Global renormalization group

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The motivation and the challenge in applying the renormalization group for systems with several scaling regimes is briefly outlined. The four-dimensional ϕ^4 model serves as an example where a nontrivial low-energy scaling regime is identified in the vicinity of the spinodal instability region. It is pointed out that the effective theory defined in the vicinity of the spinodal instability offers an amplification mechanism, a precursor of the condensation, that can be used to explore nonuniversal forces at high energies.
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I. INTRODUCTION

The idea of the renormalization group [1,2] is to view the different interactions in a hierarchical manner, by building up the more complex systems from their ''elementary'' constituents. The renormalization group should describe the manner the elementary particles rearrange themselves in forming the composite particles captured by the detectors with a given resolution, the complex structure of ordinary matter and finally the transition from the microscopic to the macroscopic physics. Because of the obvious technical difficulties of such an ambitious project the realization of this idea is severely restricted. What is usually achieved by the help of different analytical approximation methods is to analyze the dependence on the observational scale in a scaling regime where the evolution equations are linearizable or at least perturbative. The result is the renormalized trajectory, the scale dependence of the coupling constants in a regime that is dominated by a given interaction. One may call this a local analysis of the renormalization group flow, performed in the individual scaling regimes that usually but not necessarily agree with the vicinities of the fixed points. A number of important results have been derived in this manner.

The real challenge, we believe, is to describe the transmutation of one set of scaling laws into another one as the scale of the observation is changed. This requires the construction of the renormalized trajectory connecting different scaling regimes. Such a manifestly nonperturbative phenomenon will be studied in this paper in the case of a simple model with two scaling regimes, the single component ϕ^4 scalar field theory in the spontaneously broken phase. This model supports an asymptotic UV scaling regime well above the particle mass and an intermediate-energy scaling regime at the onset of the spinodal instability. Our goal is to connect the two scaling regimes and the global reconstruction of the renormalized trajectory. A preliminary account of our results has already been given in Ref. [3].

The possibility of the onset of a new scaling law has been raised by the introduction of the ''dangerously irrelevant parameters'' [4]. The usual classification of the operators in a scaling regime with respect their response to the change of the scale is based on the perturbation expansion and assumes the regular behavior of the system in the weak coupling regime. If the evolution equations develop singular behavior in the weak coupling limit then it may happen that a perturbatively irrelevant coupling constant experiences new, nonperturbative scale dependence. This mechanism can be identified in different systems with condensates [5–7], where the singular dependence of the saddle points in the coupling constants provides the mechanism to turn a perturbatively irrelevant coupling constant into an important one. It is conjectured in this paper that the condensation might serve as a quite general mechanism to generate new, important coupling constants and the dynamical renormalization group that addresses the question of the scale dependence in nonequilibrium, time-dependent phenomena is well suited to the study of such a question.

We rely in this work on the renormalization group realized by a sharp cutoff in momentum space, in the framework of the gradient expansion for the action. Such a realization of the cutoff renders the systematic gradient expansion questionable, so we restrict ourself to the lowest order, local-potential approximation [8,9]. But it should be mentioned that the usual remedies of the problem, the use of a smooth cutoff or periodic Brillouin zone are not compatible with the loop expansion for systems with a condensate at finite-momentum scales. In fact, though the loop expansion produces the action of the effective theory as a power series in $\hbar$, the saddle point, the minimum itself is not necessarily a polynom of $\hbar$. In order to preserve $\hbar$ as a small parameter, all of the stable modes have to be eliminated in the loop

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expansion before we compute the contributions of the unstable modes. Such a successive elimination of the degrees of freedom introduces automatically the strategy of the renormalization group with sharp cutoff, and renders the smooth cutoff regularization inconsistent with the loop expansion.

The organization of the paper is the following. The presence of several scaling regimes is pointed out for the theory of everything and for the BCS ground state in Sec. II. The role the Bose condensation plays in generating new scaling laws is elucidated in Sec. III. The infinitesimal, renormalization group equation is derived for the ϕ^4 model in Sec. III. The possibility of a new scaling regime in the vicinity of the spinodal instability is pointed out in Sec. IV. The numerical results for the solution of the renormalization group equation are presented in Sec. V, in the case of the four-dimensional ϕ^4 model. Finally, Sec. VI contains the summary.

II. HIDDEN COUPLING CONSTANT

The most fundamental and at the same time the most complex appearance of the multiple scaling laws can be found in the renormalized trajectory of the theory of everything (TOE). The simplest procedure to describe the manner the TOE gives rise a chain of lower-energy effective theories is the so-called matching. This method where we match two theories at their crossover scale can not give an account of the change of the scaling laws in a dynamical manner, because different sets of coupling constants are used at the two sides of the crossover. Instead, one should follow the original strategy of the Wilson-Kadanoff blocking, and put all coupling constants in the lagrangian from the very beginning that will later be generated by the blocking. Thus the coupling constant space of the TOE should contain not only the renormalizable parameters but any coupling constant we ever need in physics. For example the quark-gluon vertex or a coupling constant of the Hubbard model of the condensed matter physics have to be considered as complicated composite operators in terms of the fundamental particles of the TOE.

The renormalized trajectory, depicted schematically in Fig. 1, approaches several fixed points in its way towards the infrared limit. To understand this better consider the increase of the observational energy in the regime 1–60 GeV. The evolution of the coupling constants receive their dominant contributions from the strong interactions, from the radiative corrections of QCD, and the renormalized trajectory is in the scaling regime of QCD. Had our world contained the strong interactions only, the renormalized trajectory would have converged to the fixed point of QCD with increasing energy. But the weak interactions become important as we reach the characteristic energy of the electroweak theory, and the renormalized trajectory turns away from the QCD fixed point. This happens because the running coupling constants of the nonrenormalizable quark vertices generated by the exchange of the intermediate vector bosons increase with the energy. They saturate at the crossover between the strong and the electroweak interactions where the guidance of the evolution is taken over by the exchange of the W and the Z bosons. In a similar manner, the fixed points of all the other, renormalizable, effective theories are approached by the renormalized trajectory, but the higher-energy processes always prevent convergence as the energy is increased, except at the last fixed point, at the TOE. In the regime of solid state physics, we can influence the evolution of the running coupling constants by the environmental variables, such as the temperature or chemical potentials [10]. In this manner the renormalized trajectory may bifurcate and follow different path in different environments and finally arrive in different thermodynamical phases at the infrared fixed point.

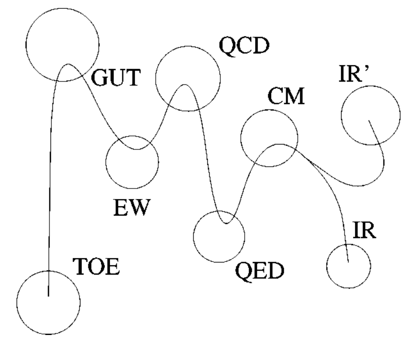


FIG. 1. The renormalized trajectory of the theory of everything (TOE). It passes by the fixed points of the grand unified models (GUT), the unified electroweak theory (EW), the strong interactions (QCD), the electromagnetic interactions (QED), certain fixed points of the solid state and condensed matter physics (CM), and finally approaches the ultimate IR fixed point. The trajectory may be influenced by the environment and reach different thermodynamical phases in the IR regime.

Facing such a complex system, the usual argument about universality appears as an oversimplification. In fact, at each scaling regime we classify the operator algebra of the model in a local manner according to the appropriate scaling laws. It may happen that an operator possesses different local classifications, and it is found relevant at one scaling regimes but becomes irrelevant at another one. The importance or unimportance of such a coupling constant must be decided in a scaling-regime-independent, global manner.

A simpler example where the global behavior is important can be found in QED containing the electron and a heavy pointlike particle with charge $+Z$ playing the role of a nucleus. Suppose that certain environment variables, such as the temperature and the baryon chemical potential are chosen in such a manner that the vacuum is a solid state lattice in the superconducting phase. We can distinguish two asymptotic scaling regimes in this model:

Asymptotic UV scaling. At energies above the nucleus mass the evolution equation is given in terms of the minimal coupling vertices and the vacuum can be considered perturbative. As indicated above we do not require the existence of a fixed point for the identification of the asymptotic scaling laws. In this manner we can ignore the possible problems arising from the nonasymptotically free character of QED, and consider the scaling laws only up to the UV Landau pole. The relevant and the marginal operators are the usual renormalizable ones, and are given in the framework of the perturbation expansion by a power-counting argument. The

TABLE I. The four classes of the coupling constants in QED.

UV	IR	Fig. 2	Example
relevant	relevant	(a)	$\tilde{m}_e \Lambda \bar{\psi}_e \psi_e$
relevant	irrelevant	(b)	$\tilde{m}_\mu \Lambda \bar{\psi}_\mu \psi_\mu$
irrelevant	relevant	(c)	$\tilde{G} \Lambda^{-2} (\bar{\psi}_e \psi_e)^2$
irrelevant	irrelevant	(d)	$\tilde{c} \Lambda^{-5} (\bar{\psi}_e \psi_e)^3$

size of the scaling regime is limited by the UV cutoff or the Landau pole.

Asymptotic IR scaling. For energies below the scale of eV the collective phenomena of the solid state lattice dominate the scaling laws. The inhomogeneity of the vacuum is a key element. No systematic classification of the scaling operators is known, but because of the massless acoustic phonons, the existence of nontrivial relevant or marginal operators cannot be excluded. The lower edge of the scaling regime is limited by the loss of quantum coherence.

