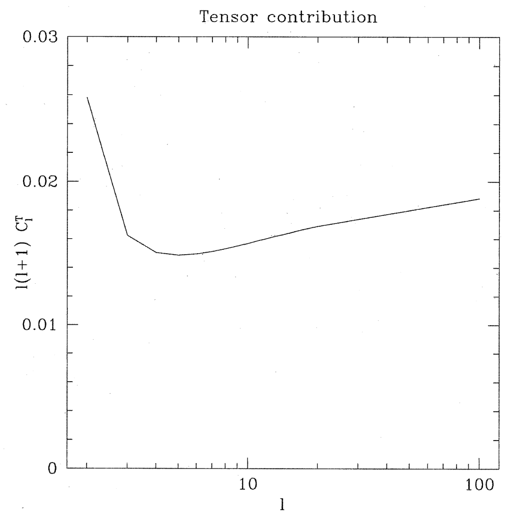

FIG. 4. Tensor anisotropy power spectrum as computed in the analytical model presented here.

$$C_l^T \sim 0.29\frac{\epsilon^3}{81}\mathcal{A}^T\frac{l^4}{2\pi}\int_0^\infty \frac{dx}{x^5}j_l^2(x) \sim 0.29\frac{\epsilon^3}{\pi}\mathcal{A}^T\frac{2}{1215l^2} + o(l^{-3}), \qquad (21)$$

where we have used

$$\int dx j_l^2(x)x^n = \frac{\pi}{2^{2-n}}\frac{\Gamma(1-n)\Gamma\left(l+\frac{1}{2}+\frac{n}{2}\right)}{\Gamma\left(1-\frac{n}{2}\right)^2\Gamma\left(l+\frac{3}{2}-\frac{n}{2}\right)}. \qquad (22)$$

We have computed the integral for the tensor contribution (20) at low l using MATHEMATICA, to check that the model reproduces the shape of $l(l+1)C_l$ seen in the plots of Fig. 1. Figure 4 confirms that this is indeed the case.

The vector integral is performed similarly, to obtain

$$C_l^V = \frac{1}{27}\mathcal{A}^V\frac{2l(l+1)}{\pi}\int_0^\infty dx x^2[(j_l(x)/x)']^2 \times\left[\frac{\epsilon^2}{2x^2} - \frac{\epsilon}{x} + \ln(1+\epsilon/x)\right], \qquad (23)$$

where prime denotes differentiation with respect to x. Approximating the expression in the square brackets with its leading large x behavior, integrating by parts and using Bessel's equation for $j_l(x)$, we obtain, at large l,

$$C_l^V \sim \mathcal{A}^V\frac{2\epsilon^3}{1215\pi l^2} + o(l^{-3}). \qquad (24)$$

The scalar contribution is evaluated using Eqs. (14) and (16), again making the large x approximation, giving

$$C_l^{scalar} = \mathcal{A}^S \frac{1}{2\pi}\int_0^\infty x^2 dx \times \left(\frac{1}{120} j_l^2 + \frac{1}{81} j_l{'}^2 + \frac{4}{217} j_l j_l''\right)(x)\frac{\epsilon^3}{2x^3}. \quad (25)$$

After integration by parts and using Bessel's equations we get

$$C_l^{scalar} \approx \frac{1}{1248}\mathcal{A}^S \frac{\epsilon^3}{\pi l^2} + o(l^{-3}). \quad (26)$$

Now the amplitudes $\mathcal{A}^S$, $\mathcal{A}^V$ and $\mathcal{A}^T$ are related via Eq. (4) in the ratio 3:1:1. Thus in the various approximations we have made, the ratio of the scalar to vector to tensor contributions to the large angle anisotropies is

$$C_l^{scalar} : C_l^V : C_l^T = 1.46:1:0.29 \quad (27)$$

which is our main result. The calculation demonstrates the relative importance of the vector and tensor modes, consistent with the numerical results shown in Fig. 1. Given the crude nature of the model used, the agreement is actually surprisingly good. The weakest point in the model is that it involves a free parameter ϵ, and the C_l's obtained are proportional to ϵ^3. It seems plausible that ϵ should be the same for the scalar, vector and tensor stresses, but we have not found any argument as to why this should necessarily be true.

