Covariant and gauge-invariant analysis of cosmic microwave background anisotropies from scalar perturbations

Anthony Challinor* and Anthony Lasenby†
MRAO, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, United Kingdom
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We present a new, fully covariant and manifestly gauge-invariant expression for the temperature anisotropy in the cosmic microwave background radiation resulting from scalar perturbations. We pay particular attention to gauge issues such as the definition of the temperature perturbation and the placing of the last scattering surface. In the instantaneous recombination approximation, the expression may be integrated up to a Rees-Sciama term for arbitrary matter descriptions in flat, open and closed universes. We discuss the interpretation of our result in the baryon-dominated limit using numerical solutions for conditions on the last scattering surface, and confirm that for adiabatic perturbations the dominant contribution to the anisotropy on intermediate scales (the location of the Doppler peaks) may be understood in terms of the spatial inhomogeneity of the radiation temperature in the baryon rest frame. Finally, we show how this term enters the usual Sachs-Wolfe type calculations (it is rarely seen in such analyses) when subtle gauge effects at the last scattering surface are treated correctly. [S0556-2821(98)05314-4]

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I. INTRODUCTION

The calculation of the primary temperature anisotropy in the cosmic microwave background radiation (CMB) resulting from density perturbations has a long history, beginning with the seminal paper by Sachs and Wolfe [1]. Since the original Sachs-Wolfe estimate, a wealth of detailed predictions for the anisotropies expected in various cosmological models have been worked out. The calculations are straightforward in principle, but, like many topics in cosmological perturbation theory, are plagued by subtle gauge issues [2].

The problems of gauge-mode solutions to the linear perturbation equations and the gauge-ambiguity of initial conditions can be eliminated by working exclusively with gauge-invariant variables, as in the widely used Bardeen approach [3] and the less well known covariant approach advocated by Ellis and co-workers [4,5]. However, gauge issues still arise in connection with the definition of the temperature perturbation and the placement of the last scattering surface [2,6]. The latter gauge issues do not arise at first-order in numerical calculations which integrate the Boltzmann equation in a perturbed universe, since the visibility function (which determines the position of the last scattering surface) multiplies first-order variables giving only a second-order error from the use of a zero-order approximation to the visibility [7]. However, this is not always the case in Sachs-Wolfe style analyses, which integrate along null geodesics back to the surface of last scattering, unless care is taken to ensure that the final result involves only first-order variables on the last scattering surface, which then only need be located to zero-order.

In this paper, we present a new expression for the CMB temperature anisotropy arising from linear scalar perturbations which is fully covariant and manifestly gauge invariant. We obtain our expression by integrating the covariant and gauge-invariant Boltzmann equation [7,8] along observational null geodesics, paying careful attention to the gauge issues discussed above. Unlike some covariant results in the literature (see, for example, [8,9]), the expression derived here can be integrated trivially, in the instantaneous recombination approximation, up to a Rees-Sciama term in universes with arbitrary matter descriptions. (The covariant results in [8,9] can only be integrated in baryon-dominated universes, thus excluding cold dark matter dominated universes, and other such models favored by observation.) We base our treatment on the physically appealing covariant and gauge-invariant formulation of perturbation theory, as described in [4,5]. In this approach, one works exclusively with gauge-invariant variables which are covariantly-defined and hence physically observable in principle. The covariant method has many advantages over other gauge-invariant approaches (such as that formulated by Bardeen [3]). Most notably, the covariant variables have transparent physical definitions which ensures that predictions are always straightforward to interpret physically. Other advantages include the unified treatment of scalar, vector and tensor modes, a systematic linearization procedure which can be extended to consider higher-order effects (the covariant variables are exactly gauge-invariant, independent of any perturbative expansion), and the ability to linearize about a variety of background models, such as Friedmann-Robertson-Walker (FRW) or Bianchi models.

For universes which are baryon-dominated at last scattering, our expression for the temperature anisotropy may be compared to other gauge-invariant analytic results in the literature. We show that, with suitable approximations, the result derived here reduces to that given by Panek [10] and corrects a similar result given by Dunsby recently [9]. For the baryon-dominated universe, we use numerical results for the covariant, gauge-invariant variables on the last scattering surface, obtained from a gauge-invariant Boltzmann code

*Email address: A.D.Challinor@mrao.cam.ac.uk
†Email address: A.N.Lasenby@mrao.cam.ac.uk

[7], to discuss the different physical contributions to the primary temperature anisotropy. In particular, we show that on intermediate and small scales, the ''monopole'' contribution to the temperature anisotropy is described by the spatial gradient of the photon energy density, in the energy-frame, on the last scattering surface. Since the (real) last scattering surface is approximately a surface of constant radiation temperature (so that recombination does occur there), the inhomogeneity of the radiation energy density in the energy-frame determines a distortion of the last scattering surface relative to the surfaces of simultaneity in the energy-frame. The extra redshift (due to the expansion of the universe) which the photons incur due to the distortion is seen as a ''monopole'' contribution to the temperature anisotropy on intermediate scales. There is a significant ''dipole'' contribution to the anisotropy on intermediate and small scales, which we discuss also.

We end with a discussion of the gauge issues inherent in the original Sachs-Wolfe calculation of the CMB anisotropy [1], focusing on the ''monopole'' contribution to the temperature anisotropy on intermediate scales, described above. This contribution is often missed in Sachs-Wolfe-type calculations through an incorrect treatment of gauge effects at the last scattering surface [6]. (Equivalently, the term is often missed through a failure to recognize the direction-dependence of the ''expected temperature'' used to define the temperature perturbation in many calculations.) This often neglected term, which is not important on large scales, is an essential component of the Doppler peaks in the CMB power spectrum.

We employ standard general relativity and use a $(+---)$ metric signature. Our conventions for the Riemann and Ricci tensors are fixed by $[\nabla_a,\nabla_b]u^c=-\mathcal{R}_{abd}{}^c u^d$, and $\mathcal{R}_{ab}\equiv\mathcal{R}_{acb}{}^c$. We use units with $c=G=1$ throughout.

II. COVARIANT COSMOLOGICAL PERTURBATIONS

In this section, we summarize the covariant approach to perturbations in cosmology [4,5] to establish our notation and conventions. We begin by choosing a velocity u^a, which is defined physically in such a manner that if the universe is exactly FRW the velocity reduces to that of the fundamental observers. This property of u^a is necessary to ensure gauge-invariance of the variables defined below. We refer to the choice of velocity as a frame choice. For most of this paper, it will not be necessary to make a frame choice. From the velocity u^a, we construct a projection tensor into the space perpendicular to u^a (the instantaneous rest space of observers whose velocity is u^a):

$$h_{ab}\equiv g_{ab}-u_a u_b, \tag{2.1}$$

where g_{ab} is the metric of spacetime. We use the symmetric tensor h_{ab} to define a spatial covariant derivative ${}^{(3)}\nabla^a$ which acting on a tensor $T^{b\cdots c}{}_{d\cdots e}$ returns a tensor which is orthogonal to u^a on every index:

$${}^{(3)}\nabla^a T^{b\cdots c}{}_{d\cdots e}\equiv h_p^a h_r^b\cdots h_s^c h_d^t\cdots h_e^u\nabla^p T^{r\cdots s}{}_{t\cdots u}, \tag{2.2}$$

where ∇^a denotes the usual covariant derivative.

