Coherence of neutrino oscillations in the wave packet approach

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The temporal and spatial coherence widths of the microscopic process by which a neutrino is detected are incorporated in the quantum mechanical wave packet treatment of neutrino oscillations, confirming the observation of Kiers, Nussinov, and Weiss that an accurate measurement of the neutrino energy in the detection process can increase the coherence length. However, the wave packet treatment presented here shows that the coherence length has an upper bound, determined by the neutrino energy and the mass-squared difference, beyond which the coherence of the oscillation process is lost. [S0556-2821(98)01415-5]

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A complete understanding of neutrino oscillations must take into account the localization of the microscopic processes by which a neutrino is produced and detected. This localization is appropriately described by a wave packet treatment of neutrino oscillations [1–9] (different treatments are discussed in [10,11]). As the authors of [9] noticed, in the quantum mechanical wave packet approach presented in [4] the dependence of the oscillation probability on the temporal and spatial coherence widths of the detection process was neglected. In this Brief Report, we wish to incorporate, in a simple and straightforward way, the temporal and spatial coherence widths of the detection process in the quantum mechanical wave packet description of neutrino oscillations and show that, as an immediate consequence of this, performing an accurate measurement of the energy of the detected neutrino leads to an increase of the coherence length for neutrino oscillations, as was noticed for the first time in [6].

Let us consider a neutrino of flavor α produced by a weak interaction process at the origin of the space-time coordinates and detected at a distance L after a time[1] T by a weak interaction process capable of detecting a neutrino of flavor β. As in [4], we describe the neutrino propagating between the production and detection processes with the time-dependent state in the Schrödinger picture:

$$|\nu_\alpha(t)\rangle = \sum_a U_{\alpha a}^* \int dp\, \psi_a(p;p_a,\sigma_{pP}) e^{-iE_a(p)t} |\nu_a(p)\rangle, \tag{1}$$

where $E_a(p)=\sqrt{p^2+m_a^2}$. This state is a superposition of mass eigenstate wave packet states (labeled by the index a) weighted by the complex-conjugated elements of the mixing matrix U of the neutrino fields. For simplicity, we consider only one space dimension in the source-detector direction and we assume that the mass eigenstate wave functions in momentum space $\psi_a(p;p_a,\sigma_{pP})$ have the following Gaussian form, which enables us to carry out an analytical calculation of the transition probability:

$$\psi_a(p;p_a,\sigma_{pP}) = (2\pi\sigma_{pP}^2)^{-1/4} \exp\left[-\frac{(p-p_a)^2}{4\sigma_{pP}^2}\right], \tag{2}$$

where p_a are the average momenta of the different mass eigenstates and σ_{pP} is the momentum width of the wave packets. The average momenta p_a are determined by the kinematics of the production process taking into account the masses m_a of the mass eigenstates. We assume that all the mass eigenstates are extremely relativistic, i.e. $p_a \gg m_a$. In this case $\Delta E \simeq \Delta p$ and σ_{pP} is determined by the *minimum* between the neutrino energy and momentum uncertainties in the production process (a possible dependence of the momentum widths from the index a can be estimated to be very small and negligible for relativistic neutrinos). Hence, from the uncertainty principle it is clear that σ_{pP} is determined by the *maximum* between the spatial and temporal coherence widths of the production process.

We assume that the Gaussian wave functions (2) are sharply peaked around the corresponding average momentum, i.e. $\sigma_{pP} \ll E_a^2/m_a$, with $E_a \equiv E_a(p_a)$. Under this condition the energy $E_a(p)$ can be approximated by $E_a(p) \simeq E_a + v_a(p-p_a)$, where $v_a = p_a/E_a$ is the group velocity of each wave packet. In this case, the neutrino wave function in coordinate space, $|\nu_\alpha(x,t)\rangle = \langle x|\nu_\alpha(t)\rangle$, is easily calculated after a Gaussian integration to be given by

$$|\nu_\alpha(x,t)\rangle = (2\pi\sigma_{xP}^2)^{-1/4} \sum_a U_{\alpha a}^* \times \exp\left[-iE_a t + ip_a x - \frac{(x-v_a t)^2}{4\sigma_{xP}^2}\right] |\nu_a\rangle, \tag{3}$$

with the orthonormal states $|\nu_a\rangle$ belonging to the mass Fock space ($\langle\nu_a|\nu_b\rangle = \delta_{ab}$). The width σ_{xP} of the mass eigenstate wave packets in coordinate space is related to the momentum width σ_{pP} by the uncertainty relation $\sigma_{xP}\sigma_{pP}=1/2$. From