Let us summarize the approximations and assumptions implicit in the ratio (27):

(1) We assumed that superhorizon modes with $k\tau < 5$ dominate, and that this cutoff is universal for each vector, scalar and tensor. This is generally observed in the simulations.

(2) We modelled the unequal time correlators as delta functions with a horizon scale cutoff.

(3) We made the approximation of pure matter domination for the background spacetime. In this approximation the C_l spectra obtained are scale invariant at large l, which is accurate for the large scale anisotropies

(4) We replaced certain functions with amplitudes times delta functions in order to perform the relevant integrations.

With all of these caveats, we feel that the model provides useful insight into the relative importance of scalar, vector and tensor contributions to the large angle anisotropies. The model explains why vector perturbations dominate over tensors, and why the combined vector and tensor contribution is comparable to that from scalars. The domination of vectors over scalars seen in the full defect simulations (Figs. 1–3) is not reproduced by this simple model. Of the assumptions stated above, the second is the weakest link. Real defects may have different coherence times for the scalar, vector and tensor modes. In the coherent eigenmode factorization [1], we indeed observe that scalar modes are less coherent than vector or tensor modes, while vector and tensor modes have very similar coherence properties. Thus we would expect the vector to tensor ratio to be an accurate prediction of this model, which is indeed observed in Fig. 2. The equal time correlators are constrained by Eq. (4). The total contribution of a component is basically proportional to its amplitude times its duration; so the shorter coherence would reduce the scalar contribution relative to the vector and tensor. This agrees qualitatively with the observed results.

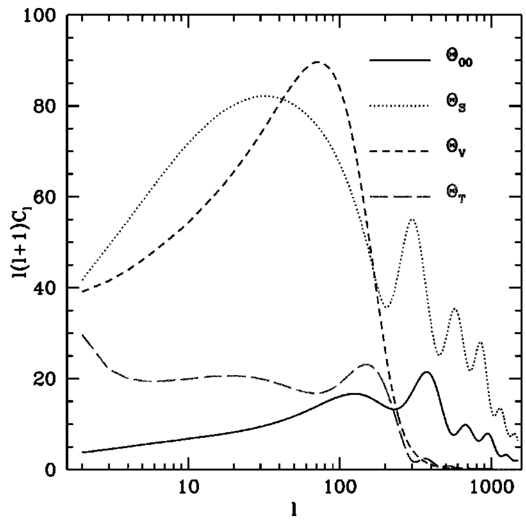

FIG. 5. Angular power spectrum of anisotropies generated by a simple ''coherent'' model of scaling sources, correctly incorporating the superhorizon constraints on the relative importance of the isotropic energy Θ^{00}, anisotropic stress Θ^S, vector Θ^V and tensor Θ^T perturbations to the source stress tensor. Each of these components is independent in this model, allowing us to map each family onto its C_l which can then be co-added.

VIII. NUMERICAL SOLUTION OF A COHERENT MODEL

As a further model we have considered the case of a completely ''coherent'' source in which the unequal time correlators of Θ_{00}, Θ^S, Θ^T, and Θ^V are all proportional to the product of Heaviside functions, for example setting

$$\langle \Theta_{00}(\mathbf{k},\tau)\Theta_{00}(-\mathbf{k},\tau')\rangle = (\tau\tau')^{-1/2}\Theta(\epsilon - k\tau)\Theta(\epsilon - k\tau'). \quad (28)$$

Note that we need to model the white noise Θ^S contribution, but as mentioned above, the cross correlator must vanish at small k. So in this model we will assume that $\langle \Theta_{00}\Theta^S\rangle$ is identically zero, and therefore solve for the Θ_{00} and Θ^S contributions separately. This model is not strictly coherent, since a fully coherent model would have unit cross correlation coefficient $\langle \Theta_{00}\Theta^S\rangle$. We use quotation marks with ''coherent'' to indicate the unequal time coherence. We define C_l^{00} and C_l^S as the respectively derived power spectra. Note that with this choice of variables, the constraints (7) and (8) are automatically satisfied.

We have used this model in the full Boltzmann code developed in [1] as usual with $\epsilon = 5$. The results are shown in Fig. 5. The anisotropic scalar, vector and tensor C_l's have been scaled so that the stress tensor white noise superhorizon amplitudes are in the correct ratios (4). Θ^{00} has been given equal normalization as Θ^S. Nevertheless, C_l^{00} is a remarkably small contribution. This may be understood by solving the equations for stress energy conservation for Θ, from which one finds that Θ is actually very small in the ''coherent'' model at horizon crossing $k\tau \sim 5$, so that from Eq. (9) its contribution to the anisotropy is small.