The asymptotic scaling regimes can be extended by embedding the model into a higher-energy, more fundamental, renormalizable theory and by approaching absolute-zero temperature. These asymptotic scaling laws are important since their relevant coupling constants may influence the physics in a strong manner. The four different possibilities with respect to these scaling laws are shown in Table I and in Fig. 2 for coupling constants whose dimension is removed by the cutoff Λ. The electron mass is a relevant parameter of the QED Lagrangian and remains relevant for the solid state lattice, too. The muon mass is relevant in the UV scaling regime but the muon-induced processes are overwhelmed by the electron-induced ones at low energy and m_μ becomes irrelevant in the IR. The four-fermion interaction is, at the same time, nonrenormalizable and represents the driving force to the BCS superconducting phase; it is marginal in the IR regime [13,14]. Recently other indications of the deviation from the Fermi-liquid behavior resulting from relevant or marginal operators of the IR regime have been found for high-T_c cuprates [16], as well. Finally the six-fermion or any higher order vertex is irrelevant in the IR regime because its effect is reduced to the multiple application of the four-fermion vertex. The four-fermion interaction plays a special role: On the one hand, it is usually left out from the microscopic Lagrangian because it is suppressed in the UV scaling regime. On the other hand, it has a key role at the IR regime in controlling the attraction between the electrons.

The qualitative behavior shown in Fig. 2(c) raises the following possibility. The suppression of the irrelevant cou-

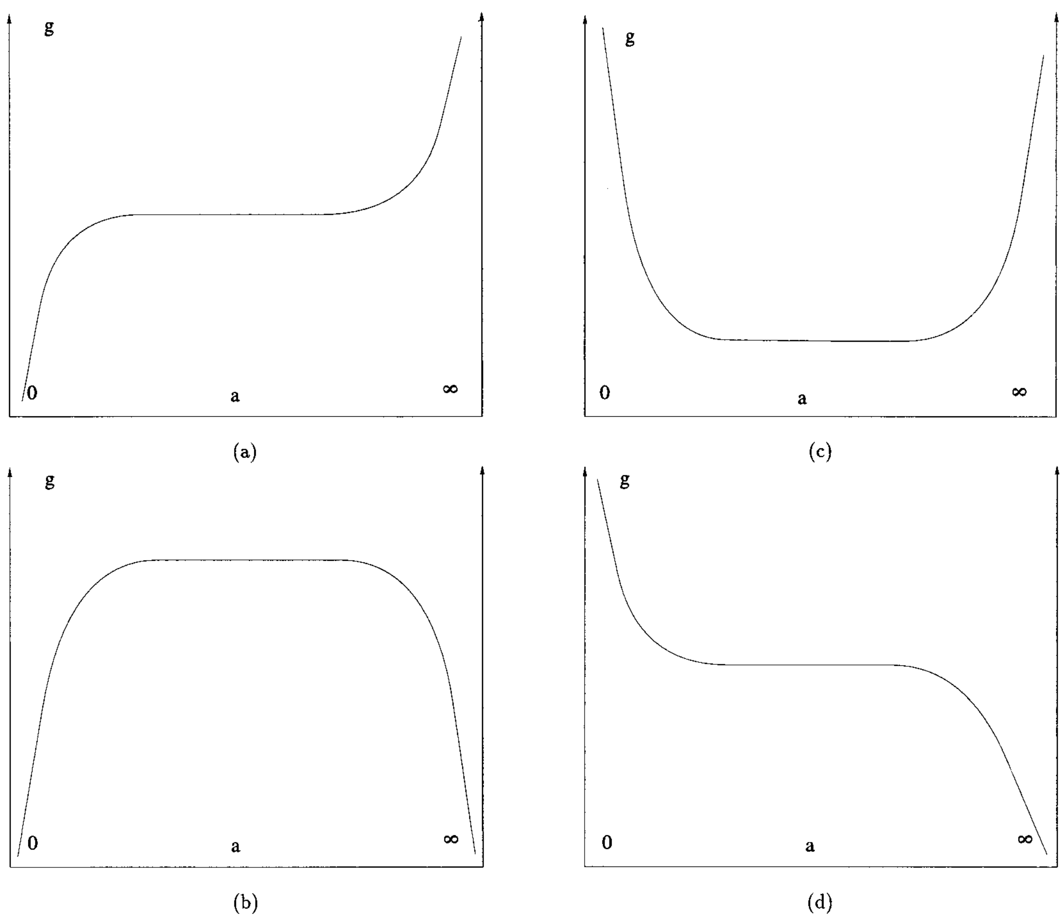


FIG. 2. The qualitative dependence of the running coupling constant of Table I as a function of the cutoff $a = 2\pi/\Lambda$. The asymptotic UV and IR scaling regimes are shown. The coupling constant is supposed to be constant in between for simplicity.

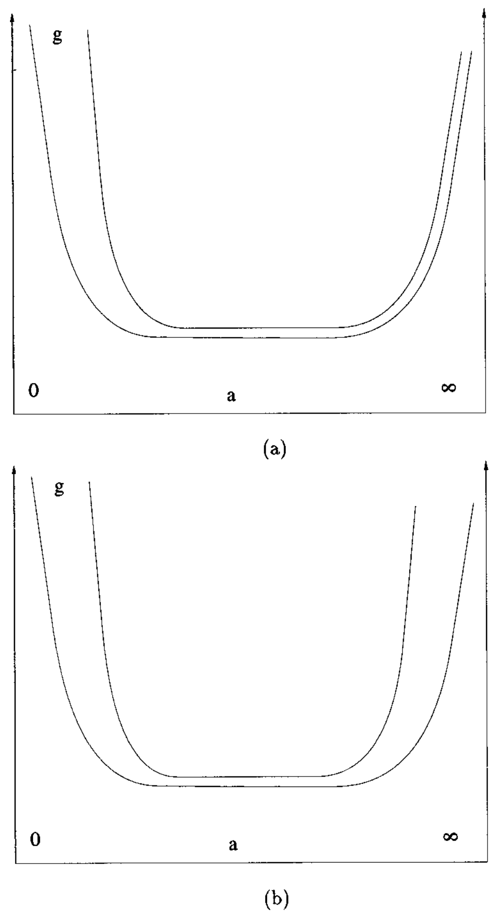


FIG. 3. Two possible behaviors of a nonrenormalizable coupling constant what is relevant at the low-energy scaling regime, case (c) in Table I. Each plot shows two curves what belong to initial conditions at the cutoff what differ in the value of the irrelevant coupling constant only.

pling constants in the UV scaling regime is used to explain the universal behavior of the models. It is certainly correct to expect that the renormalized trajectories whose initial conditions in the ultraviolet differ only in the values of the irrelevant coupling constants approach each other as we move towards the infrared direction. But if there is another scaling regime where an operator that was irrelevant in the UV scaling regime turns out to be relevant, then the resulting amplification process may undo the suppression at the UV. May one find a ''hidden coupling constant'' in this manner that has to be put into the microscopic action of the UV regime but influences the dynamics only in the IR? The answer to this question is nontrivial even if we can identify a nonrenormalizable operator that is relevant in the low-energy scaling regime. In fact, it may happen that the increasing value of this coupling constant during the lowering of the observational energy happens to be independent of its initial value at the UV cutoff. This possibility is shown schematically in Fig. 3(a). The UV scaling laws suppress the dependence on the initial values of the nonrenormalizable coupling constants. If the evolution equations have no instability or other nonanalytic features at low energies, then this suppressed sensitivity and universality is observed down to zero energy. In Fig. 3(b) the value of the coupling constant at the cutoff becomes numerically negligible at intermediate energies according to universality. However the low-energy scaling laws amplify the numerically small nonuniversal value. Depending on the critical exponents and the size of the scaling regimes the amplification of the sensitivity at the IR side may be comparable or even stronger than the suppression in the UV regime. This is a hidden coupling constant because it is undetectable at finite energies nevertheless its small value influences the IR physics in a nonuniversal manner. By rearranging a longer IR evolution we can in principle uncover the presence of nonuniversal interactions at higher energies.

It was pointed out that a relevant operator of the IR scaling regime may change the ground state either by generating bound states that condense or by driving the system to strong couplings until the growth of the coupling constants is cut off by other quantum effects [14]. Within the framework of the saddle-point expansion these two possibilities coincide, and the issue of the hidden coupling constant is that the growing coupling constants might be slowed down in a manner that contains information about the nonuniversal interactions at the microscopic scale. In order to clarify this question, namely, the sensitivity of the IR end of the renormalized trajectory on the UV initial conditions, we must obtain and solve the renormalization group equations for sufficiently many coupling constants, globally.

The coexistence of several scaling regimes have already been studied in condensed matter physics, where competing interactions are represented by the possibility of approaching different IR fixed points [15]. The difference between such systems and the TOE is that only one of the possible scaling laws are realized in the former case. On the contrary, the scaling regimes occur at different energy scales and the system visits each of them sequentially in the TOE.

The renormalized trajectory of TOE sketched in Fig. 1 reflects the usual conflict between the ''fundamental'' and ''applied'' physics. The fundamental, microscopic parameters should be determined with the help of the renormalization conditions imposed at the energy scale $\mu_{\rm fund}$, that is, within the scaling regime of the fundamental interactions, and is far from the complexity entering at lower energies. So long as the TOE is renormalizable or finite the perturbation expansion can be used to show that the resulting renormalized trajectory is independent of the choice of $\mu_{\rm fund}$, and can in a natural manner be characterized by the coupling constants observed at high energies. In this sense the fundamental laws of physics are determined by the high-energy experiments and the description of the lower-energy, complex systems requires ''only'' the capacity to apply the fundamental laws in a complicated situation.