[1]Taking into account the spatial and temporal coherence widths of the production and detection processes, $x=0$, $t=0$ and $x=L$, $t=T$ are their average space-time coordinates.

the considerations presented after Eq. (2) it is clear that the value of σ_{xP} is given by the maximum between the spatial and temporal coherence widths of the production process.

Let us consider now the detection process which takes place at a distance L and after a time T from the origin of the space-time coordinates and is capable of detecting a neutrino with flavor β. In [4] the amplitude of the flavor changing process was obtained by projecting the state $|\nu_\alpha(L,T)\rangle$, which describes the propagating neutrino, on the flavor state $|\nu_\beta\rangle = \Sigma_a U^*_{\beta a}|\nu_a\rangle$. This procedure neglects the temporal and spatial coherence widths of the detection process. In order to take them into account, the detected neutrino must be described by a wave packet state analogous to that in Eq. (1):

$$|\nu_\beta\rangle = \sum_a U^*_{\beta a} \int dp\, \psi_a(p;p_a,\sigma_{pD})|\nu_a(p)\rangle. \tag{4}$$

The value of the momentum width σ_{pD} is given, for relativistic neutrinos, by the minimum between the neutrino energy and momentum uncertainties in the detection process. The state $|\nu_\beta\rangle$ does not evolve in time because it does not represent a propagating neutrino. The average momenta p_a in the mass eigenstate Gaussian wave functions describing the detected neutrino are the same as those of the corresponding mass eigenstate Gaussian wave functions describing the neutrino propagating between production and detection. This property is a consequence of causality: after the average momenta p_a have been determined by the kinematics of the production process, the mass eigenstates propagate between the two processes and determine the kinematics in the detection process. For example, the moduli of the average momenta p_a of the mass eigenstate wave packets of a muon neutrino produced in pion decay at rest are fixed by the kinematics of the process. When this neutrino is detected, for example, by scattering with a nucleus at rest, each mass eigenstate determines with its average momentum p_a and its mass m_a a different value for the momenta of the recoil particles. Neutrino oscillations are observed if the different mass eigenstates are detected coherently, i.e. if the differences of the energies and momenta of the recoil particles corresponding to different mass eigenstates are smaller than the energy and momentum uncertainty of the detection process.

Taking into account that the detection process takes place at a distance L from the origin of the coordinates, the wave function in coordinate space describing the detected neutrino is given by

$$|\nu_\beta(x-L)\rangle = (2\pi\sigma_{xD}^2)^{-1/4} \sum_a U^*_{\beta a} \times \exp\left[i p_a (x-L) - \frac{(x-L)^2}{4\sigma_{xD}^2} \right] |\nu_a\rangle, \tag{5}$$

where σ_{xD} is defined by the uncertainty relation $\sigma_{xD}\sigma_{pD} = 1/2$. Hence, the value of σ_{xD} is given by the maximum between the spatial and temporal coherence widths of the detection process.

The amplitude of the flavor changing process is given by the overlap

$$A_{\alpha\beta}(L,T) = \int dx \langle \nu_\beta(x-L)|\nu_\alpha(x,T)\rangle. \tag{6}$$

The integral over x is Gaussian and leads to

$$A_{\alpha\beta}(L,T) = \sqrt{\frac{2\sigma_{xP}\sigma_{xD}}{\sigma_x^2}} \sum_a U^*_{\alpha a} U_{\beta a} \times \exp\left[-iE_a T + i p_a L - \frac{(L - v_a T)^2}{4\sigma_x^2} \right], \tag{7}$$

with

$$\sigma_x^2 \equiv \sigma_{xP}^2 + \sigma_{xD}^2. \tag{8}$$

This relation is very important in that it clearly shows that the width σ_x which determines the coherence of the flavor changing process depends on the spatial and temporal coherence widths of both the production and detection processes. The amplitude (7) has the same form as that in [4], with an important difference: the width σ_x in [4] was determined only by the production process, whereas in Eq. (7) also the detection process is properly taken into account.