The covariant derivative of the velocity decomposes as

$$\nabla_a u_b=\varpi_{ab}+\sigma_{ab}+\frac{1}{3}\theta h_{ab}+u_a w_b, \tag{2.3}$$

where $\varpi_{ab}=\varpi_{[ab]}$ is the vorticity, which satisfies $u^a\varpi_{ab}=0$, $\sigma_{ab}=\sigma_{(ab)}$ is the shear, which is orthogonal to u^a and traceless, $\theta\equiv\nabla^a u_a=3H$ measures the volume expansion rate (H is the local Hubble parameter), and $w_a\equiv u^b\nabla_b u_a$ is the acceleration. In an exact FRW universe the vorticity, shear and acceleration vanish identically. We regard them as first-order variables [denoted $\mathcal{O}(1)$] in an almost FRW universe, so that products of such variables may be dropped in expressions in the linearized calculations we consider here. Other first-order variables may be obtained by taking the spatial gradient of scalar quantities. Such quantities are gauge-invariant by construction since they vanish identically in an exact FRW universe. We shall make use of the comoving fractional spatial gradient of the density $\rho^{(i)}$ of a species i,

$$\mathcal{X}_a^{(i)}\equiv\frac{S}{\rho^{(i)}}{}^{(3)}\nabla_a\rho^{(i)}, \tag{2.4}$$

and the comoving spatial gradient of the expansion

$$\mathcal{Z}_a\equiv S\,{}^{(3)}\nabla_a\theta. \tag{2.5}$$

The scalar S is a local scale factor satisfying

$$\dot{S}\equiv u^a\nabla_a S=HS,\qquad {}^{(3)}\nabla^a S=\mathcal{O}(1), \tag{2.6}$$

which removes the effects of the expansion from the spatial gradients defined above. The vector $\mathcal{X}_a^{(i)}$ is a manifestly covariant and gauge-invariant characterization of the density inhomogeneity.

The matter stress-energy tensor $\mathcal{T}_{ab}$ decomposes with respect to u^a as

$$\mathcal{T}_{ab}\equiv\rho u_a u_b+2u_{(a}q_{b)}-ph_{ab}+\pi_{ab}, \tag{2.7}$$

where $\rho\equiv\mathcal{T}_{ab}u^a u^b$ is the density of matter (measured by a comoving observer), $q_a\equiv h_a^b\mathcal{T}_{bc}u^c$ is the energy (or heat) flux and is orthogonal to u^a, $p\equiv-h_{ab}\mathcal{T}^{ab}/3$ is the isotropic pressure, and the symmetric traceless tensor $\pi_{ab}\equiv h_a^c h_b^d\mathcal{T}_{cd}+ph_{ab}$ is the anisotropic stress, which is also orthogonal to u^a. In an exact FRW universe, isotropy restricts $\mathcal{T}_{ab}$ to perfect-fluid form, so that in an almost FRW universe the heat flux and isotropic stress may be treated as first-order variables.

The photons are described by a covariant distribution function $f^{(\gamma)}(E,e)$, where $E=p^a u_a$ is the energy of a photon with momentum p^a, and e^a is unit spacelike vector along the direction of propagation in the frame defined by u^a. The photon energy density $\rho^{(\gamma)}$, the heat flux $q_a^{(\gamma)}$ and the anisotropic stress $\pi_{ab}^{(\gamma)}$ are given by integrals of the three lowest angular moments of the distribution function:

$$\rho^{(\gamma)}=\int dEd\Omega\ E^3 f^{(\gamma)}(E,e), \tag{2.8}$$

$$q_a^{(\gamma)}=\int dEd\Omega\ E^3 f^{(\gamma)}(E,e)e_a, \tag{2.9}$$

$$\pi_{ab}^{(\gamma)}=\int dEd\Omega\ E^3 f^{(\gamma)}(E,e)e_a e_b+\frac{1}{3}\rho^{(\gamma)}h_{ab}, \tag{2.10}$$

where $d\Omega$ denotes an integration over solid angles. Higher-order symmetric traceless spatial tensors can be used to characterize higher moments of the distribution function (see, for example, [11]), and are useful in numerical simulations of the CMB anisotropy [7]. We use the temperature difference from the mean (the full sky average) as our definition of the temperature anisotropy $\delta_T(e)$, so that

$$4\,\delta_T(e)\equiv\frac{4\pi}{\rho^{(\gamma)}}\int dE\ E^3 f^{(\gamma)}(E,e)-1. \tag{2.11}$$

The temperature perturbation $\delta_T(e)$ is covariantly defined and gauge-invariant (it vanishes in an exact FRW universe), and is observable directly. This should be contrasted with the gauge-dependent temperature perturbation used by some authors (see Sec. V, for examples).

The final first-order gauge-invariant variables we require derive from the Weyl tensor $\mathcal{W}_{abcd}$, which vanishes in an exact FRW universe due to isotropy and homogeneity. The electric and magnetic parts of the Weyl tensor, denoted by $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$, respectively, are symmetric traceless tensors, orthogonal to u^a, which we define by

$$\mathcal{E}_{ab}\equiv u^c u^d \mathcal{W}_{acbd}, \tag{2.12}$$

$$\mathcal{B}_{ab}\equiv-\frac{1}{2}u^c u^d \eta_{ac}{}^{ef}\mathcal{W}_{efbd}, \tag{2.13}$$

where η_{abcd} is the covariant permutation tensor with $\eta_{0123}=-\sqrt{-g}$.

Linearized perturbation equations

Over the epoch of interest, the individual matter constituents of the universe interact with each other only through gravity, except for the photons and baryons (including electrons) whose dominant interaction with each other is via Thomson scattering of photons off free electrons. The variation of the gauge-invariant temperature perturbation $\delta_T(e)$ along null geodesics is given by the (linearized) covariant Boltzmann equation [7,8]:

$$\delta_T(e)'+\sigma_T n_e\delta_T(e)=\sigma_{ab}e^a e^b+w_a e^a-\left(\frac{1}{3}\theta+\frac{\rho^{(\gamma)'}}{4\rho^{(\gamma)}}\right)[1+4\,\delta_T(e)]-\sigma_T n_e\left(v_a^{(b)}e^a-\frac{3}{16\rho^{(\gamma)}}\pi_{ab}^{(\gamma)}e^a e^b\right), \tag{2.14}$$

where n_e is the free electron density (the effects of thermal motion of the free electrons is ignored), σ_T is the Thomson cross section, $v_a^{(b)}$ is the baryon velocity relative to u_a [$v_a^{(b)}u^a=\mathcal{O}(2)$], and a prime denotes differentiation with respect to the parameter λ along the null geodesic, with $(u_a+e_a)\nabla^a\lambda=1$. In Eq. (2.14) we have ignored the effects of polarization. Including polarization gives only a small correction to the collision term in the Boltzmann equation due to the polarization dependence of the Thomson cross section. Equation (2.14) is valid for any type of perturbation (scalar, vector or tensor) and for any value of the spatial curvature. The evolution of the photon density is given by

$$\dot{\rho}^{(\gamma)}+\frac{4}{3}\theta\rho^{(\gamma)}+{}^{(3)}\nabla^a q_a^{(\gamma)}=0, \tag{2.15}$$

where an overdot denotes differentiation with respect to proper time along the integral curves of u^a ($\dot{\rho}^{(\gamma)}\equiv u^a\nabla_a\rho^{(\gamma)}$). Taking the $l=1$ angular moment of the Boltzmann equation (2.14) gives a propagation equation for the photon heat flux:

$$\dot{q}_a^{(\gamma)}+\frac{4}{3}\theta q_a^{(\gamma)}+{}^{(3)}\nabla^b\pi_{ab}^{(\gamma)}+\frac{4}{3}\rho^{(\gamma)}w_a-\frac{1}{3}{}^{(3)}\nabla_a\rho^{(\gamma)}=\sigma_T n_e\left(\frac{4}{3}\rho^{(\gamma)}v_a^{(b)}-q_a^{(\gamma)}\right). \tag{2.16}$$

Taking higher-order moments of Eq. (2.14) gives a hierarchy of equations which are used in the covariant numerical calculations of CMB anisotropies described in [7].

