

## Monopole clusters in Abelian projected gauge theories

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(Received 2 December 1997; published 20 May 1998)

We show that the monopole currents which one obtains in the maximally Abelian gauge of SU(2) fall into two quite distinct classes (when the volume is large enough). In each field configuration there is precisely one cluster that permeates the whole lattice volume. It has a current density and a magnetic screening mass that scale and it produces the whole of the string tension. The remaining clusters have a number density that follows an approximate power law  $\propto 1/l^3$  where  $l$  is the length of the monopole world line in lattice units. These clusters are localized in space-time with radii which vary as  $\sqrt{l}$ . In terms of the radius  $r$  these “lumps” have a scale-invariant distribution  $\propto dr/r \times 1/r^4$ . Moreover they appear not to contribute at all to the string tension. The fact that they are scale invariant at small distances would seem to rule out an instanton origin. [S0556-2821(98)02913-0]

PACS number(s): 11.15.Ha, 12.38.Aw, 14.80.Hv

### I. INTRODUCTION

Magnetic monopole currents are the crucial degrees of freedom in the dual superconducting vacuum hypothesis for confinement in non-Abelian gauge theories [1,2]. After Abelian projection to the maximally Abelian gauge [2,3], one finds not only that the Abelian fields possess a string tension,  $\sigma$ , that (almost) equals the original SU(2) string tension (“Abelian dominance”) [4], but that this string tension is almost entirely due to the monopoles in those Abelian fields (“monopole dominance”) [5,6]. If the dual superconductor hypothesis is indeed correct, then the magnetic monopoles reflect that part of the infrared physics in the SU(2) vacuum which drives confinement. It is therefore of great interest to analyze the structure of the monopole currents so as to determine whether there are any simple or suggestive features present. This is our goal in this paper.

We shall focus on some simple properties of these monopoles. Our basic tool is to decompose the total monopole current into non-intersecting clusters. An alternative would be to decompose the current into closed loops; for example a monopole cluster might be decomposed into several closed loops that intersect. There is no obvious reason why the monopole cluster spectrum should be more revealing than the loop spectrum, and indeed in an earlier study [7] we have found that this loop spectrum does possess some interesting features. As we shall see below, however, it turns out that it is the cluster spectrum that possesses the simplest and most remarkable properties.

In the next section we briefly discuss the technical details of the calculation, including the Abelian projection, the extraction of the string tension and the parameters of the lattice simulations. Section III contains a simple analytic calculation showing how monopoles can cause Abelian Wilson

loops to decay exponentially with their area. The purpose of this simplistic but useful picture is to give some orientation as to what properties the monopoles must possess if they are to be confining. This enables us to motivate bounds on the type of monopole spectrum that can be confining.

In Sec. IV we present the evidence for our most striking result: that the monopole current contains a single “percolating” cluster that permeates the whole volume, together with a collection of smaller clusters, whose number density, as a function of length,  $l$ , is close to an inverse cubic power. Such a spectrum decays slowly enough with increasing  $l$  that it can in principle confine. Our explicit calculations show, however, that it makes no contribution to the string tension, within errors, and that it is the single largest cluster that provides the string tension. We then analyze the scaling properties of these clusters. We show that the length per unit volume of the largest cluster is constant when expressed in physical units. This is not so for the remaining clusters. We find that at large distances,  $r$ , from a monopole the magnetic flux falls exponentially with  $r$  and that the corresponding screening mass is independent of the lattice spacing. If, however, we calculate the flux at smaller values of  $r$ , where the flux is large enough to efficiently disorder appropriately positioned Wilson loops, we find that scaling is violated except if we only include the monopoles that belong to the largest cluster. As a further tool we introduce a method for locally smoothing the monopole currents. This shows us that the fact that we have a *single* huge cluster must have a dynamical origin rather than being a simple “percolation” phenomenon. Also we see that the largest cluster possesses substantial fluctuations that do not add to its confining properties.

The smaller clusters are typically localized within a 4-volume whose radius  $r \propto \sqrt{l}$ . We find that these “4-balls” possess a scale invariant distribution,  $\propto dr/r \times 1/r^4$ . If the scale-invariance of the gauge theory were not anomalous, then this is precisely the distribution one would have for instantons. Given that we know that instantons are associated with monopole loops within their cores [8,9] this would have

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provided an elegant explanation. Unfortunately the anomalous breaking of scale-invariance changes the instanton spectrum in a dramatic and calculable fashion for the small values of  $r$  where the spectrum of the ‘‘4-balls’’ is most accurately determined. Thus this seems to rule out instantons as being relevant.

In Sec. VI we provide a summary of our results and some conclusions. A brief summary of some of our results has appeared in [10]. We draw the reader’s attention to some related work that has appeared recently [11].

## II. METHODOLOGY

The first step in our calculation is to generate SU(2) lattice field configurations. We use the standard Wilson plaquette action and a standard heat bath Monte Carlo algorithm. The lattices have periodic boundary conditions. We work with  $8^4$ ,  $10^4$ ,  $12^4$  and  $16^4$  lattices at  $\beta=2.3$ , with  $10^4$ ,  $12^4$ ,  $14^4$  and  $16^4$  lattices at  $\beta=2.4$ , and with  $16^4$  lattices at  $\beta=2.5$ . The range of lattice sizes at fixed  $\beta$  is intended to provide us with control over finite volume effects. For example,  $16a \sim 6 \times 1/\sqrt{\sigma}$  at  $\beta=2.3$ : a very large length in units of the physical length scale. The range of  $\beta$  values is intended to provide us with some control over finite- $a$  corrections ( $a$  decreases by about a factor of 2 between  $\beta=2.3$  and  $\beta=2.5$ ). We typically analyze 500 configurations for each  $L$  and  $\beta$ . These configurations are typically some 25 to 50 Monte Carlo sweeps apart.

Once generated these SU(2) configurations are then fixed to the maximally Abelian gauge in the standard way: we perform gauge transformations at each site, and iterate the procedure, so as to (locally) maximize the gauge dependent functional

$$R = - \sum_{n,\mu} \text{Tr}(U_\mu(n) \cdot i\sigma_3 \cdot U_\mu^\dagger(n) \cdot i\sigma_3). \quad (1)$$

We then write the gauge fixed links in the factored form

$$U_\mu(n) = \begin{pmatrix} c_\mu(n) & w_\mu(n) \\ -w_\mu^*(n) & c_\mu(n) \end{pmatrix} \begin{pmatrix} e^{i\theta_\mu(n)} & 0 \\ 0 & e^{-i\theta_\mu(n)} \end{pmatrix}, \quad (2)$$

where  $c_\mu(n)$  is real and the  $\theta_\mu(n)$  are our Abelian link angles. We now identify the magnetic monopole currents in these Abelian fields using [12]. The currents are integer valued variables on the links of the dual lattice and they satisfy a continuity equation. So the total current can be decomposed into a number of closed current loops. In general such a decomposition is not unique since loops may intersect. If loops that intersect are concatenated into ‘‘clusters’’ then these clusters form a unique set of mutually disconnected networks and each current link may be unambiguously associated with one of these clusters.