In order to calculate the oscillation probability, the mass eigenstate energies can be approximated by

$$E_a \simeq E + \xi \frac{m_a^2}{2E}, \tag{9}$$

where E is the energy determined by the kinematics of the production process for a massless neutrino and ξ is a dimensionless quantity that is determined by energy-momentum conservation in the production process to first order in m_a^2/E^2. The quantity ξ is typically of order unity; for example, in neutrino production by pion decay at rest we have $\xi \simeq 0.2$. With this approximation we have $p_a \simeq E - (1-\xi) m_a^2/2E$ and $v_a \simeq 1 - m_a^2/2E^2$. From Eqs. (7) and (9), the probability of $\nu_\alpha \to \nu_\beta$ transitions is given by

$$P_{\alpha\beta}(L,T) \propto \sum_{a,b} U^*_{\alpha a} U_{\beta a} U_{\alpha b} U^*_{\beta b} \times \exp\left\{ -i \frac{\Delta m_{ab}^2}{2E} [\xi T + (1-\xi) L] \right\} \times \exp\left\{ - \frac{(L - v_a T)^2 + (L - v_b T)^2}{4\sigma_x^2} \right\}, \tag{10}$$

with $\Delta m_{ab}^2 \equiv m_a^2 - m_b^2$.

In principle, $P_{\alpha\beta}(L,T)$ is a measurable quantity, but in all realistic experiments the distance L is a fixed and known quantity, whereas the time T is not measured and can have any value, because the source and the detector typically operate for times much longer than the oscillation times $4\pi E/\Delta m_{ab}^2$. Therefore, the quantity that is measured in all experiments is the oscillation probability $P_{\alpha\beta}(L)$ at a fixed distance L, given by the time average of $P_{\alpha\beta}(L,T)$. After integrating over T and imposing the normalization condition $\Sigma_\beta P_{\alpha\beta}(L)=1$, we obtain

$$P_{\alpha\beta}(L)=\sum_{a,b} U_{\alpha a}^* U_{\beta a} U_{\alpha b} U_{\beta b}^* \times \exp\left[-2\pi i \frac{L}{L_{ab}^{\text{osc}}} - \left(\frac{L}{L_{ab}^{\text{coh}}}\right)^2\right] F_{ab}, \qquad (11)$$

where

$$L_{ab}^{\text{osc}}=\frac{4\pi E}{\Delta m_{ab}^2}, \quad L_{ab}^{\text{coh}}=\frac{4\sqrt{2}\sigma_x E^2}{|\Delta m_{ab}^2|} \qquad (12)$$

are the oscillation wavelengths and coherence lengths, respectively, and

$$F_{ab}=\exp\left[-2\pi^2\xi^2\left(\frac{\sigma_x}{L_{ab}^{\text{osc}}}\right)^2\right]. \qquad (13)$$

Equation (11) contains, in addition to the usual expression for the neutrino oscillation probability (the first term in the exponential with $F_{ab}=1$), two additional factors, the first being the second term in the exponential that takes into account the coherence of the contributions of different mass eigenstate wave packets and the second being the additional factor F_{ab}. The factor F_{ab} is equal to unity if $\sigma_x \ll |L_{ab}^{\text{osc}}|$, which is a necessary condition for the observation of neutrino oscillations that must be satisfied by any realistic experiment. If this condition is not satisfied, the interference among different mass eigenstate wave packets is washed out and only a constant transition probability $P_{\alpha\beta}=\Sigma_a |U_{\alpha a}|^2 |U_{\beta a}|^2$ can be observed.

It is important to notice that the integration over the oscillation time is a crucial step in the wave packet treatment of neutrino oscillations in space, because it allows us to eliminate the time degree of freedom, which is not measured in realistic experiments. This elimination is done at the classical level by integrating the probability, which is a classical quantity, and without any unphysical assumption on the energies and momenta of the mass eigenstates, which are determined by the production process [12]. An unphysical assumption would be, for example, the imposition of equal energy to the different mass eigenstates, i.e. $\xi=0$, which would eliminate the time dependence of the oscillatory term in the probability (10).