The electrons and baryons may be approximated by a tightly-coupled ideal fluid with energy density $\rho^{(b)}$, pressure $p^{(b)}$ in the rest frame of the fluid which has velocity $u_a+v_a^{(b)}$. To linear order, the stress-energy tensor of the baryons is

$$\mathcal{T}_{ab}^{(b)}=\rho^{(b)}u_a u_b+2(\rho^{(b)}+p^{(b)})u_{(a}v_{b)}^{(b)}-p^{(b)}h_{ab}, \tag{2.17}$$

which shows that the baryon heat flux is $(\rho^{(b)}+p^{(b)})v_a^{(b)}$ in the u_a frame. The conservation of photon plus baryon stress-energy gives the propagation equations for the density

$$\dot{\rho}^{(b)}+(\rho^{(b)}+p^{(b)})\theta+(\rho^{(b)}+p^{(b)})^{(3)}\nabla^a v_a^{(b)}=0, \tag{2.18}$$

and the velocity

$$(\rho^{(b)}+p^{(b)})(\dot{v}_a^{(b)}+w_a)+\frac{1}{3}(\rho^{(b)}+p^{(b)})\theta v_a^{(b)} + \dot{p}^{(b)}v_a^{(b)}-{}^{(3)}\nabla_a p^{(b)}=-\sigma_T n_e\left(\frac{4}{3}\rho^{(\gamma)}v_a^{(b)}-q_a^{(\gamma)}\right). \tag{2.19}$$

In this paper we consider only scalar perturbations. In this case, the magnetic part of the Weyl tensor $\mathcal{B}_{ab}$ and the vorticity ϖ_{ab} vanish identically. The electric part of the Weyl tensor $\mathcal{E}_{ab}$ and the shear σ_{ab} do not vanish, and satisfy the propagation equations

$$\dot{\mathcal{E}}_{ab}+\theta\mathcal{E}_{ab}+\frac{1}{6}\kappa[3(\rho+p)\sigma_{ab}+3{}^{(3)}\nabla_{(a}q_{b)} - h_{ab}{}^{(3)}\nabla^c q_c-3\dot{\pi}_{ab}-\theta\pi_{ab}]=0, \tag{2.20}$$

$$\dot{\sigma}_{ab}+\frac{2}{3}\theta\sigma_{ab}-{}^{(3)}\nabla_{(a}w_{b)} + \frac{1}{3}h_{ab}{}^{(3)}\nabla^c w_c+\mathcal{E}_{ab}+\frac{1}{2}\kappa\pi_{ab}=0, \tag{2.21}$$

where $\kappa\equiv 8\pi$, and the constraint equations

$${}^{(3)}\nabla^b\mathcal{E}_{ab}-\frac{1}{6}\kappa[2{}^{(3)}\nabla_a\rho+2\theta q_a+3{}^{(3)}\nabla^b\pi_{ab}]=0, \tag{2.22}$$

$${}^{(3)}\nabla^b\sigma_{ab}-\frac{2}{3}{}^{(3)}\nabla_a\theta-\kappa q_a=0. \tag{2.23}$$

The density, pressure, heat flux and anisotropic stress appearing in these equations are total variables obtained by summing over all matter constituents.

For scalar perturbations, the temporal and spatial aspects of the problem may be separated by expanding all first-order gauge-invariant variables in tensors derived from the scalar harmonic functions $Q^{(k)}$, which are defined covariantly as eigenfunctions of the generalized Helmholtz equation ${}^{(3)}\nabla^2 Q^{(k)}=k^2/S^2\,Q^{(k)}$ [12] satisfying $\dot{Q}^{(k)}=\mathcal{O}(1)$. Specifically, we have

$$\mathcal{X}_a^{(i)}=\sum_k k\mathcal{X}_k^{(i)}Q_a^{(k)},\quad \mathcal{Z}_a=\sum_k\frac{k^2}{S}\mathcal{Z}_k Q_a^{(k)}, \tag{2.24}$$

$$q_a^{(i)}=\rho^{(i)}\sum_k q_k^{(i)}Q_a^{(k)},\quad \pi_{ab}^{(i)}=\rho^{(i)}\sum_k \pi_k^{(i)}Q_{ab}^{(k)}, \tag{2.25}$$

$$\mathcal{E}_{ab}=\sum_k\frac{k^2}{S^2}\Phi_k Q_{ab}^{(k)},\quad \sigma_{ab}=\sum_k\frac{k}{S}\sigma_k Q_{ab}^{(k)}, \tag{2.26}$$

$$v_a^{(b)}=\sum_k v_k^{(b)}Q_a^{(k)},\quad w_a=\sum_k w_k Q_a^{(k)}. \tag{2.27}$$

The scalar expansion coefficients, such as $\mathcal{X}_k^{(i)}$, are themselves first-order gauge-invariant variables, and they satisfy ${}^{(3)}\nabla^a\mathcal{X}_k^{(i)}=\mathcal{O}(2)$. The spatial vector $Q_a^{(k)}$ and the spatial tensor $Q_{ab}^{(k)}$, which is symmetric and traceless, are defined by

$$Q_a^{(k)}\equiv\frac{S}{k}{}^{(3)}\nabla_a Q^{(k)},$$

$$Q_{ab}^{(k)}\equiv\frac{S^2}{k^2}{}^{(3)}\nabla_{(a}{}^{(3)}\nabla_{b)}Q^{(k)}-\frac{1}{3}h_{ab}Q^{(k)}. \tag{2.28}$$

Some useful properties of the scalar harmonics and derived tensors are summarized in the appendix to Bruni *et al.* [13]. This completes the definitions of quantities required in this paper. Further details of our notation and conventions may be found in [7,8].

III. A COVARIANT EXPRESSION FOR THE TEMPERATURE ANISOTROPY

The gauge-invariant CMB temperature anisotropy along a given direction is obtained by integrating the Boltzmann equation (2.14) along the null geodesic (whose tangent projects onto the given direction) through the observation point. Before integrating Eq. (2.14), it is convenient to rewrite the first-order factor multiplying $1+4\delta_T(e)$ on the right-hand side in terms of gauge-invariant variables as follows:

$$\frac{1}{3}\theta+\frac{\rho^{(\gamma)\prime}}{4\rho^{(\gamma)}}=\frac{1}{4\rho^{(\gamma)}}(e_a{}^{(3)}\nabla^a\rho^{(\gamma)}-{}^{(3)}\nabla^a q_a^{(\gamma)}), \tag{3.1}$$

where we have made use of the equation of motion of the photon density, Eq. (2.15), and $(u^a+e^a){}^{(3)}\nabla_a\lambda=1$. At this point, we specialize to scalar perturbations and introduce the harmonic expansions of the gauge-invariant variables given in the previous section. We have

$$\frac{1}{\rho^{(\gamma)}}{}^{(3)}\nabla^a q_a^{(\gamma)}=\sum_k\frac{k}{S}q_k^{(\gamma)}Q^{(k)}, \tag{3.2}$$

$$\frac{1}{\rho^{(\gamma)}}e_a{}^{(3)}\nabla^a\rho^{(\gamma)}=\sum_k\mathcal{X}_k^{(\gamma)}Q^{(k)\prime}, \tag{3.3}$$