But the fallacy of this view is clear: In the absence of hidden coupling constants where the low-energy effective parameters are determined in an autonomous manner, there is no need for the precise measurement of the high-energy parameters in order to reproduce the low-energy physics. In fact, according to the renormalized perturbation expansion and universality, the bare coupling constants are characterized in a unique manner by the renormalized ones defined at low energies, $\mu_{\rm compl} \ll \mu_{\rm fund}$. When hidden coupling constants are present then the precision of the measurement of the high-energy parameters required to predict a low-energy phenomenon with a reasonable accuracy might render the global determination of the renormalized trajectory illusory. In other words, the specification of the initial conditions with a reasonable accuracy might be unimportant for the nonlinear evolution, similarly to the nonintegrable chaotic systems. The challenge lies in comprehending the matching of the different scaling islands of Fig. 1.

III. RENORMALIZATION AND CONDENSATION

In the perturbatively implemented renormalization group we have to assume that the renormalized trajectory does not leave the vicinity of the Gaussian fixed point. If there is a small parameter assuring this, as in $4-\epsilon$ dimensions, then the set of the relevant or marginal operators is the same at each scaling regime, and the global behavior of the renormalized trajectory contains nothing new compared to the local analysis performed at the individual scaling regimes. If the renormalized trajectory explores the nonperturbative regions then the global analysis is very interesting, but we lose the general, perturbative characterization of the flow. A window of opportunity for the analytical studies opens if the nonperturbative features of the evolution can be reproduced by the only known, systematic, analytical, nonperturbative method, the semiclassical expansion.

A nontrivial saddle point corresponds to a coherent state formed by the condensate of bosons. The importance of this condensate depends on the multiplicity of the coherent state(s). We shall demonstrate this from three different points of view.

Euclidean quantum field theory for the vacuum. In the semiclassical solution to the vacuum with ferromagnetic condensate only the lowest lying excitation level with vanishing momentum is populated macroscopically.[1] When the system is placed in a finite geometry then the lowest lying state might become inhomogeneous; nevertheless the particles still condense in a single mode. The impact of the condensate is stronger if the particles form coherent states in a large number of modes. This is the case for solitons or instantons, the localized saddle points with high entropy. Another, less studied example is given by theories with higher-order derivative-terms in the action [7]. When their coupling constant is properly chosen then particles with nonvanishing momentum may condense and generate an inhomogeneous vacuum. Since the density of modes with momentum p is proportional to p^{d-1} in d dimensions there are more particle modes participating in the condensation at finite p than at $p=0$.

Real time dynamics. The condensation is triggered by the wrong sign of the forces trying to restore the equilibrium position. There are always several unstable modes which compete in the condensation when the system starts with the naive vacuum [11]. Depending on the initial conditions different slower collective modes might be formed that distribute the energy and create time-dependent condensates in several other particle modes. After sufficiently long times, the energy of the excited modes get diffused by the friction terms over infinitely many modes, without creating macroscopic population, and only the particles in the lowest lying mode remain condensed. The large-amplitude, transient excitations above the true vacuum are described by a large number of modes with condensate. This is the realm of nucleation and the spinodal decomposition [12].

The former refers to the metastability of the vacuum which decays by large amplitude fluctuations. The simplest is to assume that this takes place by the spontaneous formation of spherical droplets of the stable vacuum whose free energy is $F=4\pi R\sigma-4\pi R^3\Delta F/3$, where σ, ΔF, and R stand for the surface tension, the free energy density difference between the metastable and the stable vacuum, and the droplet radius, respectively. These droplets extend over the whole volume if they are sufficiently large, $R>R_{cr}=\sigma/\Delta F$. The spinodal decomposition is observed when there is no more finite threshold for the instability and the infinitesimal fluctuations are enough to trigger the decay of the homogeneous vacuum. This can be recognized in the framework of the static description by the appearance of the negative eigenvalues for the Euclidean propagator. The fast increase of the corresponding elementary excitation amplitudes drives an inhomogeneous separation of two stable values of the local dynamical variable.

Returning to the long-time, low-energy excitations, they experience a single condensed mode; and the static nonperturbative condensate is simpler to describe.

Saddle points of the renormalization group. When the renormalization group is implemented then we eliminate the modes in descending order in the energy, and we may encounter nontrivial saddle points in the way. If we are interested in the effective theory for the low-energy fluctuations around the true vacuum in systems without a localized saddle point, then the condensate occurs only at the last mode, at the infrared fixed point. When the effective theory is sought for large amplitude excitations, then the elimination of the modes in the presence of such a background field may induce saddle points for the blocking procedure earlier, at finite scales. The nonperturbative contribution of these saddle points may modify the direction of the renormalization group flow in a substantial manner [17]. As an example of this mechanism, the emergence of an irrelevant coupling constant in the dynamics of domain walls has already been noted in the two-dimensional nonlinear $O(2)$ model [18].

In short, the saddle points of the blocking procedure occur with higher multiplicity at finite energies, and may modify the scaling laws in a more substantial manner than the saddle points of the static system in the true vacuum. Such effects could be seen by the proper implementation of the dynamical renormalization group for large amplitude fluctuations. It is worthwhile noting that the spinodal instability region starts just at the vacuum in the presence of the Goldstone modes, and the saddle point contributions might influence the small quantum fluctuations around the true vacuum, as well [19].

[1] The degeneracy of the vacuum with spontaneously broken symmetries does not change the picture since there is only one vacuum in each ''world'' containing the states connected by local operators.

IV. RENORMALIZATION GROUP EQUATION

The study of the different scaling regimes requires the handling of a large number of coupling constants which can be achieved by converting the evolution equations referring to the individual coupling constants into a differential equation for the generating function for the coupling constants, i.e., the blocked action. We shall work with the four-dimensional ϕ^4 theory in Euclidean space-time. The cutoff k in the momentum space will be changed infinitesimally, $k\to k-\Delta k$, which generates a new small parameter $\Delta k/M$, where M is a quantity of dimension of mass made up by the dimensional parameters of the model and the cutoff k. This small parameter will be used to show that the renormalization group equation derived in the one-loop approximation becomes exact as $\Delta k/M\to 0$, because the higher loop contributions are suppressed by $\Delta k/M$.

There are two limitations to bear in mind in turning this scheme into a feasible algorithm. The first is that even thought there is a new small parameter to suppress the higher loop contributions to the evolution equations the argument presupposes the applicability of the loop expansion. Therefore the use of the resulting ''exact'' equation is questionable in the strong coupling region, beyond the validity of the loop expansion. The other limitation comes from the parametrization of the effective-action functional $S_k[\phi]$ with the cutoff k. It is usually done by relying on the gradient expansion:

$$S_k[\phi]=\int d^dx\left[Z_k[\phi(x)]\frac{1}{2}[\partial_\mu\phi(x)]^2+U_k[\phi(x)]\right]+O(\partial^4). \tag{1}$$

The assumption that terms with higher order derivatives are less important is equivalent with the belief that the action is a local functional. This is reasonable at high energies because all of the relevant operators of the short distance scaling laws should be local. But there is no reason to exclude operators which are nonlocal at the scale of the cutoff at low-energy, scaling regimes, and these terms of the action may create nonlocal effects. By holding to the assumption of locality we shall set

$$Z_k(\phi)=1 \tag{2}$$

in this work.

We eliminate the modes with momentum $k-\Delta k<p<k$ and find the blocked action

$$e^{-(1/\hbar)S_{k-\Delta k}[\phi]}=\int D[\phi']e^{-(1/\hbar)S_k[\phi+\phi']}. \tag{3}$$

The Fourier transforms of the variable $\phi(p)$ and $\phi'(p)$ are nonvanishing for $p<k-\Delta k$ and $k-\Delta k<p<k$, respectively. The functional integration is carried out in the framework of the loop expansion

$$S_{k-\Delta k}[\phi]=S_k[\phi+\phi_0']+\frac{\hbar}{2}\mathrm{tr}'\ln\,\delta^2S+O(\hbar^2), \tag{4}$$

where tr′ denotes the summation within the shell $[k-\Delta k,k]$. The positive semidefinite operator

$$\delta^2S(x,y)=\frac{\delta^2S_k[\phi+\phi']}{\delta\phi'(x)\,\delta\phi'(y)}\bigg|_{\phi'(x)=\phi_0'(x)} \tag{5}$$

and the saddle point ϕ_0',

$$\frac{\delta S_k[\phi+\phi']}{\delta\phi'(x)}\bigg|_{\phi'(x)=\phi_0'(x)}=0, \tag{6}$$

are computed by keeping the background field $\phi(x)$ fixed. Equation (4) reduces to the Wegner-Houghton equation [8] when the saddle point is trivial, $\phi_0'=0$.

Observe that the loop integration is made in a region of volume $\Omega_dk^{d-1}\Delta k$ where

$$\Omega_d=\frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{7}$$

The contributions $O(\hbar^n)$ are given in terms of n-fold loop integrals. So long as the integrands are bounded in the domain of integration the contributions $O(\hbar^n)$ contain the multiplicative factor $(\Delta k/M)^n$. The integrands are the products the propagator, the inverse of Eq. (5) evaluated within the subspace $k-\Delta k<p<k$. If the restoring force acting on the fluctuations is nonvanishing then the propagator is bounded and the higher loop contributions drop from Eq. (4) when $\Delta k/M\to 0$. The formal argument showing the suppression of the higher loop contribution to the renormalization group equation is presented in the Appendix.