A standard way to calculate the SU(2) string tension is by calculating Wilson loops,  $W(r,t)$ : i.e. the trace of the oriented product of SU(2) matrices along the rectangular  $r \times t$  contour. From these Wilson loops one can extract the static potential,  $V(r)$ :

$$aV(r) = \lim_{t \rightarrow \infty} \ln \left\{ \frac{\langle W(r,t) \rangle}{\langle W(r,t+a) \rangle} \right\}. \quad (3)$$

From the behavior of  $V(r)$  at large  $r$ ,  $V(r) \sim \sigma r$ , we can then extract the string tension,  $\sigma$ . Clearly such a calculation, involving two limits, requires large lattices and small errors. An alternative procedure is to use Creutz ratios:

$$a^2\sigma = \lim_{r \rightarrow \infty} \sigma_{\text{eff}}(r) \equiv - \lim_{r \rightarrow \infty} \ln \left\{ \frac{\langle W(r,r) \rangle \langle W(r+a,r+a) \rangle}{\langle W(r+a,r) \rangle \langle W(r,r+a) \rangle} \right\}. \quad (4)$$

In practice a useful estimate of the string tension can be extracted this way when the quality of the ‘‘data’’ does not permit the preceding, more complete analysis.

Once we have gauge fixed and extracted our Abelian fields, we can obtain the Abelian string tension in exactly the same way. We simply calculate the Wilson loops using the Abelian fields  $u_\mu(n) = \exp\{i\theta_\mu(n)\}$  rather than the SU(2) matrices  $U_\mu(n)$ . The fact that this Abelian string tension turns out to be close to the full SU(2) string tension [4], has provided much of the motivation for the current interest in the maximally Abelian gauge.

To calculate the monopole contribution to a Wilson loop let us consider contours that are purely space-like, e.g.,  $W(x,y)$ . (Since space-time is Euclidean, this involves no loss of generality.) The integral of the Abelian gauge potential around the contour will simply equal the magnetic flux,  $B(x,y)$ , through a surface spanning the Wilson loop contour, so the value of the Abelian Wilson loop will be given by

$$W(x,y) = \exp[iB(x,y)]. \quad (5)$$

In principle the surface chosen should be one over which the Abelian potential is non-singular. But since the flux through any other surface will differ by an integer multiple of  $2\pi$  (Dirac strings), we are free to choose whichever surface is the most convenient — which will usually be the minimal surface. The monopole Wilson loop is obtained by using that part of the magnetic flux that is generated by the monopole charges. This is just the dual of the electric flux that would be generated by the corresponding electric charges. We calculate this flux by solving the dual Maxwell equations with the given monopole currents. This is done by an iterative procedure and for the particular periodic four-volume under consideration. Once one has the dual 4-potential, it is trivial to generalize the calculation to non-space-like Wilson loops. Calculating Wilson loops in this way we can extract the monopole potential and string tension, using Eqs. (3) or (4).

In the same way one can, if one wishes, calculate the string tension due to some specified subset of monopole clusters. One simply calculates the dual potential due to that subset of monopole currents.

If we were working with a U(1) theory then we would expect the whole of the Abelian string tension to be due to monopoles [13,14]. In the present case, however, the Abelian fields are not generated by a (semi-)local Abelian action but are obtained in a complicated way from the non-Abelian fields. It is therefore possible that the resulting vacuum con-

TABLE I. The effective string tension obtained from Creutz ratios of size  $r$ ; as obtained from the U(1) fields, the monopole clusters and from the difference of the U(1) and monopole fluxes. All are obtained from an ensemble of 500 configurations on  $16^4$  lattices at  $\beta=2.4$ .

$r$	$\sigma_{U(1)}(r)$	$\sigma_{\text{mon}}(r)$	$\sigma_{\text{diff}}(r)$
2	0.1561 (18)	0.0673 (5)	0.0894 (14)
3	0.1103 (28)	0.0651 (6)	0.0348 (20)
4	0.0983 (132)	0.0649 (12)	0.0132 (41)
5	0.0259 (354)	0.0628 (21)	0.0040 (80)
6		0.0621 (37)	-0.0056 (172)

tains confinement-inducing, disordering fluctuations other than monopoles. For example, if the vacuum were to contain finite-width tubes of magnetic flux, with the flux, say, equal to  $\pi$ , and if these loops were to be arbitrarily long (a “condensate”) then this would typically produce a non-zero string tension. Thus it is important to ask whether it is the case that within the ensemble of Abelian fields obtained by Abelian projection from the SU(2) fields, confinement is indeed generated entirely by monopoles. A first step is to calculate both the Abelian and monopole string tensions and to compare them. Several investigations of this type suggest that they are indeed quite similar [5,6]. To go further than this we need to directly compare the confining Abelian and monopole fluctuations. To do this we calculate on each field configuration the difference between the total magnetic flux and that due to the monopoles. Using this “difference” flux we then calculate the corresponding Wilson loops and potentials. If the string tension that we extract from this potential is zero, then we will have shown that the confining fluctuations in the Abelian fields are entirely due to the monopoles. We have performed such calculations and display a typical set of results in Table I. The effective string tension,  $\sigma_{\text{eff}}(r)$ , has been obtained from Creutz ratios, as in Eq. (4). We observe that within errors the “difference” string tension is indeed consistent with being zero. This provides direct evidence that confinement is entirely driven by monopoles in these U(1) fields.

The reader will note something rather peculiar about the numbers in Table I. It is apparent that the monopole Creutz ratios show very much smaller statistical fluctuations than those from the U(1) fields. We would therefore expect that the difference string tension should have statistical errors that are at least as large as those in the U(1) measurement. In fact, as we see, they are much smaller. This clearly requires a strong correlation between the fluctuations in the U(1) fields and in the monopole currents: as we expect to be the case from monopole dominance. This is not in itself sufficient to explain the pattern of fluctuations, however. We note also that the small- $r$ , Coulombic deviation of the difference potential away from the purely linear asymptotic form is much greater than in the pure monopole calculation. This is actually something we can rather easily understand, as we shall see in Sec. IV.

Finally two cautionary asides. The first concerns Gribov copies. The gauge fixing described above is not unique. The

gauge functional has many maxima: the well known Gribov copy problem. These copies are, of course, identical for gauge invariant quantities but differ for gauge variant quantities such as the Abelian fields and monopole currents. Since there is currently no convincing criterion for which maximum is the “best,” we shall simply ignore this ambiguity. A practical justification for doing so is the demonstration [15] that while the monopole content of different Gribov copies of the same SU(2) gauge field can be very different, the long distance monopole physics that produces confinement is in fact very similar for typical Gribov copies. Since confinement is what we are mainly interested in here, this reassures us that our qualitative conclusions should not be affected by the Gribov copy problem.