so that

$$\frac{1}{4\rho^{(\gamma)}}({}^{(3)}\nabla^a q_a^{(\gamma)}-e_a{}^{(3)}\nabla^a\rho^{(\gamma)}) =-\frac{1}{3}\sum_k\left(\frac{k}{S}\mathcal{Z}_k-\frac{S}{k}\theta w_k\right)Q^{(k)}-\frac{1}{4}\sum_k(\mathcal{X}_k^{(\gamma)}Q^{(k)})', \tag{3.4}$$

where we have used the equation

$$\frac{k}{S}q_k^{(\gamma)}=-\frac{4}{3}\frac{k}{S}\mathcal{Z}_k+\frac{4}{3}\frac{S}{k}\theta w_k-\mathcal{X}_k^{(\gamma)\prime}, \tag{3.5}$$

which follows from taking the spatial gradient of Eq. (2.15) and harmonically expanding the result, to eliminate $q_k^{(\gamma)}$ in favor of $\mathcal{Z}_k$ and the acceleration. Integrating the Boltzmann equation (2.14) along the null geodesic connecting the reception point R (where $\lambda=\lambda_R$) and a point in the distant past (where $\lambda=\lambda_i$), we find

$$(\delta_T(e))_R=-\frac{1}{4}\sum_k(\mathcal{X}_k^{(\gamma)}Q^{(k)})_R+\sum_k\int_{\lambda_i}^{\lambda_R}e^{-\tau}\left[\frac{k}{S}\sigma_k e^a e^b Q_{ab}^{(k)}-\frac{1}{3}\left(\frac{k}{S}\mathcal{Z}_k-\frac{S}{k}\theta w_k\right)Q^{(k)}+w_k e^a Q_a^{(k)}\right]d\lambda$$
$$+\sum_k\int_{\lambda_i}^{\lambda_R}-\tau' e^{-\tau}\left[\frac{3}{16}\pi_k^{(\gamma)}e^a e^b Q_{ab}^{(k)}-v_k^{(b)}e^a Q_a^{(k)}+\frac{1}{4}\mathcal{X}_k^{(\gamma)}Q^{(k)}\right]d\lambda, \tag{3.6}$$

where $(M)_R$ denotes the value of the quantity M evaluated at the point R, and $\tau(\lambda)$ is the optical depth along the line of sight, defined by

$$\tau(\lambda)\equiv\int_\lambda^{\lambda_R}n_e\sigma_T\,d\lambda. \tag{3.7}$$

On angular scales larger than $8'$ we may approximate the visibility function $-\tau' e^{-\tau}$ by a δ function whose support defines the last scattering surface (the instantaneous recombination approximation). With this approximation, Eq. (3.6) integrates to

$$(\delta_T(e))_R=\sum_k\left(\frac{1}{4}\mathcal{X}_k^{(\gamma)}Q^{(k)}+\frac{3}{16}\pi_k^{(\gamma)}e^a e^b Q_{ab}^{(k)}-v_k^{(b)}e^a Q_a^{(k)}\right)_A$$
$$+\sum_k\int_{\lambda_A}^{\lambda_R}\left\{\frac{k}{S}\left[\sigma_k\left(\frac{S}{k^2}(SQ^{(k)\prime})'+\frac{1}{3}Q^{(k)}\right)-\frac{1}{3}\mathcal{Z}_k Q^{(k)}\right]+\frac{S}{k}[w_k Q^{(k)\prime}+Hw_k Q^{(k)}]\right\}d\lambda, \tag{3.8}$$

where the point A (where $\lambda=\lambda_A$) is the point of intersection of the null geodesic with the last scattering surface, and we have used the result

$$e^a e^b Q_{ab}^{(k)}=\frac{S}{k^2}(SQ^{(k)\prime})'+\frac{1}{3}Q^{(k)}. \tag{3.9}$$

We have dropped a direction independent (monopole) term evaluated at R from Eq. (3.8) since it will eventually be cancelled by other monopole terms in the integral. Note that the integrand in Eq. (3.8) contains only kinematic gauge-invariant variables (the shear, the spatial gradient of the expansion θ and the acceleration), which simplifies the next stage in the integration. In [8] we gave a more general expression for the anisotropy, valid for all perturbation types, but the integrand involved the spatial gradient of the baryon density which could only be replaced by the spatial gradient of the total density if the universe is baryon dominated at recombination. Expression (3.8) proves to be more convenient for the discussion of CMB anisotropies in multicomponent universes where only scalar perturbations are present.

Integrating the last term in Eq. (3.8) by parts twice, we find that

$$\int_{\lambda_A}^{\lambda_R}\frac{k}{S}\left[\sigma_k\left(\frac{S}{k^2}(SQ^{(k)\prime})'+\frac{1}{3}Q^{(k)}\right)-\frac{1}{3}\mathcal{Z}_k Q^{(k)}\right]+\frac{S}{k}[w_k Q^{(k)\prime}+Hw_k Q^{(k)}]d\lambda$$
$$=\left[\sigma_k e^a Q_a^{(k)}-\frac{S}{k}(\dot{\sigma}_k-w_k)Q^{(k)}\right]_A^R+\int_{\lambda_A}^{\lambda_R}\left[\left(\frac{S}{k}\sigma_k'\right)'+\frac{1}{3}\frac{k}{S}(\sigma_k-\mathcal{Z}_k)-\frac{S}{k}w_k'\right]Q^{(k)}d\lambda. \tag{3.10}$$

The integrand on the right-hand side of Eq. (3.10) may be simplified by using the linearized propagation and constraint equations for the shear and the electric part of the Weyl tensor. The harmonic expansions of Eqs. (2.20) and (2.21) give

$$\left(\frac{k}{S}\right)^2\left(\dot{\Phi}_k+\frac{1}{3}\theta\Phi_k\right)+\frac{1}{2}\frac{k}{S}\kappa\rho(\gamma\sigma_k+q_k)+\frac{1}{6}\kappa\rho\theta(3\gamma-1)\pi_k-\frac{1}{2}\kappa\rho\dot{\pi}_k=0, \tag{3.11}$$

$$\frac{k}{S}\left(\dot{\sigma}_k+\frac{1}{3}\theta\sigma_k-w_k\right)+\left(\frac{k}{S}\right)^2\Phi_k+\frac{1}{2}\kappa\rho\pi_k=0, \tag{3.12}$$

where γ is defined by $p=(\gamma-1)\rho$, and the constraint equation (2.23) gives

$$\frac{2}{3}\left(\frac{k}{S}\right)^2\left[\mathcal{Z}_k-\left(1-\frac{3K}{k^2}\right)\sigma_k\right]+\kappa\rho q_k=0. \tag{3.13}$$

In these equations, the scalar variables q_k and π_k are the harmonic expansion coefficients of the total heat flux and anisotropic stress. They are related to the component variables $q_k^{(i)}$ and $\pi_k^{(i)}$ [defined by Eq. (2.25)] by

$$\rho q_k=\sum_i \rho^{(i)}q_k^{(i)}, \quad \rho\pi_k=\sum_i \rho^{(i)}\pi_k^{(i)}, \tag{3.14}$$

where the sums are over individual components i.