The only quantity needed in the local potential approximation, Eqs. (1),(2), is the potential $U_k(\Phi)$ so it is natural to choose for its determination a homogeneous background field $\phi(x)=\Phi$ for which the kinetic energy is vanishing. An additional bonus of this choice is that the saddle point is nontrivial, $\phi_0'\neq 0$, just in the spinodal unstable phase. Outside of the spinodal unstable phase one can set $\phi_0'=0$ which yields

$$\begin{aligned}\exp-S_{k-\Delta k}[\Phi]&=\exp-L^dU_{k-\Delta k}(\Phi)\\&=\exp-S_k[\Phi+\phi_0']-\frac{\hbar}{2}\mathrm{tr}'\ln\delta^2S+O(\hbar^2)\\&=\exp-L^dU_k(\Phi)-\frac{\hbar}{2}L^d\int'\frac{d^dp}{(2\pi)^d}\\&\quad\times\ln[p^2+U_k''(\Phi)]+O(\hbar^2),\end{aligned} \tag{8}$$

where the plane wave elementary excitations of the homogeneous vacuum were used to compute the trace in the third line and $\int' d^4p=\int_{k-\Delta k<p<k}d^4p$. In this manner one arrives at the finite difference equation

$$U_k(\Phi)-U_{k-\Delta k}(\Phi)=-\frac{\hbar}{2}\int\frac{d^dp}{(2\pi)^d}\ln[p^2+U_k''(\Phi)]\times\left[1+O\left(\frac{\hbar\Delta k}{M}\right)\right], \tag{9}$$

where

$$U_k''(\Phi)=\frac{\partial^2 U_k(\Phi)}{\partial\Phi^2}. \tag{10}$$

By taking the limit $\Delta k\to 0$ we find the differential equation

$$k\frac{\partial}{\partial k}U_k(\Phi)=-\hbar\frac{\Omega_d k^d}{2(2\pi)^d}\ln[k^2+U_k''(\Phi)] \tag{11}$$

with the help of the polar coordinates in the momentum space.

The evolution equation (11) is actually the one-loop resummation of the perturbation expansion. This can be seen by expanding in the non-Gaussian pieces of the potential,

$$k\frac{\partial}{\partial k}U_k(\Phi)=-\hbar\frac{\Omega_d k^d}{2(2\pi)^d}\Bigg[\ln(k^2+m_k^2)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{U_k''(\Phi)-m_k^2}{k^2+m_k^2}\right)^n\Bigg], \tag{12}$$

where $m^2(k)=U_k''(0)$. The evolution equation for the effective vertices at zero momentum, i.e., the leading order of the gradient expansion is now the explicit sum of the one-loop graphs defined by the free propagator $G^{-1}(k)=k^2+m_k^2$. If we ignore the k dependence in the right hand side then $U_0(\Phi)$ is the usual one-loop effective potential obtained in the bare or the renormalized perturbation expansion for

$$m^2(k)\to m_B^2,\quad U_k''(\Phi)-m_k^2\to\frac{\lambda_B}{2}\Phi^2 \tag{13}$$

or

$$m^2(k)\to m_R^2,\quad U_k''(\Phi)-m_k^2\to\frac{\lambda_R}{2}\Phi^2, \tag{14}$$

respectively. The evolution generated by the renormalization group provides a partial resummation of the perturbation expansion by piling up the effects of the modes which have already been eliminated in the k dependence of the running coupling constants, $U_k(\Phi)$ in our case. In the usual renormalization group method, which is based on the renormalized perturbation expansion, only the renormalizable coupling constants $\partial^2 U_k(0)/\partial\Phi^2$, $\partial^4 U_k(0)/\partial\Phi^4$ pile up the effects of the elimination. In solving the differential equation (12) the nonrenormalizable coupling constants, i.e., the higher order derivatives are evolved, as well. Such an extension of the usual scheme is necessary to find the eventual hidden coupling constants.

It is easy to perform the direct numerical integration of Eq. (11) with an initial condition $U_\Lambda(\Phi)$ given at $k=\Lambda$ towards the infrared direction after sufficient attention is paid to the instabilities arising from the finite Φ and k resolution. However, our goal is to arrive at a description of the modification of the scaling laws which requires that the restoring force for the fluctuations, the argument of the logarithm function in Eq. (11), be vanishing at $k=k_0(\Phi)\neq 0$. The singular behavior of the differential equation made the numerical quadratures unstable and forced us to resort to methods other than the discretization of the Φ space. Instead of discretizing the variable Φ we truncated the Taylor expansion of $U_k(\phi)$ at finite order.

To this end we introduce the coupling constants defined at $\Phi=\Phi_0$ by

$$g_n(k)=\frac{\partial^n U_k(\Phi_0)}{\partial\Phi^n}. \tag{15}$$

The result is

$$U_k(\Phi+\Phi_0)=\sum_n\frac{g_n(k)}{n!}\Phi^n. \tag{16}$$

The β functions are defined as

$$\beta_n=k\frac{\partial}{\partial k}g_n(k)=\frac{\partial^n}{\partial\Phi^n}k\frac{\partial}{\partial k}U_k(\Phi)_{|\Phi=\Phi_0}, \tag{17}$$

where the analyticity of the potential in k and Φ was assumed. They are obtained by taking the successive derivatives of Eq. (11),

$$\beta_n=-\hbar\frac{\Omega_d k^d}{2(2\pi)^d}\mathcal{P}_n(G_1,\ldots,G_{n+2}), \tag{18}$$

where

$$G_n=\frac{g_n}{k^2+g_2} \tag{19}$$

and

$$\mathcal{P}_n=\frac{\partial^n}{\partial\Phi^n}\ln[k^2+U_k''(\Phi)] \tag{20}$$

is a polynom of order $n/2$ in the variables G_j, $j=2,\ldots,n+2$.

In order to exploit the simplification offered by the symmetry $U_k(-\Phi)=U_k(\Phi)$ we set $\Phi_0=0$, which cancels the odd vertices and yields, for the even ones,

$$\begin{aligned}\mathcal{P}_2&=G_4,\\ \mathcal{P}_4&=G_6-3G_4^2,\\ \mathcal{P}_6&=G_8-15G_6G_4+30G_4^3,\ \ldots\ .\end{aligned} \tag{21}$$

One can verify diagrammatically that the system of equations (20),(21) is a compact rewriting of the one-loop contribution to the β functions of the effective vertices.

It is useful to obtain the evolution equations for the coupling constants whose dimension is removed by the cutoff:

$$\begin{aligned}\tilde\beta_n&=\tilde k\frac{\partial}{\partial\tilde k}\tilde g_n(\tilde k)=\frac{\partial^n}{\partial\tilde\Phi^n}\tilde k\frac{\partial}{\partial\tilde k}\tilde U_{\tilde k}(\tilde\Phi),\\ &=\left[n\left(\frac{d}{2}-1\right)-d\right]\tilde g_n-\hbar\frac{\Omega_d}{2(2\pi)^d}\mathcal{P}_n(\tilde G_2,\ldots,\tilde G_{n+2}),\end{aligned} \tag{22}$$

where

$$k=\Lambda\tilde{k},\quad U=k^d\tilde{U},\quad \Phi=k^{d/2-1}\tilde{\Phi},\quad g_n=k^{n(1-d/2)+d}\tilde{g}_n, \tag{23}$$

and

$$\tilde{G}_n=\frac{\tilde{g}_n}{1+\tilde{g}_2}. \tag{24}$$

V. SCALING REGIMES

The ϕ^4 model has two scaling regimes in the symmetrical phase, separated by a crossover at the mass, $k_{\rm cr}^2=g_2(k_{\rm cr}^2)$. For $k\gg k_{\rm cr}$,

$$G_n=\frac{g_n}{k^2}\left[1+O\left(\frac{g_2}{k^2}\right)\right],\quad \tilde{G}_n=\tilde{g}_n[1+O(\tilde{g}_2)], \tag{25}$$

and the evolution of the coupling constants is that of a massless theory with the only scale of the cutoff. Below the crossover scale we have

$$G_n=\frac{g_n}{g_2}\left[1+O\left(\frac{k^2}{g_2}\right)\right],\quad \tilde{G}_n=\frac{\tilde{g}_n}{\tilde{g}_2}[1+O(\tilde{g}_2^{-1})], \tag{26}$$

and the evolution comes to a halt due to the factor k^d in right-hand side of Eq. (11). This is as expected since a theory with mass gap has the quadratic mass term as relevant operator and all non-Gaussian coupling constants are irrelevant. The β functions are always dominated by the highest order coupling constant which is linear in the coupling constant in question. In fact, this term contains the lowest power of $1/k^2$ at high energies. At the IR side this term is the most important because the others contain the product of more non-Gaussian coupling constants which are supposed to be small.