The second aside concerns positivity. Since the SU(2) action is local (i.e. it extends over only a fixed number of lattice units) there is a corresponding positive-definite Hamiltonian in the continuum limit and it makes sense to talk of masses, potentials etc. (For non-zero lattice spacing there might be peculiar effects for masses on the order of the cut-off.) There is no guarantee, however, that the ensemble of Abelian fields possesses such an underlying Hamiltonian — because the Abelian fields depend in a completely non-local manner on the original SU(2) fields — and so we cannot be certain that it makes sense to talk of Abelian potentials and string tensions. This applies even more so to the ensemble of monopole fields; and even more to situations where we consider only subsets of monopole currents. Having said this, one finds in practice that the Abelian and monopole Wilson loops usually do behave as if there were an underlying transfer matrix, and the extraction of the potential seems to be largely unambiguous. So we will follow previous work and ignore possible problems with positivity. That these problems do exist becomes immediately apparent if one tries to “modernize” the calculation using smearing/blocking techniques. The correlation functions of smeared operators badly break positivity. This undermines the usual variational approach and means that we can only be confident that we have obtained the lightest mass if we have a clear, extended effective mass plateau. These problems have occasionally arisen in our calculations, but not in those that are reported upon in this paper.

### III. MONOPOLES AND CONFINEMENT

Before moving on to our results concerning the distribution of monopole currents it is interesting to ask whether there are any restrictions or bounds that such a distribution should satisfy if it is to have any possibility of producing confinement. To do so it will first be useful to outline how monopoles produce confinement in Abelian theories. The focus here will be on identifying the essential features of the phenomenon and will involve a variety of simplifying approximations to the exact calculations [13,14].

#### A. A simple picture

To begin with we consider the simpler case of the 3-dimensional U(1) theory. Here the monopoles are really instantons, but because the fields are identical to time-sliced

fields from the static 4-dimensional U(1) theory, it is customary and appropriate to refer to them as monopoles and to talk of the fields as being magnetic. Suppose, then, that we consider a Wilson loop on an  $R \times T$  contour. The contribution of the monopoles to the value of the Wilson loop is just

$$\langle W(R, T) \rangle = \langle \exp[iB_{\text{mon}}(R, T)] \rangle \quad (6)$$

where the average is over all field configurations and  $B_{\text{mon}}(R, T)$  is the total magnetic flux through the  $R \times T$  rectangle that arises from the monopoles in each field configuration.

How do we calculate  $B_{\text{mon}}(R, T)$ ? One might try to neglect the monopole interactions as a first approximation, so that we just have a random gas of monopoles. This leads to arbitrarily large energy densities, however, and so the system prefers to trade off some entropy and form a screened plasma of magnetic charges instead. Let the screening length be  $\xi$ . We shall treat our system as being, to a first approximation, a random gas of monopoles with a screened magnetic flux that decreases with distance  $r$  as  $b_{\pm}(r) = \pm 2\pi \exp(-r/\xi)$  (the sign being chosen at random). Consider now the total flux  $\Phi$  through our  $R \times T$  contour. Given the exponential drop in the flux a reasonable approximation for  $R, T \gg \xi$  is to assume that if a monopole lies within a ‘‘slab’’ of thickness  $\xi$  either side of the  $R \times T$  rectangle then half of its flux, i.e.  $\Phi = \pi$ , will pass through the rectangle while if it is outside the slab then the flux is suppressed to zero. This obviously neglects various perimeter effects, but we do not care because these will not contribute to the interesting area term. In this approximation then

$$B_{\text{mon}}(R, T) = n_+ \pi - n_- \pi \quad (7)$$

where  $n_+$  ( $n_-$ ) is the number of positively (negatively) charged monopoles above the Wilson rectangle plus the number of negatively (positively) charged monopoles below — counting only those within the slab of course. Clearly once  $R, T \gg \xi$  the average number of monopoles within the slab must be proportional to its volume  $\langle n_+ \rangle = \langle n_- \rangle = c\xi RT \equiv \bar{n}$ . If the gas in the slab is random then  $\langle n_{\pm} \rangle$  should be Poisson distributed with mean  $\bar{n}$ . We can now calculate our Wilson loop average:

$$\begin{aligned} \langle W(R, T) \rangle &= \langle \exp[iB_{\text{mon}}(R, T)] \rangle \\ &= \sum_{n_+=0}^{\infty} e^{in_+\Phi} \frac{\bar{n}^{n_+}}{n_+!} e^{-\bar{n}} \\ &\quad \times \sum_{n_-=0}^{\infty} e^{-in_-\Phi} \frac{\bar{n}^{n_-}}{n_-!} e^{-\bar{n}} \\ &= e^{-2\bar{n}(1-\cos\Phi)} = e^{-4c\xi RT} \end{aligned} \quad (8)$$

using  $\Phi = \pi$  in the last line. Thus the monopole magnetic flux causes the Wilson loops to decay exponentially with the loop area. This means that the monopoles lead to a non-zero string tension:  $\sigma = 4c\xi$ , in the above approximation.

The mechanism here is very simple. Only a monopole within a distance  $\xi$  will significantly affect the Wilson loop because of screening. Its contribution to the flux is about  $\Phi \sim \pi$  and so it flips the sign of the loop:  $e^{i\pi} = -1$ . That is to say, these monopoles maximally disorder the loop. Their number is obviously proportional to the area and this immediately translates into an area decay and a corresponding string tension.

We can easily do a bit better. If we consider a monopole a distance  $r$  above a large Wilson loop, the screened flux through that loop is

$$\Phi(r) = \pi \int_0^1 dy \cdot e^{-r/y\xi}. \quad (9)$$

This assumes that  $\xi \ll R, T$  so that we are only interested in  $r \ll R, T$ , in which case the flux through the Wilson loop is (almost) the same as the flux through the whole plane in which the loop lies. Using our previous expression for the average screened flux, we readily obtain the above  $\Phi(r)$ . Multiplicatively combining the disordering effects of an infinite tower of slabs, each infinitesimally thick, we obtain by analogy to Eq. (8):

$$\langle W(R, T) \rangle = e^{-2c\xi RT \int_0^{\infty} dr \left\{ 1 - \cos \left( \pi q \int_0^1 dy e^{-r/y} \right) \right\}}. \quad (10)$$

We have now introduced a general electric charge  $q$  for the Wilson loop. Since we have chosen the magnetic charge to be unity the usual Dirac quantization relation tells us that  $q$  must be an integer (otherwise the Dirac strings become ‘‘visible’’). For  $q = \pm 1$  Eq. (10) is an inessential improvement. For, say,  $q = 2$ , however, the argument of the previous paragraph gives no confinement since a flux of  $\pi$  does not disorder a doubly charged Wilson loop:  $e^{\pm i2\pi} = 1$ . So in this case it is Eq. (10) that must used and we obtain a string tension

$$\sigma(q) = 2c\xi \int_0^{\infty} dr \left\{ 1 - \cos \left( \pi q \int_0^1 dy e^{-r/y} \right) \right\} \quad (11)$$

for the potential between static sources of charge  $q$ .