Evaluating the integrand in Eq. (3.10) by differentiating the shear propagation equation (3.12), substituting for q_k and $\mathcal{Z}_k$ from equations (3.11) and (3.13), and using the zero-order Friedmann equation

$$H^2+\frac{K}{S^2}=\frac{1}{3}\kappa\rho, \tag{3.15}$$

we find the result

$$\left(\frac{S}{k}\sigma_k'\right)'+\frac{1}{3}\frac{k}{S}(\sigma_k-\mathcal{Z}_k)-\frac{S}{k}w_k'=-2\dot{\Phi}_k, \tag{3.16}$$

which is true for scalar perturbations, independent of the matter description and spatial curvature. With this, we obtain our final result for the temperature anisotropy (which is exact in linear theory on angular scales where instantaneous recombination is valid):

$$\begin{aligned}(\delta_T(e))_R=&\sum_k\left(\left[\frac{1}{4}\mathcal{X}_k^{(\gamma)}+\frac{S}{k}(\dot{\sigma}_k-w_k)\right]Q^{(k)}\right)_A\\&-\sum_k([v_k^{(b)}+\sigma_k]e^aQ_a^{(k)})_A\\&+\frac{3}{16}\sum_k(\pi_k^{(\gamma)}e^ae^bQ_{ab}^{(k)})_A\\&-2\sum_k\int_{\lambda_A}^{\lambda_R}\dot{\Phi}_kQ^{(k)}\,d\lambda,\end{aligned} \tag{3.17}$$

where we have dropped a (frame-dependent) dipole term evaluated at R since such a term cannot be distinguished from a first-order peculiar velocity of the observer at R. The final term in Eq. (3.17) describes the Rees-Sciama effect, which only makes a small contribution to the anisotropy in $K=0$ models that are matter dominated at recombination (for $K=0$, Φ_k is approximately constant while a mode is outside the horizon and during the matter dominated era on all scales). The third term on the right-hand side of Eq. (3.17) represents a small contribution to the anisotropy from photon anisotropic stress at last scattering. The sum of the first and second terms dominates the CMB anisotropy in a $K=0$ universe, with the relative importance of each term being dependent on Ω_b and H_0. Expression (3.17) is a generalization of the result given by Dunsby in Sec. V of [9] which was valid only for universes that are fully baryon dominated at last scattering and are spatially flat. We shall see in Sec. IV how the result in [9] (actually a corrected version of it) may be obtained from Eq. (3.17) in the limit of baryon domination at recombination. A similar result to Eq. (3.17) is derived in [14] in terms of Bardeen's gauge-invariant variables.

In deriving Eq. (3.17), we have not made an explicit choice for the velocity u^a. Each of the four terms on the right-hand side is frame-independent, which follows from the fact that under a change of frame $u_a\mapsto u_a+v_a$, where v_a is a first-order relative velocity [$u^av_a=\mathcal{O}(2)$], $\mathcal{E}_{ab}$, $\dot{\mathcal{E}}_{ab}$ and π_{ab} are invariant, while $v_a^{(b)}\mapsto v_a^{(b)}-v_a$ and

$$\sigma_{ab}\mapsto\sigma_{ab}+{}^{(3)}\nabla_{(a}v_{b)}-\frac{1}{3}h_{ab}{}^{(3)}\nabla^cv_c. \tag{3.18}$$

The frame-invariance of the right-hand side of Eq. (3.17) is necessary since $\delta_T(e)$ is invariant in linear theory, up to the dipole terms that we have dropped from Eq. (3.17).

The nonintegral terms on the right-hand side of Eq. (3.17) are evaluated at the point A on the last scattering surface, which lies on a null geodesic through the observation point R. However, it is only necessary to locate A to zero-order since the displacement from the ''true'' position is first-order, which leads to only a second-order error when evaluating a first-order variable. We shall return to this point in Sec. V where we discuss some of the gauge issues associated with the placement of the last-scattering surface in the standard calculations of the Sachs-Wolfe effect.

IV. CMB ANISOTROPY IN A UNIVERSE DOMINATED BY BARYONS AT RECOMBINATION

In cosmological models that are baryon dominated at recombination, the covariant result for the temperature anisotropy, Eq. (3.17), may be cast in a more familiar form, which aids direct comparison with other such results in the literature (for example [9]) and physical interpretation.

Using the propagation equations (3.12) and (3.11) for the shear and the electric part of the Weyl tensor, and the harmonic expansion of the constraint (2.22):

$$2\left(\frac{k}{S}\right)^3\left(1-\frac{3K}{k^2}\right)\Phi_k-\frac{k}{S}\kappa\rho\left(\mathcal{X}_k+\left[1-\frac{3K}{k^2}\right]\pi_k\right)-\kappa\rho\theta q_k=0, \tag{4.1}$$

we find in the limit

$$\rho\to\rho^{(b)},\quad p\to 0,\quad \mathcal{X}_k\to\mathcal{X}_k^{(b)},\quad q_k\to v_k^{(b)},\quad \pi_k\to 0, \tag{4.2}$$

at last scattering that

$$\begin{aligned}\frac{1}{4}\,\mathcal{X}_k^{(\gamma)}+\frac{S}{k}(\dot{\sigma}_k-w_k)\to&\frac{1}{4}\,\mathcal{X}_k^{(\gamma)}-\frac{1}{3}\,\mathcal{X}_k^{(b)}\\&-\frac{1}{3}\Phi_k-\frac{2(6K-k^2)}{3\kappa\rho S^2}\,\Phi_k\\&+\frac{2}{\kappa\rho}H\dot{\Phi}_k,\end{aligned} \tag{4.3}$$

where we have added and subtracted $\mathcal{X}_k^{(b)}/3$ making use of Eq. (4.1), and

$$-(\sigma_k+v_k^{(b)})\to\frac{2k}{\kappa\rho S}\,(\dot{\Phi}_k+H\Phi_k). \tag{4.4}$$

It follows that in a universe which is baryon dominated at last scattering, the temperature anisotropy from scalar perturbations in the instantaneous recombination approximation becomes

$$\begin{aligned}(\delta_T(e))_R=&\sum_k\left(\left[\frac{1}{4}\,\mathcal{X}_k^{(\gamma)}-\frac{1}{3}\,\mathcal{X}_k^{(b)}-\frac{1}{3}\,\Phi_k\right.\right.\\&\left.\left.-\frac{2(6K-k^2)}{3\kappa\rho S^2}\Phi_k+\frac{2}{\kappa\rho}H\dot{\Phi}_k\right]Q^{(k)}\right)_A\\&+\sum_k\left(\frac{2k}{\kappa\rho S}\,[\dot{\Phi}_k+H\Phi_k]e^aQ_a^{(k)}\right)_A\\&-2\sum_k\int_{\lambda_A}^{\lambda_R}\dot{\Phi}_kQ^{(k)}\,d\lambda.\end{aligned} \tag{4.5}$$

The first set of terms in square brackets on the right-hand side of Eq. (4.5) give the ''monopole'' contribution (at last scattering) to the temperature anisotropy. The term $(\mathcal{X}_k^{(\gamma)}/4-\mathcal{X}_k^{(b)}/3)Q^{(k)}$ arises from entropy perturbations, which may be characterized covariantly by a vector $\mathcal{S}_a^{(\gamma b)}$ where

$$\mathcal{S}_a^{(\gamma b)}\equiv\frac{3}{4\rho^{(\gamma)}}{}^{(3)}\nabla_a\rho^{(\gamma)}-\frac{1}{\rho^{(b)}}{}^{(3)}\nabla_a\rho^{(b)}. \tag{4.6}$$

For adiabatic initial conditions, entropy perturbations vanish at last scattering on large scales, where tight-coupling between the baryons and photons still holds. The second ''monopole'' term, $-\Phi_kQ^{(k)}/3$, is the usual Sachs-Wolfe contribution to the temperature anisotropy [1], whose effect is modified on small scales by the third and fourth ''monopole'' terms. As noted by Ellis and Dunsby [6], the third term is rarely seen in analytic calculations of the Sachs-Wolfe effect, although it is present in Panek's result [10]. The omission arises from subtle gauge effects at the last scattering surface which we discuss in Sec. V. The final monopole term is a small correction arising from the nonstationarity of the potential Φ_k. The terms under the second summation in Eq. (4.5) make a ''dipole'' contribution to the temperature anisotropy, and are important on small angular scales.