Qualitatively new scaling proprieties can be established if the β functions are dominated by other terms or receive comparable contributions from different terms. This is certainly the case when the restoring force

$$D(k)=k^2+g_2(k),\quad \tilde{D}(\tilde{k})=1+\tilde{g}_2(\tilde{k}) \tag{27}$$

is vanishing. Suppose that a theory specified at the cutoff by $U_\Lambda(\Phi)$ is in the symmetrical phase, i.e., $g_2(0)>0$ for the choice $\Phi_0=0$. The mass gap decorrelates the field variables which are well separated in the space-time, and the central limit theorem asserts that $U_k(\Phi)$ approaches a quadratic form and $g_2(k)$ is monotonically increasing as $k\to 0$. Perturbation expansion gives a more detailed picture: $g_2(k)$ is a monotonically decreasing ($\beta_2<0$) and $D(k)>0$ is a monotonically increasing function for all values of k. In order to find the new scaling laws we need either a massless or symmetry broken theory, where $D(k)$ reaches zero at $k=0$ or at $k=k_0\neq 0$, respectively. In the former case the Coleman-Weinberg mechanism [20] generates a different vacuum where the theory manages to develop a mass gap, thereby preserving the usual IR scaling laws. In the latter case the vanishing of the inverse propagator $D(k)$ indicates an instability of the homogeneous vacuum in the effective theory with the cutoff $k\leqslant k_0$ with respect to small fluctuations and the presence of the spinodal instability. Thus new scaling laws might be found as the precursor of the spinodal phase separation. We should bear in mind that within the spinodal-unstable region the saddle point is nontrivial, $\phi_0'\neq 0$, and we need a different renormalization group equation.

VI. RENORMALIZED TRAJECTORY

We want to follow the renormalized trajectory of the coupling constants

$$g_{2n}(k)=\frac{\partial^{2n}U_k(\Phi_0)}{\partial\Phi^{2n}}_{|\Phi_0=0}, \tag{28}$$

in $d=4$, i.e., we seek solutions of the set of coupled equations

$$k\frac{\partial}{\partial k}g_n(k)=\beta_n(g_1,\ldots,g_{n+2}), \tag{29}$$

where

$$\beta_n=-\hbar\frac{\Omega_4 k^4}{2(2\pi)^4}\frac{\partial^n}{\partial\Phi^n}\ln[k^2+U_k''(\Phi)]_{|\Phi_0=0}, \tag{30}$$

with the initial condition

$$U_\Lambda(\Phi)=U_B(\Phi) \tag{31}$$

in the stable region

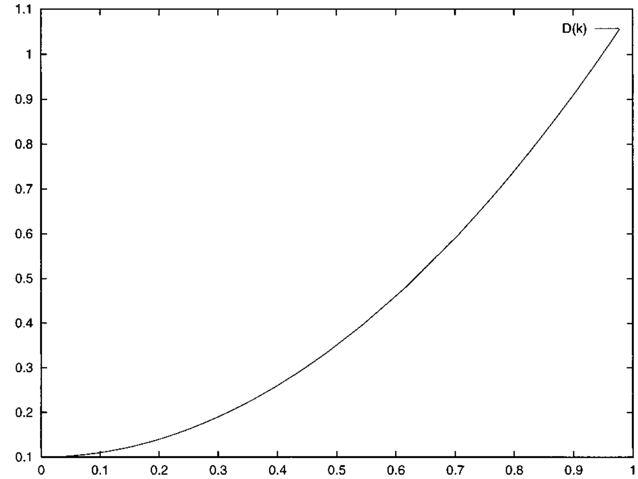
$$k^2>k_0^2(\Phi)=-U''_{k_0(\Phi)}(\Phi). \tag{32}$$

We set $\hbar=1$ in the numerical work. We truncated the potential

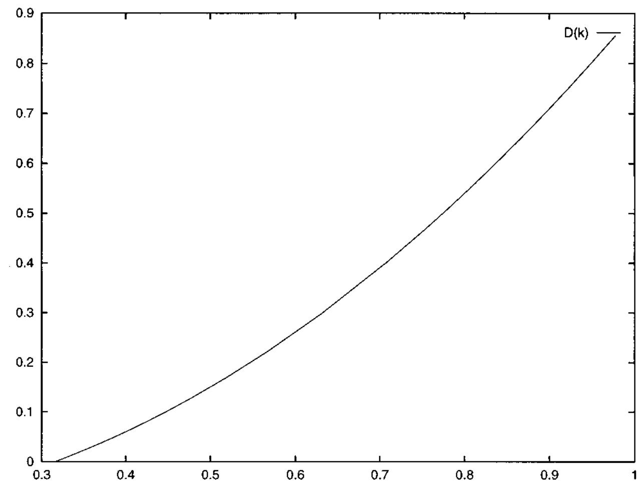
$$U_k(\Phi)=\sum_{n=1}^{N}\frac{g_{2n}(k)}{(2n)!}\Phi^{2n}, \tag{33}$$

and solved the resulting equations numerically. Special care is needed in the vicinity of the spinodal line $k\approx k_0(\Phi)$, where the β functions are the sum of large numbers with different sign. We used third and fourth order Runge-Kutta methods with a dynamically determined value of Δk. Quadruple precision numbers were used when necessary to make sure that the roundoff errors in the β functions were less than 10^{-8} times the actual value of the β functions.

The behavior of the restoring force for the fluctuations $D(k)$ was found to be in qualitative agreement with the perturbation expansion; two typical cases are shown in Fig. 4. Such an agreement is expected in the UV scaling regime where the kinetic energy dominates the action. Our interest will be to see the detailed behavior of the flow in the vicinity of the critical curve $k>k_0(\Phi)$. The curves $k_0(\Phi)$ obtained from the tree-level solution and the numerical integration of the renormalization group equations are shown in Fig. 5. The quantum fluctuations help the disorder and drive the saddle points to zero at the weakly unstable regime of the tree-level scaling relations. As a result the curve obtained by solving



(a)



(b)

FIG. 4. The evolution of the inverse propagator $D(k)$ obtained by $2N=22$. (a) Symmetrical phase, $\tilde{g}_2(1)=0.1$, $\tilde{g}_4(1)=0.01$; (b) Symmetry broken phase, $\tilde{g}_2(1)=-0.1$, $\tilde{g}_4(1)=0.01$. In both cases $\tilde{g}_{2n}(1)=0$, for $n>2$.

the renormalization group equations is inside of the spinodal-unstable region of the tree-level solution, except for the roundoff effect at small k. It is remarkable that the radiative corrections are rather small.

The numerical integration of the renormalization group equations produces oscillation for the coupling constants $g_n(k)$ for $n>4$, with increasing amplitude as we approach the instability. We found two qualitatively different behaviors as far as the vicinity of the unstable line is concerned.

Focusing. When the potential is truncated up to $2N=20$ then $g_4(k)$ stays positive and approaches zero. The values of the higher order coupling constants drop significantly after several, large-amplitude oscillations and approach zero. This indicates that the blocking transformations has an attractive fixed point,

$$\tilde{U}_{\tilde{k}_0(0)}(\tilde{\Phi})=-\frac{1}{2}\tilde{\Phi}^2. \tag{34}$$

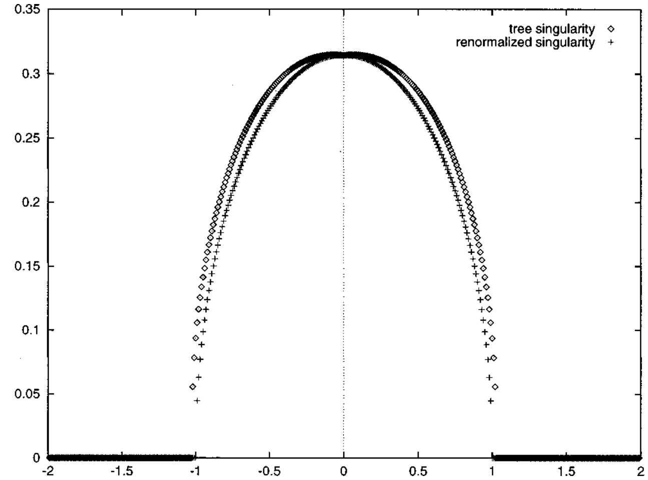


FIG. 5. The line of singularity of the renormalized trajectory on the plane $(\tilde{\Phi},\tilde{k})$ for $\tilde{g}_2(1)=-0.1$, $\tilde{g}_4(1)=0.2$. The diamond and the cross show the tree level and the renormalization group results.

The coupling constants $g_n(k)$ for $n>4$ produce large fluctuations, but after a while start to all fall and the quadratic potential (34) is approached. It is found that $g_4(k)\to 0^+$ and the higher order coupling constants drop after undergoing large amplitude oscillations as $k\to k_0$. The evolution of $\ln|g_{20}(k)|$ is depicted in Fig. 6. We find a cusp where the sign of $g_{20}(k)$ changes with finite altitude due to the finite resolution of the k values. Note that the negative coefficient of ϕ^2 causes no problem with the stability of the vacuum because the potential $U_k(\Phi)$ of the effective theory recovers the perturbative form for large values of the field, far away from the spinodal-unstable region.

The approach to the Gaussian potential can be made plausible by inspecting $\mathcal{P}_4$. By assuming that the coupling constants remain finite at the critical line we have $g_4[k_0(0)]=0$, since β_4 diverges otherwise. Once $g_4=0$ is accepted, the vanishing of the higher order coupling constants is plau-

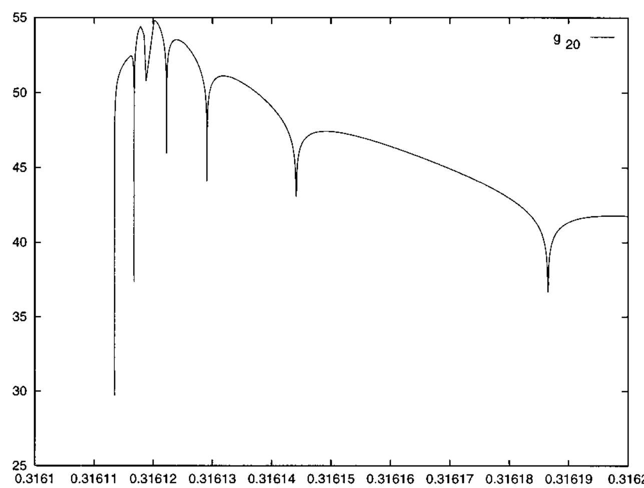


FIG. 6. The evolution of the coupling constant $\tilde{g}_{20}(\tilde{k})$ for $2N=20$, with the initial conditions $\tilde{g}_2(1)=-0.1$, $\tilde{g}_4(1)=0.01$, and $\tilde{g}_{2n}(1)=0.0$ for $n=3,\ldots,10$.