It is crucial, if we are to obtain confinement, that screening is something that occurs only on the average — it is a statistical phenomenon. If, for example, we were to consider a gas of magnetic dipoles — a non-statistical form of screening — then we would get no confinement: the net flux through our very large Wilson loop is essentially zero if the dipole is well within the perimeter of the loop and a distance  $\ll R, T$  from the surface of the loop. The fact that screening is statistical means that the fluctuations around the mean screened flux will be important. The Wilson loop is of course sensitive to all fluctuations — it is, after all, a phase — and so we are making an uncontrolled approximation in replacing the monopole fluxes by their mean, screened values. This is the only serious approximation that we have made. We shall return to the link between the confining properties of the monopoles, the monopole current density and the screening length in the next section.

Although our treatment of screening is very approximate, this does not undermine the simple picture we gave above of how monopoles maximally disorder Wilson loops, and so maximize the interaction between electric charges. Indeed suppose we ignore screening entirely and calculate an  $R \times R$  Wilson loop, say, within a completely random gas of monopoles. The calculation is now actually much easier since there is no screening length to bring in an extra scale. We can therefore just scale out the scale  $R$  and we obtain

$$\langle W(R,R) \rangle \propto e^{-cR^3} \quad (12)$$

where  $c$  is proportional to the density of monopoles. The cubic power of  $R$  arises on simple dimensional grounds. So the potential grows faster than linearly: a random gas of monopoles over-confines. This is not possible in a quantum field theory: the associated unbounded energy densities will break down through particle production. Screening is the way the theory regulates itself and in the process weakens the over-confining potential to the linear form that is the fastest growth possible for a proper field theory [16].

The above simple and, no doubt, well-known picture contains the essential features of how monopoles drive linear confinement in 3 dimensions and, for static monopoles, in 4 dimensions as well. Of course we are interested here in the non-static case. Since space-time is Euclidean we lose no generality by considering only space-like Wilson loops. In that case it continues to be the magnetic flux that disorders the Wilson loop, exactly as above. It is still the case that the net magnetic flux from a monopole will be  $2\pi$ . Of course this flux will no longer be spherically symmetric but will depend on the motion of the monopoles. The generic effect of this asymmetry is to weaken the string tension but only by a finite factor that should not be far from unity on the average. Thus the qualitative physics is unchanged. If we time-slice monopole loops that are much smaller than our Wilson loop, they will look like dipoles and will not disorder the Wilson loop sufficiently to confine. The same should apply to Wilson loops that are long in one direction but short in another. The qualitative conclusion is that confinement on a scale  $R$ , requires monopole loops that are large compared to the corresponding  $R \times R$  Wilson loops. (A numerical confirmation of this may be found in [5], where it is seen that small monopole loops do not contribute to the string tension.) One could try to go further but we shall stop here and see what we can infer from this rather general constraint.

### B. Bounds on a confining monopole spectrum

So we now ask what conditions  $N(l)$ , the number of clusters of length  $l$ , must satisfy if we are to have confinement. We shall take the lattice spacing to be fixed so that the only quantity we vary is the lattice volume:  $L^4$  in lattice units.

We start with the simplifying assumption that the monopole cluster spectrum,  $N(l)$ , falls off as a power of  $l$ :

$$N(l) = \frac{C(L)}{l^\gamma}. \quad (13)$$

Our arguments can be straightforwardly adapted to other functional forms but we choose to focus on a power law because we already know that the spectrum of monopole loops decreases approximately as  $\sim 1/l^3$  [7]. Moreover, as we shall see in the next section, the cluster spectrum also possesses such a component. Once the volume is large compared to the physical length scale, we expect the  $L$ -dependence of  $C(L)$  to be simply  $C(L) = cL^4$ . The first bound then arises if we make the reasonable assumption that the density of monopole current must be finite, i.e.

$$\lim_{L \rightarrow \infty} \frac{cL^4 \int l \frac{dl}{l^\gamma}}{4L^4} \neq \infty. \quad (14)$$

This equation immediately implies that

$$\gamma \geq 2 \quad (15)$$

as long as the maximum length of those clusters which are associated with the power law,  $l_{\max}(L)$ , grows  $\rightarrow \infty$  when  $L \rightarrow \infty$ . In general this must be the case. Indeed simple random walk arguments would suggest that  $l_{\max}(L) \propto L^2$ .

We now have a lower bound on  $\gamma$ . Confinement should provide us with some upper bound: after all if  $\gamma$  is large enough then there will be essentially no large monopole clusters to disorder large Wilson loops. Let us be more specific. Consider Wilson loops of size  $\epsilon L \times \epsilon L$ , on an  $L^4$  lattice. If the theory is confining then it is reasonable to expect that it should be confining on scales  $\epsilon L$  where  $\epsilon$  can be chosen arbitrarily small but is then fixed. This will require monopole clusters that extend over distances of order  $\epsilon L$  at least. Let  $L$  be so large that  $\epsilon L$  is large compared to the physical length scale. Then we expect from random walk arguments that the length of such a cluster should be at least  $\propto (\epsilon L)^2$ . This should certainly apply to the coarse-grained length (the length of the cluster after the smallest ultraviolet fluctuations in the current have been removed by smoothing or blocking up to the physical length scale). Now, let the fraction of configurations with clusters that are this long, i.e.  $l \geq (\epsilon L)^2$ , be  $f(l)$ . Clearly if  $f(l) \rightarrow 0$  as  $L \rightarrow \infty$  then we will have lost confinement on the size scale  $\sim \epsilon L$ . So we require

$$\lim_{L \rightarrow \infty} f(l) \propto \lim_{L \rightarrow \infty} cL^4 \int_{-(\epsilon L)^2}^{\infty} \frac{dl}{l^\gamma} \neq 0 \quad (16)$$

which immediately implies

$$\gamma \leq 3. \quad (17)$$

We note that our discussion assumes, as seems reasonable, that the clusters are essentially independent of each other, i.e. that there are no strong long-range correlations between different clusters. Obviously a highly ordered set of small clusters can simulate the effects of a large cluster, and this would undermine our above arguments and bounds.

Thus as long as the monopoles possess some very general physical properties, the exponent characterizing the number density is limited to the narrow range

$$2 \leq \gamma \leq 3 \quad (18)$$

if the monopoles are to provide the disordering fluctuations that drive confinement. By making more specific assumptions one can try to narrow this range, but one then increasingly relies on arguments of decreasing plausibility. We shall not pursue this here.

Our above arguments have thus led us to the conclusion that a spectrum of the form  $N(l) \propto 1/l^3$  might be confining. This is intriguing: one finds just such a distribution for monopole loops [7] and, as we shall shortly see, for monopole clusters as well. Moreover it has been suggested [17] that for large  $l$  such a distribution could arise from instantons.