In a $K=0$ universe, Eq. (4.5) may be written in the form

$$\begin{aligned}(\delta_T(e))_R=&\sum_k\left(\left[\frac{1}{4}\,\mathcal{X}_k^{(\gamma)}-\frac{1}{3}\,\mathcal{X}_k^{(b)}-\frac{1}{3}\Phi_k\right.\right.\\&\left.\left.+\frac{2}{3}H^{-1}\dot{\Phi}_k+\frac{2}{9}\,\mathcal{H}_k^{-2}\Phi_k\right]Q^{(k)}\right)_A\\&+\sum_k\left(\frac{2}{3}\,\mathcal{H}_k^{-1}[\Phi_k+H^{-1}\dot{\Phi}_k]e^aQ_a^{(k)}\right)_A\\&-2\sum_k\int_{\lambda_A}^{\lambda_R}\dot{\Phi}_kQ^{(k)}\,d\lambda,\end{aligned} \tag{4.7}$$

where $\mathcal{H}_k\equiv SH/k$ is the ratio of proper wavelength to the Hubble radius. This result corrects that given by Dunsby in Sec. V of [9] [his equation (61); note also the difference in metric signature from that adopted here]. Note, in particular, that the dominant contribution to the CMB anisotropy from adiabatic perturbations on large scales is $-\Phi_k/3$, which is a factor of 3 smaller than the result in [9].

In Fig. 1 we plot the first two summands in Eqs. (3.17) and (4.7) as a function of k, with $S=1$ at the present, in a $K=0$ universe, in the limit $\Omega_b=1$, where Ω_b is the present-day baryon density in units of the critical density. We take $H_0=50$ km s^{-1} M pc^{-1} and consider adiabatic perturbations with only the fastest growing mode present. At early times, we take Φ_k to be independent of k. The actual conditions on the last scattering surface are obtained from the transfer functions of Fig. 1 by multiplying by the initial values of Φ_k (which are Gaussian random variables in most inflationary theories). The gauge-invariant variables on the last scattering surface are obtained from an accurate Boltzmann code employing covariant, gauge-invariant variables, with adiabatic initial conditions [7]. The agreement between the approximate expression (4.7) and the ''exact'' expression (3.17) is good over the full range of k depicted. The small discrepancy between the sets of curves is due to the small but nonzero

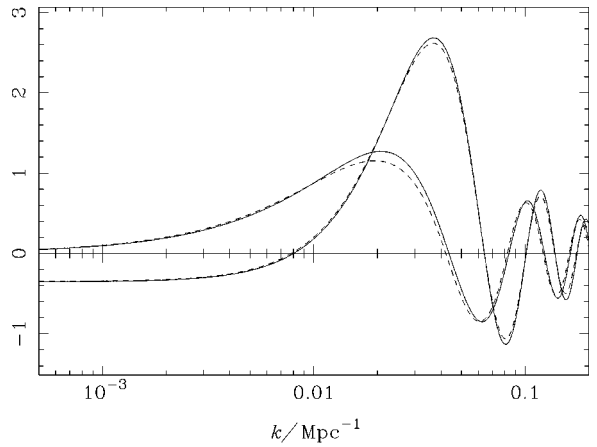

FIG. 1. Contributions to the CMB temperature anisotropy from the conditions on the last scattering surface ($z\simeq1050$) in a $K=0$ universe with $\Omega_b=1$, and $H_0=50$ km s^{-1} M pc^{-1}. Only adiabatic scalar perturbations are considered, and Φ_k is independent of k initially (so that we are plotting transfer functions). The curves should be multiplied by the initial Φ_k to give the actual conditions on the last scattering surface. The solid lines are calculated from Eq. (3.17); the ''monopole'' contribution (negative on large scales) is $\frac{1}{4}\mathcal{X}_k^{(\gamma)}+(S/k)\,(\dot{\sigma}_k-w_k)$, and the ''dipole'' contribution (positive on large scales) is $-\sigma_k-v_k^{(b)}$. The dashed lines are given by the approximate expression (4.7); the ''monopole'' contribution (negative on large scales) is $\frac{1}{4}\mathcal{X}_k^{(\gamma)}-\frac{1}{3}\mathcal{X}_k^{(b)}-\frac{1}{3}\Phi_k+\frac{2}{3}H^{-1}\dot{\Phi}_k+\frac{2}{9}\mathcal{H}_k^{-2}\Phi_k$, and the ''dipole'' contribution is $\frac{2}{3}\mathcal{H}_k^{-1}(\Phi_k+H^{-1}\dot{\Phi}_k)$. The small discrepancy between the solid and dashed curves is due to the small but non-zero values of $\rho^{(\gamma)}/\rho$ and $\rho^{(\nu)}/\rho$ at last scattering.

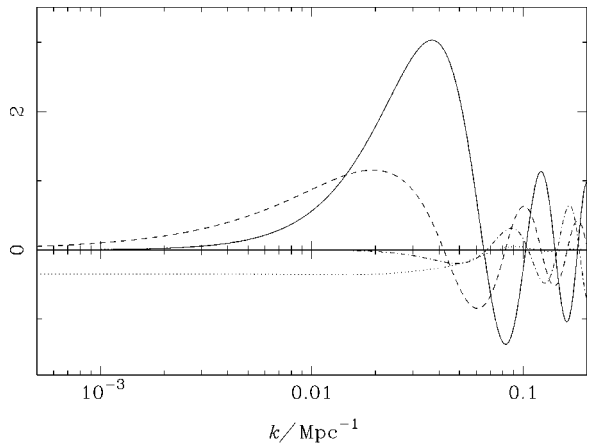

FIG. 2. Approximate contributions to the CMB temperature anisotropy from the conditions on the last scattering surface ($z\simeq1050$) in a $K=0$ universe with $\Omega_b=1$, and $H_0=50$ k ms^{-1} M pc^{-1}. Only adiabatic scalar perturbations are considered. On large scales the usual Sachs-Wolfe term (dotted line) $-\Phi_k/3$ dominates. For adiabatic initial conditions, the effect of entropy perturbations (dashed-dotted line) is only important on small scales where the tight-coupling approximation ceases to hold and adiabaticity is broken. The coherent sum of the entropy perturbations and the term $\frac{2}{9}\mathcal{H}_k^{-2}\Phi_k$ (solid line), as well as the ''dipole'' term $\frac{2}{3}\mathcal{H}_k^{-1}(\Phi_k+H^{-1}\dot{\Phi}_k)$ (dashed line) determine the structure of the Doppler peaks.

values of $\rho^{(\gamma)}/\rho$ and $\rho^{(\nu)}/\rho$ on the last scattering surface (which is located at $z\simeq1050$ in this model).