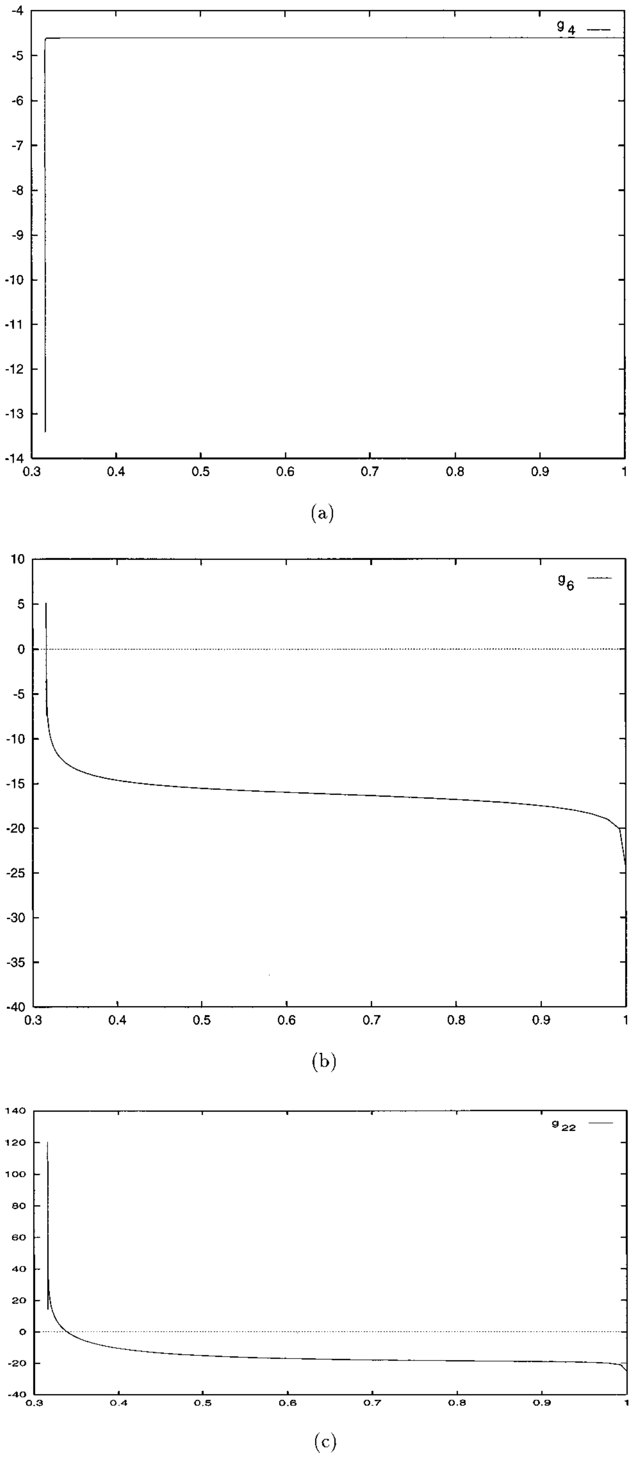


FIG. 7. The evolution of (a) $\ln|\tilde{g}_4(\tilde{k})|$, (b) $\ln|\tilde{g}_6(\tilde{k})|$, and (c) $\ln|\tilde{g}_{22}(\tilde{k})|$ at $2N=22$, with the initial conditions $\tilde{g}_2(1)=-0.1$, $\tilde{g}_4(1)=0.01$, and $\tilde{g}_{2n}(1)=0.0$ for $n=3,\ldots,11$. The coupling constants oscillate for $n>4$ with increasing amplitude and changing sign so $\ln|g_n(\tilde{k})|$ is plotted.

sible. The only finite parameter, $g_2[k_0(0)]$, is fixed by the condition $D[k_0(0)]=0$. The existence of a single fixed point, that all finite set of coupling constants runs into Eq. (34), can be called focusing. The potential turns out to be Eq. (34) in the whole unstable region that appears as a ''fixed region'' [19].

Divergence. This fixed point turned out to be an artifact of the truncation of the potential, a feature which has already been noted in other cases, as well [21]. When the truncation is made beyond $2N=20$ then the accumulation of the contributions of the higher order vertices in the β functions make the term G_6 more important in β_4, which in turn helps g_4 to decrease faster with k in the approach of the critical line. Once the sign of g_4 flips, the further decrease is not limited by zero and g_4 quickly approaches $-\infty$. It is not then so surprising that all of the other coupling constants start to diverge at the same time. The typical flow is depicted in Fig. 7 for $2N=22$; the further increase of N makes no further qualitative changes in the flow. The coupling constant undergoes oscillations with increasing amplitude as $g_4(k)\to 0^+$ and start to diverge as $g_4(k)$ flips sign. The loop expansion naturally ceases to be applicable in the vicinity of the critical line, and all we can say is that the modes with momentum slightly above $k_0(\Phi)$ appear strongly coupled, and our solution is no longer reliable. Though there is a marked difference in the behavior of the renormalization group flow for $2N<22$ and $2N\geqslant 22$, we should not forget that this difference shows up after a strong coupling regime where the high order non-Gaussian coupling constants develop extremely large values. So it is not clear if the difference between two cases in the vicinity of the unstable line is indeed so large.

Universality. The increase of the coupling constants at $k\approx k_0(0)$ indicates the existence of new relevant operator(s) in this scaling regime. One suspects that this operator is nonlocal since the value of the cutoff is finite. Can this operator modify the usual universality argument [5]? According to universality, the introduction of the irrelevant operators at the cutoff modify only the scale parameter of the theory. The dimensionless quantities, such as the β functions, are supposed to be independent of the value of the irrelevant coupling constants at the cutoff. To verify this scenario we computed

$$\frac{\partial\tilde{\beta}_n(\tilde{k})}{\partial\tilde{g}_6(1)} \tag{35}$$

numerically. The result, plotted in Fig. 8, shows clearly the coexistence of two different scaling regimes, the UV one where this quantity is suppressed and the precursor of the spinodal instability where we find an increasing value.

The comparison of Figs. 7 and 8 contains an important lesson confirmed by Fig. 9 where Eq. (35) is plotted as the function of the appropriate coupling constant. Namely, the violation of universality, the increase of Eq. (35) already takes place when the coupling constants are weak. It is reasonable to assume the ansatz

$$\frac{\partial\tilde{\beta}_n(\tilde{k})}{\partial\tilde{g}_6(1)}\approx F^2\left(\frac{\tilde{k}-\tilde{k}_0(0)}{\tilde{k}_0(0)}\right)\tilde{k}_0^2(0), \tag{36}$$

where the term $\tilde{k}_0^2(0)$ that is proportional to Λ^{-2} represents the suppression of the UV scaling laws, and the amplification

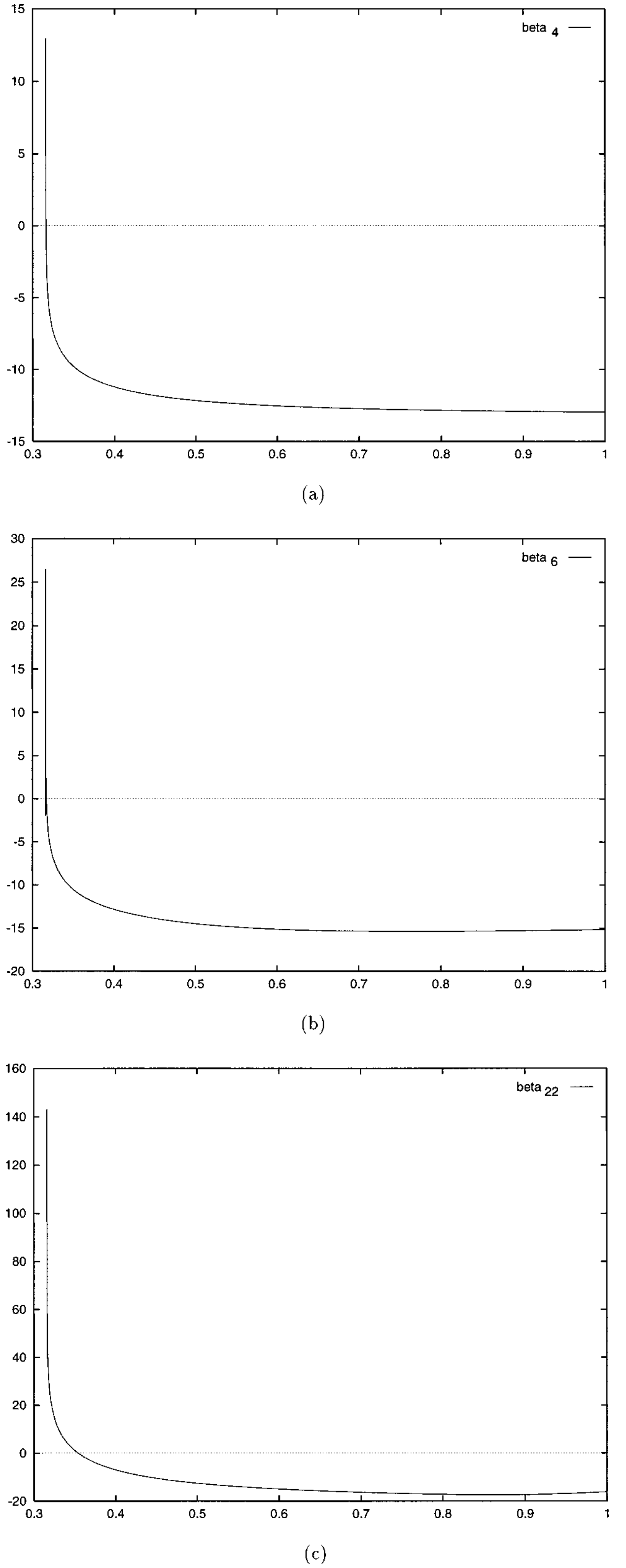


FIG. 8. The evolution of $\ln|\partial\tilde{\beta}_n(\tilde{k})/\partial\tilde{g}_6(1)|$ with $2N=22$. (a): $n=4$, (b): $n=6$, (c): $n=22$.