The ''coherent'' model provides a useful comparison to the previous incoherent model. The broad agreement between the two models suggests that our main result (27) is actually insensitive to the detailed nature of the source. An advantage of the ''coherent'' model is that we can more easily incorporate the matter-radiation transition, giving rise to departures from scale invariance in the C_l spectrum qualitatively similar in character to those observed in realistic source calculations. And as seen in Fig. 5 the model gives a reasonable impression of the main features of the realistic calculations in Fig. 1, at least on large angular scales.

IX. CONCLUSIONS

In this paper we have developed a set of physically reasonable models for the perturbations generated on superhorizon scales by causal sources. We gave some rigorous and some approximate arguments that the large angular scale anisotropies due to the scalar and vector plus tensor modes are in general similar in magnitude. The physical basis is Eq. (4), where we showed that causality and analyticity forces the superhorizon stress-energy sources to be of similar magnitude. If the vector and tensor contributions to the large angle anisotropies are large, the scalar normalization is lower and the Doppler peaks due to scalar perturbations are small compared to the large angle Sachs-Wolfe plateau.

Let us close by mentioning some loopholes in the above arguments, which make it possible to circumvent the conclusion that causal sources are unlikely to have large Doppler peaks.

If subhorizon modes with $k\tau > 5$ dominate the anisotropies, then our arguments do not apply. Sources in this category have been explored by Durrer and Sakellariadou [8].

If the shape of the unequal time correlators, parametrized in our models by ϵ, is different for the scalar, vector and tensor components, then the C_l contributions could be strongly affected, since $C_l \propto \epsilon^3$. One could imagine a model where ϵ for scalars was larger than for vectors and tensors, but even here one would probably not find sharp Doppler peaks, since increasing ϵ is likely to increase the incoherence of the source and thus smooth out the Doppler peaks (this is apparent in Figs. 2 and 3).

One could consider sources like those in [5] in which the anisotropic stresses are by construction zero outside the horizon. In such a model the superhorizon constraint (4) is satisfied with all terms being zero. In the model of [5] this is true because the real space stress energy master functions were taken to be *spherically symmetric*, clearly a special case.

We have assumed perfect scaling of the sources and matter domination. There is some violation of scaling due to the matter-radiation transition, but this is a small effect on large angular scales. Stronger departures from scaling would result from a non-minimal-coupling mass term $R\theta^2$ for the Goldstone bosons [9]. We are currently exploring this possibility.

We have assumed that the scalar, vector and tensor modes are equally incoherent, and have tested numerically the case that they are equally ''coherent.'' In simulations, however, we find that the vector and tensor components are about equally coherent, while the scalar components are less coherent. We would expect our results to apply to the vector and tensor components, and to overestimate the scalar, which is indeed observed. In order to change the qualitative results, one could try to design a defect model wherein the scalars are more coherent than vectors or tensors.

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[1] U. Pen, U. Seljak, and N. Turok, Phys. Rev. Lett. **79**, 1611 (1997); U. Seljak, U. Pen, and N. Turok, *ibid.* **79**, 1615 (1997).
[2] U. Pen, U. Seljak and N. Turok (in preparation).
[3] B. Allen, R. R. Caldwell, S. Dodelson, L. Knox, E. P. S. Shellard, and A. Stebbins, Phys. Rev. Lett. **79**, 2624 (1997).
[4] U. Pen, D. Spergel, and N. Turok, Phys. Rev. D **49**, 692 (1994).
[5] N. Turok, Phys. Rev. D **54**, 3686 (1996); Phys. Rev. Lett. **77**, 4138 (1996).
[6] A. Albrecht, D. Coulson, P. Ferreira, and J. Magueijo, Phys. Rev. Lett. **76**, 1413 (1996).
[7] R. Crittenden and N. Turok, Phys. Rev. Lett. **75**, 2642 (1995).
[8] R. Durrer and M. Sakellariadou, Phys. Rev. D **56**, 4480 (1997).
[9] U. Pen, in Fundamental Parameters in Cosmology, Proceedings of the XXXIIIrd Recontres de Mariond, Les Arcs, France, 1998, astro-ph/9804083.