#### IV. INFRARED BEHAVIOR

As described above, for each field configuration we extract the associated monopole current,  $\{j_\mu(n)\}$ . The current is integer valued and conserved. Therefore it can be decomposed into continuous closed loops of non-zero current. Such a decomposition is ambiguous when loops cross. If we now form ‘‘clusters’’ of monopole currents by saying that two loops belong to the same cluster if and only if they intersect, then the decomposition into clusters is clearly unique. In this paper we shall focus on clusters rather than loops. (For an investigation of the latter see [7].) In addition to being constrained to form closed loops, the currents must satisfy a further constraint due to the periodic boundary conditions. Periodicity demands that in any given time-slice the total magnetic charge must be zero. Contractible monopole loops automatically satisfy this requirement. A loop can also satisfy current conservation, however, by closing through one of the boundaries. Periodicity then requires that such winding loops be matched by other non-contractible loops so that the net charge is zero however we time-slice the lattice. We mention this fact since it will become important when we attempt to calculate the string tension that arises from a subset of the clusters.

Suppose we have a particular cluster  $C$ . Then we define the length of the cluster to be

$$l_C = \sum_{\{n,\mu\} \in C} |j_\mu(n)|. \quad (19)$$

In practice, if one is outside the strong-coupling region of the SU(2) theory then the current is almost always  $\pm 1$  when it is non-zero. Thus our definition almost coincides with the number of links in the cluster.

##### A. Cluster decomposition

Our first step is to calculate the length of each cluster. This reveals that the clusters fall into two quite distinct classes. First there is a *single* cluster that is very much longer than any of the other clusters (at least if the volume is large enough). For example, of the  $500 \times 16^4$  configurations that we analyzed at  $\beta=2.3$  there was not a single case where we observed two large clusters rather than just one. Secondly

TABLE II. The total length of the current,  $l_{\text{tot}}$ , and its scaling with the lattice volume. Ditto for the largest cluster,  $l_{\text{max}}$ . The length of the second largest cluster is also listed.

$\beta$	$L$	$l_{\text{tot}}$	$l_{\text{tot}}/L^4$	$l_{\text{max}}$	$l_{\text{max}}/L^4$	$l_{2\text{nd}}$
2.3	8	840 (4)	0.205 (1)	624 (5)	0.152 (11)	35 (10)
	10	2054 (5)	0.205 (1)	1158 (58)	0.156 (6)	42 (11)
	12	4230 (7)	0.204 (1)	3200 (110)	0.154 (5)	49 (10)
	16	13394 (18)	0.204 (1)	10168 (141)	0.155 (2)	67 (9)
2.4	10	1100 (5)	0.110 (1)	584 (70)	0.058 (7)	83 (26)
	12	2288 (12)	0.110 (1)	1277 (104)	0.062 (5)	116 (39)
	14	4228 (10)	0.110 (1)	2441 (141)	0.064 (4)	121 (48)
	16	7184 (18)	0.110 (1)	4187 (177)	0.064 (3)	125 (38)
2.5	16	3583 (16)	0.055 (1)	1339 (123)	0.020 (2)	255 (40)

there are the remaining, smaller clusters. These possess a spectrum which follows an approximate power-law,  $N(l) = C(L)/l^\gamma$ , with  $\gamma \approx 3$ .

That the very large cluster is not simply the largest cluster of a smooth distribution of clusters can be established as follows. First, simply at the qualitative level, we note that on the  $12^4$  lattice at  $\beta=2.3$  (for example) the average length of the largest cluster is  $\sim 3200$ , while the average length of the second largest is only  $\sim 49$ . Moreover on none of the 500 analyzed configurations is the second largest cluster ever larger than 220 or the largest cluster ever smaller than 2300. On  $16^4$  the distinction is even more marked, with the average length of the largest cluster  $\sim 10169$ , while that of the second largest is only  $\sim 67$ . We can of course be less impressionistic than this. First we remark (see Table II) that the total length of all the clusters,  $l_{\text{tot}}$ , is proportional to the volume,  $L^4$ , as is the length of the largest cluster,  $l_{\text{max}}$ . So, as one would expect, the normalization of the spectrum is  $C(L) = cL^4$ . (One would expect this, because the number of clusters of length  $l$  should be proportional to the volume once the volume is large enough.) On the average a field configuration will contain  $\int N(l) dl$  clusters that belong to this spectrum. The largest of these clusters will, roughly speaking, be sampled from the tail of  $N(l)$  that integrates to unity:

$$cL^4 \int_{l_0} \frac{dl}{l^\gamma} = 1 \quad (20)$$

From this we can estimate the average length of this largest cluster to be

$$\langle l_1 \rangle = \frac{\int_{l_0} \frac{l dl}{l^\gamma}}{\int_{l_0} \frac{dl}{l^\gamma}} \propto L^{4(\gamma-1)}. \quad (21)$$

That is to say, because this largest cluster is sampled from a falling spectrum its length increases much more slowly than  $L^4$ . This is in contrast to the observation in Table II that the length of the largest cluster in fact increases as  $L^4$ . Therefore this cluster does not belong to the observed continuous spec-

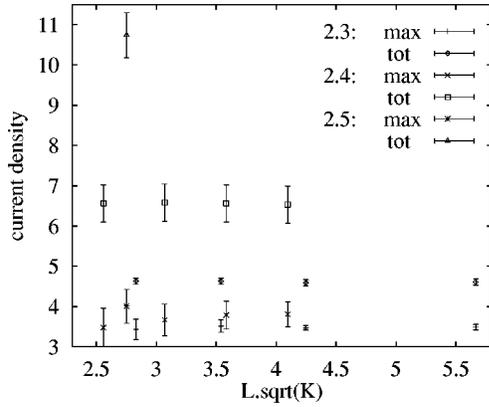


FIG. 1. The total current density,  $\rho_{\text{tot}}$ , and that of the largest cluster,  $\rho_{\text{max}}$ , as functions of the physical lattice size and the lattice spacing; for  $\beta=2.3, 2.4$  and  $2.5$ .

trum of clusters. One can also analyze the probability of observing such a large cluster, if it is sampled from an extrapolation of our observed spectrum. This probability is negligibly small. Indeed it is the second largest cluster that appears to be the largest cluster that is drawn from the continuous part of the spectrum. We list its length as a function of  $L$  in Table II and we can see that it increases weakly with  $L$  — just the behavior we argued for above.

### B. Scaling and the largest cluster

We have seen that at fixed  $a$  the length of the largest cluster increases linearly with the volume. This means that it will spread throughout the space-time volume in the thermodynamic limit. This is, qualitatively at least, exactly the kind of monopole cluster that might give us confinement on arbitrarily large scales. If it is to do so, however, then its structure must encode the physical length scale and not just the lattice scale. Consider then the length of this largest cluster in physical units: this will be  $l_{\text{max}}\sqrt{K}$ , where  $l_{\text{max}}$  is the length in lattice units and  $K \equiv a^2\sigma$  is the SU(2) lattice string tension [18]. Similarly the lattice volume will be  $(L\sqrt{K})^4$  in these physical units. So the monopole current density, in these nonperturbatively defined physical units, is

$$\rho_{\text{max}} = \frac{l_{\text{max}} \cdot \sqrt{K}}{(L \cdot \sqrt{K})^4} = \frac{l_{\text{max}}}{L^4 (\sqrt{K})^3} \quad (22)$$

for the largest cluster. Similarly we define  $\rho_{\text{tot}}$  for the total monopole current. We plot these densities against the physical lattice size,  $L\sqrt{K}$ , in Fig. 1. We first note that the points at fixed  $\beta$  are constant for both  $\rho_{\text{tot}}$  and  $\rho_{\text{max}}$ . This tells us that for fixed  $a$  both the total current and that from the largest cluster increase linearly with the volume; something we have noted already. Comparing now the points corresponding to different values of  $\beta$  we see that  $\rho_{\text{max}}$  is independent of the variation in  $a$ . That is to say, the length of the largest cluster is proportional to the volume if everything is expressed in physical units. By contrast we observe that this is certainly not the case for the total current. Since a large part of the total current resides in the largest cluster, this tells us

that there are very strong scaling violations arising from the currents of the smaller clusters.