effect of the instability is manifest in the behavior of $F(x)$ which is supposed to diverge after large oscillations as $x\rightarrow 0$. For any finite value of the cutoff we can find a value of k sufficiently close to the instability where Eq. (36) is unity,

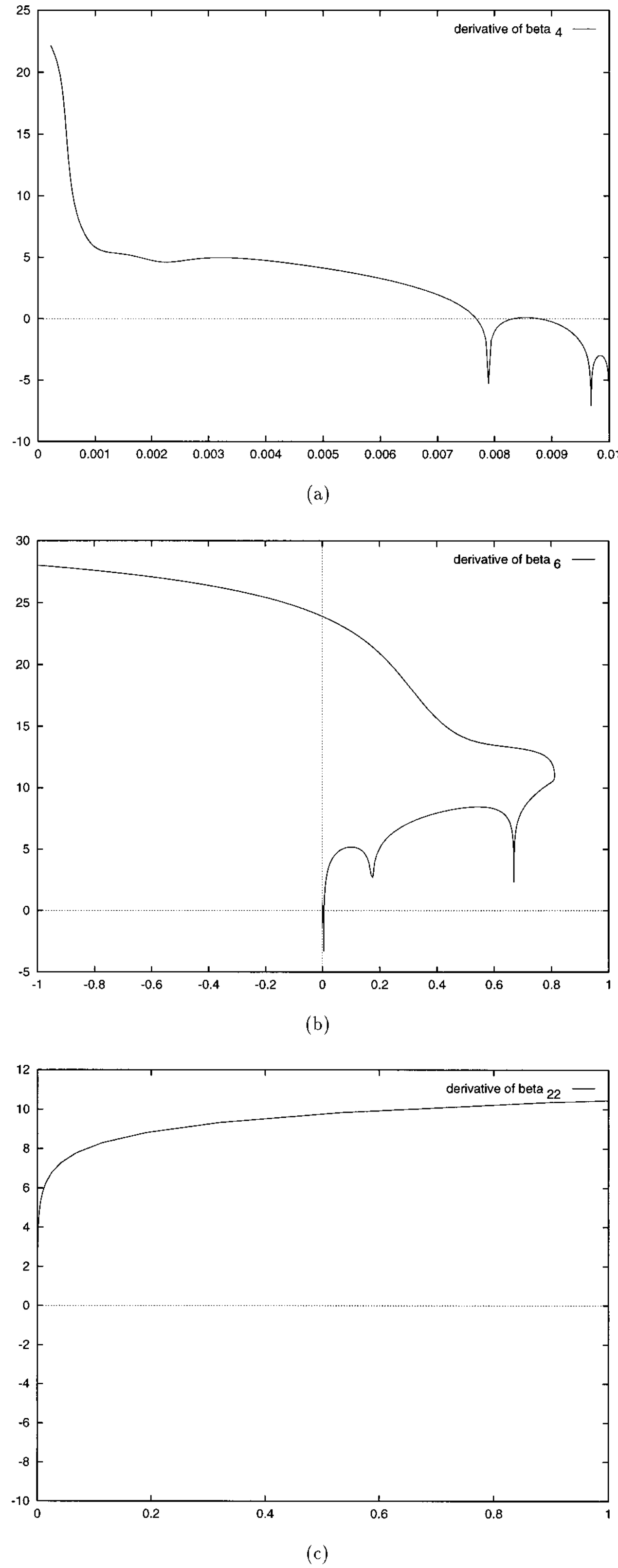


FIG. 9. $\ln|\partial\tilde{\beta}_n(\tilde{k})/\partial\tilde{g}_6(1)|$ plotted against $\tilde{g}_n(\tilde{k})$ for (a): $n=4$, (b): $n=6$, (c): $n=22$, $0.317<\tilde{k}<1$. As the scale parameter k decreases we move towards the left or right for (a) or (c), respectively. The derivative of the β function shows the onset of the low-energy, nonuniversal scaling already when the coupling constants are still weak enough to rely on the loop expansion.

$$\Lambda \approx k_0(0) F\left(\frac{\tilde{k}-\tilde{k}_0(0)}{\tilde{k}_0(0)}\right), \tag{37}$$

and the cutoff effects of the nonrenormalizable coupling constant become visible because the suppression of the UV scaling regime is compensated for by the amplification of the instability region.

If the environment of the system, represented by the insertion of the constraint

$$\delta\left(\Phi-\frac{1}{V}\int d^4x\,\phi(x)\right), \tag{38}$$

where V is the four volume into the path integration is chosen in such a manner that the spinodal instability occurs, i.e., Φ is within the spinodal unstable region, then the amplification makes the effective coupling strength nonuniversal around the instability. In other words, the coupling constants of the effective theory for the system subject to this constraint can pick up the values of certain nonrenormalizable coupling constants at high energies, and allow us to investigate the nonuniversal interactions at high energy if the cutoff of the effective theory is brought close to $k=k_0(\Phi)$. By this method one could in principle increase the energy regime we can access experimentally, and may get closer to the ''last'' important scale, the onset of the asymptotic UV scaling of the TOE. This mechanism can be called a ''renormalization group microscope'' since the amplification offered by the instabilities is similar to the usual microscope, except that it is achieved by the renormalization group flow in the space of the coupling constants.

According to the numerical results the strength of the singularity and the value of Eq. (35) approach zero at the critical line, as Φ is increased towards the edge of the unstable region $k=k_0(\Phi)$. This is the result of the factor k^d in the renormalization group equations, the decreasing entropy of the modes with weak restoring force as $k\to 0$.

Renormalized perturbation expansion. In statistical physics one usually follows the evolution of the bare coupling constants $g_n(k)$ as functions of the cutoff

$$k\frac{\partial}{\partial k} g_n(k)=\beta_n^{(B)}(\{g\}), \tag{39}$$

where the explicit dependence on k drops from the bare β functions in the UV region. We studied the evolution of these coupling constants in our work, too.

In particle physics one introduces the renormalized running coupling constants $\lambda_n(\mu)$, $n=1,\ldots,n_r$, where n_r is the number of renormalizable parameters in the theory. The irrelevant coupling constants are neglected because the cutoff is sent sufficiently far from the observational energy μ. The running coupling constants are defined with the help of the scattering amplitudes or Green functions and their scale dependence is described by the renormalization group equations

$$\mu\frac{\partial}{\partial \mu}\lambda_n(\mu)=\beta_n^{(R)}(\{\lambda\}), \tag{40}$$

involving the renormalized β functions.

So long as the perturbation expansion is applicable we can establish a one-to-one mapping between these schemes,

$$g_n(k)=\lambda_n(k)+\cdots, \tag{41}$$

where the terms omitted are higher orders in the supposedly small coupling constants or are $O(k/\Lambda)$. In this manner the qualitative features of the renormalization group flow agree in the two schemes, i.e., the running coupling constant becomes the bare one when the scale reaches the cutoff. How can we reconcile the importance of certain nonrenormalizable coupling constants at low energy with the evolution of the running coupling constants $\lambda_n(\mu)$, which are introduced to keep track of the renormalizable coupling constants only?

The independence of the theory from the observational scale

$$\mu\frac{d}{d\mu}\Gamma_B(\{\lambda_B\},\Lambda)=0 \tag{42}$$

is reached by the readjustment of the coupling constants $\lambda_n(\mu)$. According to the multiplicative renormalization scheme the number of adjustable parameters is just n_r+1. In particular, one chooses n_r+1 independent observables $\Gamma^{(m)}$, $m=1,\ldots,n_r+1$,

$$\Gamma_B^{(m)}(\{\lambda_B\},\Lambda)=Z^{\ell_m}\left(\{\lambda(\mu)\},\frac{\Lambda}{\mu}\right)\Gamma_R^{(m)}(\{\lambda(\mu)\},\mu), \tag{43}$$

and imposes

$$0=\mu\frac{d}{d\mu}\left[Z^{\ell_m}\left(\{\lambda(\mu)\},\frac{\Lambda}{\mu}\right)\Gamma_R^{(m)}(\{\lambda(\mu)\},\mu)\right] \tag{44}$$

$$=\left(\mu\frac{\partial}{\partial\mu}+\beta_n^{(R)}(\{g\})\frac{\partial}{\partial g_n}+\ell_m\gamma^{(R)}(\{g\})\right)\Gamma_R^{(m)} \times(\{\lambda(\mu)\},\mu). \tag{45}$$

The renormalized renormalization group functions $\gamma^{(R)}$ and $\beta_n^{(R)}(\{\lambda\})$ can be found by inverting

$$0=\mu\frac{d}{d\mu}\left[Z^{\ell_m}\left(\{\lambda(\mu)\},\frac{\Lambda}{\mu}\right)\Gamma_R^{(m)}(\{\lambda(\mu)\},\mu)\right] \tag{46}$$

$$=\left(\mu\frac{\partial}{\partial\mu}+\beta_n^{(R)}(\{g\})\frac{\partial}{\partial g_n}+\ell_m\gamma^{(R)}(\{g\})\right)\Gamma_R^{(m)} \times(\{\lambda(\mu)\},\mu) \tag{47}$$

for the $\gamma^{(R)}$ and the $\beta^{(R)}$ functions. The perturbative renormalization assures that these functions are well defined, i.e., are independent of the choice of the observables $\{\Gamma\}$.