On large angular scales (small k), the ''monopole'' terms dominate the CMB anisotropy, giving the familiar Sachs-Wolfe plateau for a scale-invariant spectrum of initial conditions. On smaller angular scales the ''monopole'' term oscillates in k due to the coherent acoustic oscillations in the photon-baryon plasma that occur for modes inside the (sound) horizon. These oscillations, along with the oscillations in the ''dipole'' on small scales, determine the structure of the Doppler peaks in the CMB power spectrum. For adiabatic perturbations in a $K=0$ universe, the first zero in the ''monopole'' contribution to the anisotropy (the initially lower curves in Fig. 1) occurs where $\mathcal{H}_k\approx\sqrt{(2/3)}$. This follows from Eq. (4.7), the fact that initially adiabatic perturbations are still adiabatic at last scattering on such scales (see Fig. 2), and the approximate stationarity of the potential Φ_k in the matter dominated era of a $K=0$ universe. This effect was noted recently by Ellis and Dunsby [6], and was also discussed by Hu and Sugiyama [14]. Ellis and Dunsby suggested that this zero in the ''monopole'' contribution should be observable as a zero in the CMB power spectrum on angular scales $\simeq50'$. However, as noted by Hu and Sugiyama and evident in Fig. 1, the ''dipole'' contribution is already significant on these angular scales, with its first maximum occurring close to the zero in the ''monopole.'' This tends to wash out the zero (actually a dip once the k modes are summed over) in the power spectrum, as shown by Hu and Sugiyama [14] in their Fig. 4.

In Fig. 2 we plot the individual contributions to the CMB anisotropy in the baryon dominated limit from equation (4.7). On large scales (small k), the dominant contribution is from the usual Sachs-Wolfe term $-\Phi_k/3$. The effect of entropy perturbations is negligible on large scales since the baryon-photon fluid is still tightly-coupled at last scattering on these scales, and our initial conditions are adiabatic. However, on small scales the entropy perturbations add coherently with the term $2\mathcal{H}_k^{-2}\Phi_k/9$ to reduce the ''monopole'' contribution to the anisotropy. These terms along with the ''dipole'' term determine the structure of the Doppler peaks.

We introduced $\mathcal{X}_k^{(b)}$ into Eq. (4.3) to show explicitly the contribution from entropy perturbations at last scattering, and to facilitate comparison with other results in the literature. However, this decomposition into an entropy perturbation term and terms involving the potential Φ_k is rather unnatural, and can be replaced by an expression which allows a more physical interpretation of the temperature anisotropy on intermediate and small scales. We use Eq. (4.1) to eliminate $\mathcal{X}_k^{(b)}$ and choose u^a to coincide with the baryon velocity ($v_a^{(b)}=0$; if the universe is not baryon dominated at recombination, then the energy-frame should be employed instead of the baryon-frame):

$$\frac{1}{4}\,\mathcal{X}_k^{(\gamma)}+\frac{S}{k}(\dot{\sigma}_k-w_k)\rightarrow\frac{1}{4}\,\overline{\mathcal{X}}_k^{(\gamma)}-\frac{1}{3}\,\Phi_k-\frac{2}{\kappa\rho}\,\frac{K}{S^2}\,\Phi_k+\frac{2H}{\kappa\rho}\,\dot{\Phi}_k, \quad (4.8)$$

where $\overline{\mathcal{X}}_k^{(\gamma)}$ is the harmonic expansion coefficient of the spatial gradient of the photon energy density in the energy-frame. This result actually holds in the energy-frame of any model that is matter dominated at last scattering, and so includes standard CDM and variants. We see that in a $K=0$ universe, which is matter dominated at recombination, the ''monopole'' contribution to the anisotropy consists of the Sachs-Wolfe term, $-\Phi_k Q^{(k)}/3$, which is dominant on large scales, a term $\overline{\mathcal{X}}_k^{(\gamma)} Q^{(k)}/4$ arising from inhomogeneity of the radiation temperature in the energy-frame, which is important on intermediate and small scales, and the small term $2H^{-1}\dot{\Phi}_k Q^{(k)}/3$ arising from nonstationarity of the potential at last scattering. Since the last scattering surface is well approximated by a surface of constant radiation temperature, the spatial gradient of the radiation energy density, in the energy frame, across the last scattering surface describes a distortion of the last scattering surface relative to the surfaces of simultaneity of the energy-frame (which are defined by the average motion of all matter in the universe). The distortion of the last scattering surface causes photons to incur extra redshift, due to the expansion of the universe, during propagation from last scattering to the point of observation.

Finally, if we consider initially adiabatic perturbations in a $K=0$ baryon dominated universe on large enough scales that entropy perturbations may be neglected, we may replace $\overline{\mathcal{X}}_k^{(\gamma)}/4$ by the spatial gradient of the baryon energy density in the baryon-frame, $\overline{\mathcal{X}}_k^{(b)}/3$. Replacing Φ_k with $3\mathcal{H}_k^2\overline{\mathcal{X}}_k^{(b)}/2$ [from the constraint (4.1) with $K=0$] and neglecting terms involving $\dot{\Phi}_k$, we obtain the covariant equivalent of Panek's result [10]:

$$(\delta_T(e))_R = \sum_k \left(\frac{1}{3}\,\overline{\mathcal{X}}_k^{(b)} \left[1 - \frac{3}{2}\,\mathcal{H}_k^2 \right] Q^{(k)} \right)_A + \sum_k \left(\overline{\mathcal{X}}_k^{(b)} \mathcal{H}_k e^a Q_a^{(k)} \right)_A . \tag{4.9}$$

V. COMPARISON WITH THE SACHS-WOLFE RESULT

It is instructive to compare the preceding analysis with the original calculation of Sachs and Wolfe [1]. Assuming that the radiation is nearly isotropic in the frame of the baryons at last scattering, the temperature of the radiation at R (in the baryon-frame) is given in terms of the radiation temperature at the point A in the last scattering surface as

$$T_R = \frac{T_A}{1+z}, \tag{5.1}$$

where the redshift z along the null geodesic connecting A and R is given by [1]

$$1+z = \frac{S_R}{S_A}\left[1 - \frac{1}{2}\int_{\eta_A}^{\eta_R} (\partial_\eta h_{ij} e^i e^j + 2\partial_\eta h_{i0} e^i)\, d\eta \right], \tag{5.2}$$

where $S_R \equiv S(\eta_R)$ is the background scale factor at R and similarly for S_A, $h_{\mu\nu}$ is the metric perturbation in the comoving gauge (with $h_{00}=0$), η is conformal time, and e^i $(i=1,2,3)$ is the direction of photon propagation in the background model. Differencing the temperature between two directions on the sky we find

$$\left(\frac{\Delta T}{T}\right)_R = \left(\frac{\Delta T}{T}\right)_E + \left(\frac{\Delta S}{S}\right)_E + \frac{1}{2}\Delta\left[\int_{\eta_E}^{\eta_R} (\partial_\eta h_{ij} e^i e^j + 2\partial_\eta h_{i0} e^i) d\eta \right], \tag{5.3}$$

where $(\Delta M)_E \equiv (M)_A - (M)_B$. The $(\Delta S)_E$ term on the right-hand side of Eq. (5.3) is gauge-dependent [first-order gauge transformations of the form $\eta \mapsto \eta + f(x^i)/S(\eta)$, which preserve the gauge-conditions implicit in Eq. (5.2), move the field S around in the real universe], while the term $(\Delta T)_E$ is gauge-invariant. It follows that the sum of the first two terms on the right-hand side of Eq. (5.3) is also gauge-dependent. [A compensating gauge-dependence in the integral term preserves the necessary gauge-invariance of $(\Delta T)_R$.] This sum may be expressed in terms of the gauge-dependent photon density perturbation $\delta^{(\gamma)}$, defined by

$$\frac{1}{4}\delta^{(\gamma)} \equiv \frac{T - T^{(0)}}{T^{(0)}}, \tag{5.4}$$

where $T^{(0)} \equiv T^{(0)}(\eta)$ is the background radiation temperature. Using the constancy of $T^{(0)}S$, we find

$$\left(\frac{\Delta T}{T}\right)_E + \left(\frac{\Delta S}{S}\right)_E = \frac{1}{4}\Delta(\delta^{(\gamma)})_E . \tag{5.5}$$