The simple scaling property of the largest cluster is quite remarkable. Of course, it reassures us that the largest cluster does indeed encode the physical length scale, but it goes further than that. Realistically one could only hope for a suitably coarse-grained cluster length to satisfy scaling. Naively one would expect the length defined in terms of the links to be an ultraviolet length whose relationship to the physical coarse-grained length would involve anomalous dimensions that would lead to a violation of scaling. This would arise from the fact that the monopole world line has fluctuations on all length scales. Instead what we infer is that the largest monopole cluster does not really seem to encode any information concerning the ultraviolet length scale.

### C. The monopole potential and string tension

Having seen that the largest cluster fills space-time in a way that scales in physical units, we now ask whether it does in fact contribute to confinement, i.e. does it generate a potential between static sources that has a non-zero string tension. As described previously, we can do this by first calculating the magnetic field arising from the largest monopole cluster and then, from this, calculating the values of space-like Wilson loops. (In practice we calculate the dual 4-potential and all orientations of Wilson loops.) From the Wilson loops we extract the monopole potential,  $V(r)$ , as in Eq. (3), and obtain the string tension by fitting it with the generic form  $V(r) = a + b/r + \sigma r$ .

First an aside about the  $b/r$  term. It has been noted before, e.g. [7], that  $b$  is very small for the full monopole current ensembles. We find that the potential from the largest cluster alone is even more linear in form. Naively we would expect two contributions  $\propto 1/r$ : a Coulomb interaction and the universal Lüscher correction to the flux tube energy. Are they both absent or are they cancelling each other? The latter possibility is not as implausible as it might at first appear. We know that in the Villain model Wilson loops exactly factorize into spin-wave and monopole pieces [13,14]. Hence the total potential is a sum:  $V(r) = V_{\text{sw}}(r) + V_{\text{mon}}(r)$ , using an obvious notation. Suppose we are in the confining phase. Then  $V(r)$  has a linear piece  $\sigma r$  and, in addition, a Coulomb piece,  $V_C = -\alpha/r$ , at small  $r$  and a Lüscher term,  $V_L = -\pi/12r$ , at large  $r$ . These have the same sign and there is no possibility of a cancellation. In any case, since there is no massless gluon, the Coulomb piece will be screened at large  $r$ , typically  $V_C(r) \approx -(\alpha/r)\exp\{-r/\xi\}$ , and since the flux tube has a finite width, the Lüscher term will be “screened” at small  $r$ , crudely  $V_L(r) \sim -(\pi/12r)(1 - \exp\{-r/\xi'\})$ . We expect the two scales,  $\xi$  and  $\xi'$ , to be similar, so in practice the Coulomb and Lüscher terms will hardly overlap. In contrast to this the spin-wave potential does possess a massless photon and no linear piece, i.e.  $V_{\text{sw}}(r) \approx -\alpha/r$  for all  $r$ . Therefore  $V_{\text{mon}}(r) = V(r) - V_{\text{sw}}(r)$  will have the form

$$V_{\text{mon}}(r) \approx c - \frac{\alpha}{r} \exp\left[-\frac{r}{\xi}\right] - \frac{\pi}{12r} \left(1 - \exp\left[-\frac{r}{\xi'}\right]\right) + \sigma r + \frac{\alpha}{r}. \quad (23)$$

We note that at small  $r$  the two Coulomb terms cancel and the Lüscher term is negligible: so there is no significant  $1/r$  piece. At large  $r$  the screened Coulomb term is negligible and so  $V_{\text{mon}}(r)$  will be the difference of the Lüscher and unscreened Coulomb terms. If these have a similar magnitude, as in fact they do in typical SU(2) calculations, then they will largely cancel. So  $V_{\text{mon}}(r)$  has no significant  $1/r$  contribution at large  $r$  either. Of course all this has only been demonstrated in the Villain model. It seems plausible to us, however, that an approximate version of this mechanism should hold more generally, and that it provides the explanation for the observed lack of a significant  $1/r$  piece in the monopole potentials.

In calculating the potential from the largest monopole cluster there is one significant problem. As we remarked earlier periodicity implies that the total magnetic charge in any time-slice must be zero. So it needs to be the case that the ‘pruned’ configurations formed by the removal of certain clusters have no net winding number in any direction: e.g.

$$\sum_{x,y,z} j_4(x,y,z,t=1) = 0. \quad (24)$$

Without this, Gauss’ law and the magnetic flux are ill-defined. In practice the largest cluster does sometimes have a net winding in one or more directions. In these cases we implement the following ‘fix.’ We manually correct the winding number to zero by the addition of straight lines of magnetic current of appropriate charge that wrap around the lattice; structures analogous to the Polyakov line of gauge links. The position of such a static monopole world line is chosen randomly. In practice this problem is only significant at  $\beta=2.5$ . The reason is that the total current necessarily has zero winding. So the largest cluster will only wind if some other smaller cluster has a compensating winding. To have some chance of winding around a lattice this secondary cluster cannot be too small. Only at  $\beta=2.5$  are the secondary clusters large enough to make this a frequent occurrence. (This in part arises because of their scaling properties—as discussed in the next section.) We estimate the bias induced by our winding fix as follows. We place the same static monopole lines used to correct the winding number onto otherwise empty lattices. To each of these, we add a similar number of oppositely charged lines, also at random positions, and calculate the string tension. One half of this is an estimate of the bias inherent in our correction method. (Of course, as we showed earlier, a random monopole gas overconfines; here we are simply obtaining an effective string tension at an appropriate distance.) This bias is found to be completely negligible at  $\beta=2.3$  and 2.4. At  $\beta=2.5$  it is  $\pm 0.002$  in lattice units. This is only an estimate, so the message is that some caution should be attached to the string tensions we calculate at  $\beta=2.5$ .

In Table III we show the string tensions calculated using: first the total current, second the largest cluster and third all the clusters except the largest. We also indicate the proportion of current in the largest cluster by quoting the value of the percolation parameter [19]

TABLE III. The monopole string tensions from all the clusters, the largest and from the remainder. The second error is the systematic bias from correcting the winding number.  $P$  is the percolation parameter.