Observe that there are two steps in this procedure which prevent the detection of a hidden coupling constant. One is that we have already committed ourselves to use as many coupling constants to compensate the modification of μ, as there are renormalizable operators in the system. The second is that the cutoff is formally removed from the renormalized perturbation expansion. The hidden coupling constant represents a coupling between the ultraviolet and the infrared modes that can be seen by keeping the UV and IR cutoffs finite. When the UV cutoff is removed, the contributions $O(\mu/\Lambda)$ are ignored. The renormalization group used in this work resums these contributions which yield the new nontrivial scaling laws. Thus the observables obtained by the improved renormalized perturbation expansion using the renormalization group for n_r coupling constants will, by construction, never show any indication of the eventual, hidden parameters. What we have found is that our partial resummation of the perturbation expansion, which keeps track of nonrenormalizable operators as well, indicates the presence of hidden coupling constants and suggests that the low-energy dynamics is parametrized by more than n_r coupling constants. Consequently more than n_r+1 observables must be used in Eq. (47) to obtain the evolution of the parameters that can compensate the change of the observational scale.

In order to find the number of real parameters, we need a controllable method to study the low-energy scaling behavior. Because of the limitation of the gradient expansion based on local operators we cannot at the present stage clarify this point.

VII. SUMMARY

The renormalization group is traditionally used to follow the scale behavior in the vicinity of a fixed point of the blocking transformation. We showed in the case of the ϕ^4 model that, after paying the price of following the mixing of a large number of operators during the blocking, the investigation of the manner by which the different scaling regimes give rise to each other is feasible.

A new finite energy scaling regime of the ϕ^4 model with spontaneously broken symmetry is generated by the spinodal instability. It was found, by the numerical integration of the Wegner-Houghton equation in the local-potential approximation, that the spinodal instability generates new relevant operators, and may undo the suppression of the nonrenormalizable operators at the UV scaling regime. A nonrenormalizable operator gives rise to a hidden coupling constant if the operator in question is relevant in the low-energy scaling regime, and the initial, high-energy value of its coupling constant influences the low-energy physics. This raises the possibility of the eventual use of this instability as a renormalization group microscope to detect the nonuniversal physics at high energy by going sufficiently close to the unstable region.

The instability studied in this work appears in the Euclidean effective theory at finite-momentum scales. It is conjectured that the simplest manner to observe the effects of the instability is in the framework of the dynamical renormalization group, applied for large amplitude fluctuations. The saddle-point structure of the effective theory, for the real time dependence, is needed to make a more definite proposal concerning the whereabouts of nonuniversal phenomena. The investigation of the renormalization group equation with nontrivial condensate within the spinodal phase separation [19], and the search for the effects of the instability in the vacuum with Goldstone modes, are in progress.

ACKNOWLEDGMENTS

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APPENDIX: WEGNER-HOUGHTON EQUATIONS

The heuristic derivation of the renormalization group equation (11) indicates that the higher loop contributions to the equation are suppressed. This is not obvious from the derivation presented above because by placing the system into a finite quantization box the spectrum of the momentum becomes discrete and we may eliminate the modes one-by-one. What is the small parameter in this case?

The expansion of the action around the constant background Φ in powers of the Fourier components of fluctuations ϕ' is

$$S_k[\Phi+\phi']=S_k[\Phi]+\sum_p \phi'_p \frac{\partial S_k}{\partial \phi_p}\Big|_\phi + \frac{1}{2}\sum_{p,q} \phi'_p\phi'_q \frac{\partial^2 S_k}{\partial\phi_p\partial\phi_q}\Big|_\phi+\cdots . \tag{A1}$$

We have

$$\frac{\partial S_k}{\partial \phi_p}\Big|_\phi = U_k^{(1)}(\Phi)L^d\delta(p),$$

$$\frac{\partial^2 S_k}{\partial\phi_{p_1}\partial\phi_{p_2}}\Big|_\phi=[U_k^{(2)}(\Phi)+p^2]L^d\delta(p_1+p_2),$$

$$\frac{\partial^n S_k}{\partial\phi_{p_1}\cdots\partial\phi_{p_n}}\Big|_\phi=U_k^{(n)}(\Phi)L^d\delta(p_1+\cdots+p_2), \tag{A2}$$

where the subscripts stand for the derivatives with respect to Φ. Since $k-\Delta k<|p|<k$, the first derivative of the action does not contribute,

$$S_k[\Phi+\phi']=S_k[\Phi]+\frac{L^d}{2}\sum_p \phi'_p\phi'_{-p}[U_k^{(2)}(\Phi)+p^2]+\cdots . \tag{A3}$$

If the minimum value of Δk is $2\pi/L$, where L is the length of the quantization box, the number of modes to eliminate in the shell $k-\Delta k<|p|<k$ is

$$\mathcal{N}_d=\frac{\Omega_d k^{d-1}\Delta k}{(2\pi/L)^d}=\frac{\Omega_d}{2\pi}\left(\frac{k}{2\pi}\right)^{d-1}L^d\Delta k, \tag{A4}$$

where Ω_d is the solid angle in dimension d. The integration over degrees of freedom ϕ' will be done after the expansion of the exponential around the free action. The only terms contributing in the integration are those for which the Fourier components of ϕ' are combined in pairs $\phi'_p\phi'_{-p}$,

$$\exp-\frac{L^d}{\hbar}(U_{k-\Delta k}(\Phi)-U_k(\Phi))$$
$$=\int\mathcal{D}[\phi']\exp-\frac{1}{2\hbar}\sum_p\,[k^2+U_k^{(2)}(\Phi)]\phi'_p\phi'_{-p}$$
$$\times\left(1-\frac{U_k^{(4)}(\Phi)}{2L^d\hbar}\sum_{p,q}\,\phi'_p\phi'_{-p}\phi'_q\phi'_{-q}+\cdots\right),\qquad(A5)$$

where $\mathcal{D}[\phi']=\Pi_{k-\Delta k<|p|<k}d\ \mathrm{Re}(\phi'_p)d\ \mathrm{Im}(\phi'_p)$. The Gaussian integrations lead to

$$e^{-(L^d/\hbar)[U_{k-\Delta k}(\Phi)-U_k(\Phi)]}$$
$$=\left(\frac{\hbar\pi}{k^2+U_k^{(2)}(\Phi)}\right)^{\mathcal{N}_d/2}\left[1-\frac{\hbar}{L^d}\frac{U_k^{(4)}(\Phi)}{[k^2+U_k^{(2)}(\phi)]^2}\right.$$
$$\left.\times\mathcal{N}_d\left(\frac{\mathcal{N}_d}{2}+1\right)+\cdots\right].\qquad(A6)$$

We now introduce the small variable

$$\hbar^{n-1}\frac{|U_k^{(2n)}(\phi)|}{[k^2+U_k^{(2)}(\phi)]^2}\frac{(\mathcal{N}_d)^n}{(L^d)^{n-1}}\ll 1\qquad(A7)$$

which is proportional to the ratio of the number of degrees of freedom eliminated in a blocking to those left in the effective theory. We must keep in mind that first we have to take the thermodynamical limit $L\to\infty$, and after $\Delta k\to 0$, to make sure that the higher loop contributions are small. In principle Δk has a lower bound $2\pi/L$ but we can imagine that we make an interpolation on the renormalized trajectory so that the value of Δk we use to derive the renormalization group equation is as small as we wish. In this manner Δk and L are independent and the small parameter is $\Delta k/M$. If we want to keep the lower bound for Δk nonvanishing then we must assume that the derivatives of the potential are small enough according to Eq. (A7) and therefore the existence of another small parameters hidden in the potential.

By assuming that Eq. (A7) is valid and taking the logarithm of both sides of Eq. (A6) we obtain

$$U_k(\Phi)-U_{k-\Delta k}(\Phi)$$
$$=C(k)-\Delta k\frac{\hbar\Omega_d k^{d-1}}{2(2\pi)^d}\ln[k^2+U_k^{(2)}(\Phi)]$$
$$+\frac{(\Delta k)^2}{2}\left(\frac{\hbar\Omega_d k^{d-1}}{(2\pi)^d}\right)^2\left[\frac{U_k^{(4)}(\Phi)}{(k^2+U_k^{(2)}(\Phi))^2}\right]+\cdots\qquad(A8)$$

which finally yields the Wegner-Houghton equation

$$\partial_k U_k(\Phi)=-\frac{\hbar\Omega_d k^{d-1}}{2(2\pi)^d}\ln\left(\frac{k^2+U_k^{(2)}(\Phi)}{k^2+U_k^{(2)}(0)}\right),\qquad(A9)$$

where the denominator in the logarithm function was inserted to cancel the potential at $\Phi=0$.

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