The significance of this result is that it is a first-order quantity. To evaluate $\Delta(\delta^{(\gamma)})_E$ to first-order, we need only locate the last scattering surface to zero-order, since locating the last scattering surface correctly amounts to a first-order displacement in a first-order quantity. The implicit choice made by many authors is to evaluate Eq. (5.5) on the background last scattering surface (over which η is constant). With this choice ΔS and $\Delta T^{(0)}$ are both equal to zero, and trivially we obtain the result $\Delta(\delta^{(\gamma)})_E$ evaluated on the background last scattering surface. Note that Sachs and Wolfe [1] explicitly made this choice, although they also made the (gauge-dependent) assumption that the radiation temperature was constant on the background last scattering surface. Although it is certainly necessary to locate last scattering correctly to calculate $\Delta(T)_E$ and $\Delta(S)_E$ separately, this is not true for the difference $\Delta(TS)_E$, which is all that is required for CMB calculations. Physically, this effect arises from the compensation effect: the extra redshift along a given direction due to the difference between the scale factor on the true last scattering surface and at the point of intersection of the geodesic with the background (zero-order) last scattering surface cancels out the difference in temperature between the same points on the real and true last scattering surface. Note that

such problems do not arise in the kinetic theory calculation of Sec. III, since the optical depth (which determines the position of the last scattering surface) multiplies only first-order variables, and so is needed only to zero-order.

The term $\Delta(\delta^{(\gamma)})_E/4$ is often omitted from Sachs-Wolfe type analyses (for example, [15]), effectively being absorbed into a (direction-dependent) ''expected'' temperature, $\overline{T} \equiv T_A S_A/S_R$ (see [2,10] for alternative choices). Along with the observed temperature T_R, this defines a first-order temperature variation $\delta T/T$ by

$$\frac{\delta T}{T} \equiv \frac{T_R - \overline{T}}{\overline{T}}. \tag{5.6}$$

This should not be confused with the gauge-invariant temperature fluctuation from the mean $\delta_T(e)$, used in the rest of this paper. The quantity $\delta T/T$ is gauge-dependent through the gauge-arbitrariness in the scale factor S. Differencing between two directions on the sky, one finds

$$\Delta(\delta T/T)_R = \Delta\left[\int_{\eta_E}^{\eta_R} (\partial_\eta h_{ij} e^i e^j + 2\partial_\eta h_{i0} e^i) d\eta\right]. \tag{5.7}$$

Of course, this result is entirely equivalent to Eq. (5.3), one needs only the result

$$\Delta\left(\frac{\delta T}{T}\right)_E = \left(\frac{\Delta T}{T}\right)_E - \frac{\Delta(TS)_E}{(TS)_E}, \tag{5.8}$$

which follows from the definition of $\overline{T}$. This appears to be the step where confusion sometimes arises in Sachs-Wolfe-type analyses, with authors comparing the gauge-dependent $\delta T/T$ with the results of observation, instead of the physically relevant $(\Delta T/T)_R$ [or $\delta_T(e)$ which is easily derived from the former]. As noted in [2], it is not enough just to difference $\delta T/T$; in general, the result will not be gauge-invariant and will not include the contribution from $\delta^{(\gamma)}$ at last scattering.

Finally, we demonstrate how the gauge-invariant contribution $\overline{\mathcal{X}}_k^{(\gamma)} Q^{(k)}/4$ to the anisotropy arises in the Sachs-Wolfe calculation. If we approximate the (vorticity-free) baryon motion as being geodesic through recombination to the present, we can choose a comoving gauge with $h_{i0}=0$ (the synchronous gauge, in which the surfaces of constant η are surfaces of simultaneity for the baryons). The result given in Eq. (5.3) (with $h_{i0}=0$) still holds in this gauge, and the conformal time η satisfies $\nabla_a \eta = u_a^{(b)}/S(\eta)$, where $u_a^{(b)}$ is the baryon velocity. It follows that the background scale factor satisfies ${}^{(3)}\nabla_a S = 0$ and $\dot{S} = \partial_\eta S/S = \theta S/3 + \mathcal{O}(1)$, where $\theta = \nabla^a u_a^{(b)}$ is the covariant expansion of the baryon-frame. With this gauge-condition, we have removed the gauge-freedom to perform the transformation $\eta \mapsto \eta + f(x^i)/S$ [the gauge-condition ${}^{(3)}\nabla_a \eta = 0$ forces $f(x^i)$ to be a constant], with the result that $\Delta S/S$ is gauge-invariant. Under these conditions, we may express the first two terms on the right-hand side of Eq. (5.3) in terms of covariant, gauge-invariant variables as follows:

$$\begin{aligned}\left(\frac{\Delta T}{T}\right)_E + \left(\frac{\Delta S}{S}\right)_E &= \frac{1}{4}\int_B^A \left(\frac{\nabla_a \rho^{(\gamma)}}{\rho^{(\gamma)}} + 4\frac{\nabla_a S}{S}\right) dx^a \\ &= \frac{1}{4}\int_B^A \left(\frac{\nabla_a \rho^{(\gamma)}}{\rho^{(\gamma)}} + \frac{4}{3}\theta u_a^{(b)}\right) dx^a \\ &= \sum_k \Delta\left(\frac{1}{4}\overline{\mathcal{X}}_k^{(\gamma)} Q^{(k)}\right)_E,\end{aligned} \tag{5.9}$$

where A and B are the points of intersection of the two geodesics through the observation point R with the last scattering surface, dx^a lies in the last scattering surface $[dx^a u_a^{(b)} = \mathcal{O}(1)]$, and we have replaced $3\dot{S}/S$ by the covariantly defined θ which is correct to the required order. In this manner, we recover the $\overline{\mathcal{X}}_k^{(\gamma)} Q^{(k)}/4$ contribution to the temperature anisotropy.

VI. CONCLUSION

Starting from a covariant and gauge-invariant formulation of the Boltzmann equation, we have derived a new expression for the CMB temperature anisotropy under the instantaneous recombination approximation, valid for scalar perturbations in open, closed and flat universes. Our expression uses only covariantly defined variables, and is manifestly gauge invariant. The result is more useful in multicomponent models with scalar perturbations than earlier covariant results [6,8,9]. In the case of a universe which is baryon-dominated at recombination, we find a simple expression for the anisotropy which corrects a similar result by Dunsby [9]. By making use of numerical solutions to the perturbation equations, we have discussed the conditions on the last scattering surface and their contributions to the characteristic features of the CMB power spectrum. We ended with a discussion of the original Sachs-Wolfe calculation for the temperature anisotropy. We have discussed why it is not necessary to locate accurately the last scattering surface in such calculations (because of the compensation effect), and how the extra term in the Sachs-Wolfe calculation, reported recently by Ellis and Dunsby [6], is missed in many calculations which employ a gauge-dependent ''expected temperature,'' since the angular dependence of this temperature is often overlooked. For a universe which is matter dominated at recombination, but not necessarily adiabatic, the extra term is the spatial gradient of the radiation energy density in the energy-frame, $(\overline{\mathcal{X}}_k^{(\gamma)} Q^{(k)}/4)_A$. For models with isothermal surfaces of last scattering, this inhomogeneity describes a distortion of the last scattering surface relative to the surfaces of simultaneity of the energy-frame. The extra redshift incurred by this distortion is a significant component of the temperature anisotropy on intermediate and small scales.

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