$L=16$ :	$a^2\sigma_{\text{tot}}$	$a^2\sigma_{\text{max}}$	$a^2\sigma_{\text{rest}}$	$P$
$\beta=2.3$ :	0.128 (5)	0.124 (3) (0)	0.000 (1)(0)	0.75 (1)
$\beta=2.4$ :	0.067 (2)	0.058 (2) (0)	0.01 (1)(0)	0.57 (1)
$\beta=2.5$ :	0.034 (2)	0.024 (2) (2)	$<0.005$ (-)(2)	0.37 (1)

$$P = \frac{n_{\text{max}}}{n_{\text{tot}}}, \quad (25)$$

where  $n_{\text{max}}$  is the number of (dual) sites connected by current links from the largest cluster and  $n_{\text{tot}}$  the number connected in all clusters. We note that at  $\beta=2.3$ , where the volume is largest in physical units, the string tension is given entirely by the largest cluster and there is no contribution from the secondary clusters, despite the fact that the latter carry some 25% of the total monopole current. At  $\beta=2.4$  the secondary clusters still do not provide any confining force even though their contribution to the current is now approaching half of the total. At  $\beta=2.5$  the situation is not so clear-cut but that is not surprising: the volume is now becoming quite small in physical units, the distinction between the largest and second largest clusters begins to blur, and the winding correction has become important. There is also some indication that  $a^2\sigma_{\text{tot}}$  and  $a^2\sigma_{\text{max}}$  — and indeed  $a^2\sigma_{\text{tot}} + a^2\sigma_{\text{max}}$  — are not quite the same at  $\beta=2.4$ . We believe that this is directly related to our observation that as the volume decreases (at fixed  $\beta$ ) there is a growing disparity between  $a^2\sigma_{\text{tot}}$  and  $a^2\sigma_{\text{max}} + a^2\sigma_{\text{rest}}$ . We are not at present sure whether this indicates a significant correlation between the largest and smaller clusters on smaller volumes, or whether it is an artifact of the difficulty of extracting extended effective mass plateaux on small lattices.

#### D. Smoothing the monopole fields

We have just seen that the largest cluster is the source of all the interesting confining physics. Given its importance it is worth probing its structure in more detail. In particular we return to our earlier observation of scaling and the question of whether this cluster fluctuates on the scale of the lattice spacing or not. To address this question we locally ‘smooth’ the monopole currents and see what effect this has on the length of the cluster. We have employed two methods. The first is simply to ‘cool’ the Abelian fields by locally changing the fields so as to maximize the value of the sum of plaquettes. (That is, we cool using a plaquette ‘action.’) This directly smoothes the Abelian fields and therefore, indirectly, the monopole currents as well. The second method involves ‘smoothing’ the monopole currents directly: we examine each (dual) plaquette in turn, and superimpose a  $1 \times 1$  current loop with a charge chosen to minimize the total magnetic current on the lattice. This constitutes one smoothing sweep. It directly removes the kinks (‘staples’) in the current. The two methods give simi-

TABLE IV. The evolution under  $s$  monopole smoothing sweeps of the string tension, the proportion of current remaining, the percolation parameter, the length of the two largest clusters and the number of clusters. Also given are the last two quantities when the largest cluster is excluded from consideration. Results labelled  $\dagger$  are from  $L = 16$ .

$\beta = 2.3, L = 12$								
$s$	$a^2\sigma$	% curr.	$P$	$l_{\max}$	$l_{2\text{nd}}$	$n_C$	$l'_{2\text{nd}}$	$n'_C$
0	0.128 $\dagger$ (5)	100.0 (0)	75.7 (3)	3200 (22)	49 (10)	156.9 (7)	49 (10)	156.9 (7)
1	0.124 (6)	56.7 (2)	78.5 (4)	1882 (65)	85 (32)	56.5 (3)	29 (3)	36.2 (3)
2	0.122 (5)	40.7 (1)	76.2 (8)	1314 (94)	141 (48)	27.3 (2)	22 (3)	13.6 (2)
3	0.124 (5)	32.6 (1)	74.0 (8)	1021 (74)	163 (44)	16.3 (2)	18 (2)	6.8 (1)
5		24.2 (1)	70.8 (9)	725 (64)	160 (39)	8.9 (2)	11 (2)	2.7 (1)
10		15.5 (1)	68.4 (11)	449 (50)	125 (27)	4.4 (1)	3 (2)	1.2 (1)
$\beta = 2.4, L = 14$								
$s$	$a^2\sigma$	% curr.	$P$	$l_{\max}$	$l_{2\text{nd}}$	$n_C$	$l'_{2\text{nd}}$	$n'_C$
0	0.067 $\dagger$ (2)	100.0 (0)	56.8 (3)	2441 (141)	121 (48)	249.3 (5)	121 (48)	249.3 (5)
1	0.064 (2)	52.7 (2)	59.6 (6)	1345 (107)	202 (44)	75.1 (4)	73 (22)	61.8 (4)
2	0.063 (2)	37.6 (2)	60.2 (7)	965 (90)	216 (41)	34.2 (2)	60 (19)	24.9 (3)
3	0.062 (2)	30.3 (2)	61.0 (8)	788 (84)	207 (41)	19.6 (2)	51 (15)	12.1 (2)
5	0.059 (2)	22.9 (1)	61.8(10)	603 (75)	182 (34)	9.9 (2)	39 (12)	5.4 (1)
10	0.055 (2)	15.5 (1)	64.5 (9)	423 (50)	136 (24)	4.9 (1)	25 (10)	2.2 (1)

lar results, but the latter one has the important advantage for us that, in addition to being more transparent, it enables us to smooth individual clusters if we so wish.

The result of smoothing the monopole fields, by the second method, is summarized in Table IV. The first thing we observe is that the string tension shows very little variation with cooling. This is as it should be: the ultraviolet fluctuations of the monopole current should not affect its confining properties. Secondly we note that, as we cool, there is a rapid decrease in the total length of the largest cluster. This tells us that it does contain fluctuations on the size scale of the ‘‘cut-off’’ even if these are not strong enough to destroy the scaling of the total length. We also find something else that is very interesting: the largest cluster frequently breaks up into more than one cluster even after just one smoothing sweep. To see this we display in Table IV not just the results of smoothing all the monopole clusters, but also what happens if we exclude the original largest cluster from consideration and smooth just the secondary clusters. We observe that already after just one smoothing sweep the largest of the latter, labeled  $l'_{2\text{nd}}$ , is on average much smaller than the second largest cluster, labeled  $l_{2\text{nd}}$ , obtained when we smooth all the clusters. (Note that for these quantities the numbers in brackets are not the errors but the one standard deviation variations.) Thus the second largest cluster must have hived off from the largest cluster. Since  $l_{2\text{nd}}$  increases with smoothing (initially) it is clear that the largest cluster is hiving off a substantial number of clusters during the first few smoothing sweeps. This raises a puzzle. These hived-off clusters are, as we can see, typically much larger than the second largest cluster that one observes prior to smoothing. This implies that this hiving off never occurs during the Monte Carlo generation. It certainly would occur if we were applying a

Monte Carlo directly to the U(1) or monopole fields — after all the smoothing is just a particular move that would be part of the Monte Carlo choice. The implication appears to be that there is something in the SU(2) dynamics that ensures the presence of just one large monopole cluster. Despite initial appearances this largest cluster cannot be understood as a simple U(1) monopole percolation phenomenon.