From Eqs. (8) and (12) one can see that the coherence length L_{ab}^{coh} is proportional to $\sigma_x \equiv \sqrt{\sigma_{xP}^2+\sigma_{xD}^2}$. Hence, L_{ab}^{coh} is dominated by the largest among the temporal and spatial coherence widths of the production and detection process. In particular, a precise determination of the neutrino energy (or momentum) in the detection process implies a small σ_{pD} and a large σ_{xD}, leading to a large coherence length $L_{ab}^{\text{coh}} \simeq 4\sqrt{2}\sigma_{xD}E^2/|\Delta m_{ab}^2|$ (if $\sigma_{pD} \ll \sigma_{pP}$). Hence, as noted in [6], the coherence length can be increased by measuring accurately the energy of the detected neutrino. However, even if, at least in principle, the coherence length can be increased without limit by an extremely precise measurement of the neutrino energy in the detection (and/or production) process, the oscillations lose the coherence for $\sigma_x \gtrsim L_{ab}^{\text{osc}}$, because the factor F_{ab} becomes important and has the effect of suppressing the interference of ν_a and ν_b for $\sigma_x \gg L_{ab}^{\text{osc}}$. Physically this is due to the fact that the spatial or temporal width of the detection (or production) process becomes larger than the oscillation length, leading to the washout of the oscillations.[2] In order to understand the reason for this washing-out let us consider, for example, the case in which the detection process has a small spatial coherence width and a large temporal coherence width that dominates the total coherence width σ_x. As shown by Eq. (10), in this case the interference between the mass eigenstates a and b is not suppressed if the average detection time T does not differ from L/v_a and L/v_b by more than $\sim\sigma_x$. This is due to the fact that the modes of the detection process corresponding to the different mass eigenstates oscillate coherently during a time interval of width σ_x around T, even when the mass eigenstate wave packets are far from L.[3] Then it is clear that if σ_x is larger than the oscillation length, the phase of the interference term depends crucially on the value of T. In this case, the average of the transition probability over T washes out the interference. This result is in agreement with the fact that if the neutrino energy is known with an accuracy smaller than the energy difference between the two mass eigenstates ν_a and ν_b implied by Eq. (9), $\Delta E_{ab}=\xi\Delta m_{ab}^2/2E=2\pi\xi/L_{ab}^{\text{osc}}$, only one of the two mass eigenstates contributes to each event and the interference of the contributions of the two mass eigenstates that produces neutrino oscillations is absent (see [3]).

Therefore, using Eq. (12) one can see that the coherence length has the upper bound

$$L_{ab}^{\text{coh}} \lesssim E(L_{ab}^{\text{osc}})^2=\frac{16\pi^2E^3}{(\Delta m_{ab}^2)^2}, \qquad (14)$$

beyond which the coherence length loses its meaning, because the coherence of the process is lost for any value of the distances L.

In conclusion, we have shown that the temporal and spatial coherence widths of the detection process can easily be

[2]Notice that a similar effect is obtained if the distance L from the source is not known and the time-dependent probability $P_{\alpha\beta}(T)$ is obtained by averaging $P_{\alpha\beta}(L,T)$ over L. Such a situation is realized, for example, in the calculation of the effects of neutrino oscillations in the early universe.

[3]Notice that if σ_{xD} is dominated by the temporal coherence width of the detection process, the Gaussian approximation in Eq. (5) for the wave function of the detected neutrino introduces an unphysical interaction for times smaller than the time of arrival of the propagating neutrino to the detector. In this case a more realistic approximation can be obtained by inserting in Eqs. (5) and (6) a $\theta(x-L+\tilde{\sigma}_{xD})$, where $\tilde{\sigma}_{xD}$ is the spatial coherence width of the detection process. This approximation requires a numerical solution of the integral in Eq. (6) and will be discussed in detail elsewhere [13].

incorporated in the quantum mechanical wave packet treatment of neutrino oscillations. As a result, we confirm the observation presented in [6] that an accurate measurement of the neutrino energy (or momentum) in the detection process can increase the coherence length. However, the wave packet treatment presented here shows that the coherence length cannot be increased beyond the upper bound given by Eq. (14) without losing the coherence of the oscillation process.

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