### E. Screening lengths

As we saw in Sec. III, a plasma of monopoles is confining and the resulting string tension is proportional to the product of the monopole density and the screening length. Since we have found that the ensemble of monopole currents that we generate by going to the maximally Abelian gauge is confining, it would be interesting to show explicitly that these monopoles do form such a plasma. Indeed, since the confinement is entirely driven by the largest cluster, it is the monopoles in this cluster that should provide a realisation of our simple picture in Sec. III. We should also be able to see in what way the monopoles belonging to the non-confining secondary clusters do not constitute such a plasma.

The fact that the string tension has a finite continuum limit, means that both the screening length and the monopole density should also scale, i.e. that they should be constant when expressed in physical units, up to lattice corrections that vanish as  $a \rightarrow 0$ . [The reader may be aware that this is not what happens in the D=2+1 U(1) theory, but the peculiarities of that theory are not relevant to us here.] In this subsection we shall study the scaling properties of the screening length and we shall do so separately first for all the monopoles, then for the largest cluster alone, and finally for the secondary clusters alone. (We have already shown that

the monopole density of the largest cluster scales while that of the secondary clusters does not.) The more subtle question of whether we really have a plasma rather than, say, a distribution of dipoles, is something we shall not touch upon here.

Before moving to the details of our calculations we need to reconsider how our simple monopole plasma picture might be modified in a realistic context. First an aside: we shall calculate the screening length in an approximation where we neglect the non-static character of our monopoles. A more substantial point is that there will exist excitations of the lightest screening mass. Thus the magnetic flux from a monopole will not decay as a simple exponential in  $r$ . What we call the screening mass,  $m_s$ , will show up in the asymptotic exponential decay, as  $r \rightarrow \infty$ , of the flux:

$$B(r) \propto \exp\left\{-\frac{r}{l_s}\right\}, \quad (26)$$

where  $l_s = 1/am_s$  is the screening length and  $r$  is the distance from the monopole, all in lattice units. At very large  $r$ , however, this flux is very small and will have a negligible effect upon Wilson loops. So what is relevant to confinement is not this asymptotic screening mass but rather the effective screening mass that governs the decay of the flux at those distances where the flux is still sufficiently large to efficiently disorder Wilson loops. This effective screening mass will be some combination of the lightest screening mass and its excitations. It is this that we would like to see scale.

Another complication will arise when we consider the screening properties of a subset of all the clusters. Although we have assumed that these clusters are independent, it is unlikely that this is really the case. If we were in a U(1) field theory then a monopole would, through the (dual) Coulomb interaction, affect other monopoles whether they belonged to the same cluster or not. That is to say, all monopoles participate in the screening of all other monopoles. If we focus on the screening of the monopoles within some subset of clusters, and if only the monopoles in that subset are allowed to participate in the screening, then we will in general obtain an incorrect screening length. And if the total fraction of the monopoles that are excluded does not scale (as will be the case, for example, when we consider only the largest cluster) then the extracted ‘‘screening length’’ might well not scale either. This, as we shall shortly see, is what occurs in our case, despite the fact that we have no reason to think that our U(1) fields are governed by a simple U(1) effective action. Note that although the Coulomb interaction between individual monopoles in different clusters might be important for screening, it is a weak high-order multipole interaction between well separated clusters. Since the secondary clusters are compact objects (*vide* the next section) this interaction should be weak enough not to affect our derivation of Eq. (18).

Suppose, then, that  $B(r)$  is the flux from a monopole. We expect that for large enough  $r$  Eq. (26) will hold. If we now plot  $-\ln B(r)$  against  $r\sqrt{K}$ , then we expect to see a linear rise at large  $r$  whose slope is just the inverse of the screening length in physical units,  $\xi_s = l_s\sqrt{K}$ . Moreover if the screening

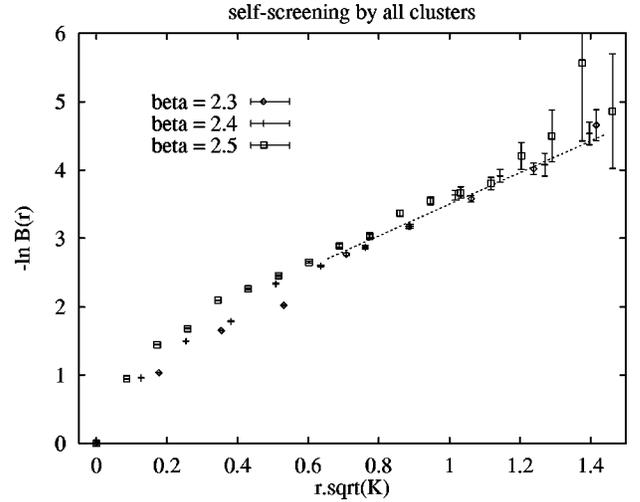


FIG. 2. Screening of flux by all monopoles as a function of distance in physical units for monopoles at  $\beta=2.3, 2.4$  and  $2.5$ : with a linear fit shown.

length is constant in physical units, then this slope should be independent of  $\beta$ . (All this assumes we are in an infinite volume. In a finite periodic volume one needs to make an obvious finite volume correction and this we shall do.)

In Fig. 2 we produce such a plot using all the monopoles on the lattice. (Note that we normalize the monopole flux to unity.) We see that indeed there is a linear rise at large  $r$ , and that the slope is independent of  $\beta$ , within statistical errors. That is to say, we find a scaling screening mass. If we now fit the combined data with a single scaling mass, we obtain  $m_s \approx 2.30(10)\sqrt{\sigma}$ .

As we can see in Fig. 2 this scaling screening mass only governs the decay of the magnetic flux at large distances where the flux is small. If instead we look at the effective screening mass in the range of  $r$  where the flux is still large enough to disorder Wilson loops, say  $1/e \leq B(r) \leq 1$ , then we see that it does not scale. This should not be a great surprise given that we have seen that the total monopole density does not scale either, and that there is a substantial number of monopoles, those in the secondary clusters, which do not appear to contribute to confinement.

Since we have found that it is the largest cluster that generates all of the string tension, and that the remaining clusters generate none of it, it is interesting to repeat the analysis separately for these two subsets of the total monopole current. This we do in Fig. 3. We first note that in both cases the screening mass does not scale—to the extent that one can identify a linear rise at large  $r$ . Moreover the large- $r$  screening is much weaker than that obtained when we include all the clusters. This ‘‘under-screening’’ is what we would expect if there were Coulomb interactions between all the monopoles, as discussed previously. For the secondary clusters the lack of scaling persists down to the smaller values of  $r$  which are relevant for confinement. For the largest cluster, on the other hand, the small- $r$  effective screening masses do scale and we extract a value  $m_s = 2.5(1)\sqrt{\sigma}$ , which is similar to the screening mass at large  $r$  from all the clusters. This and the fact that the density of monopoles in the largest

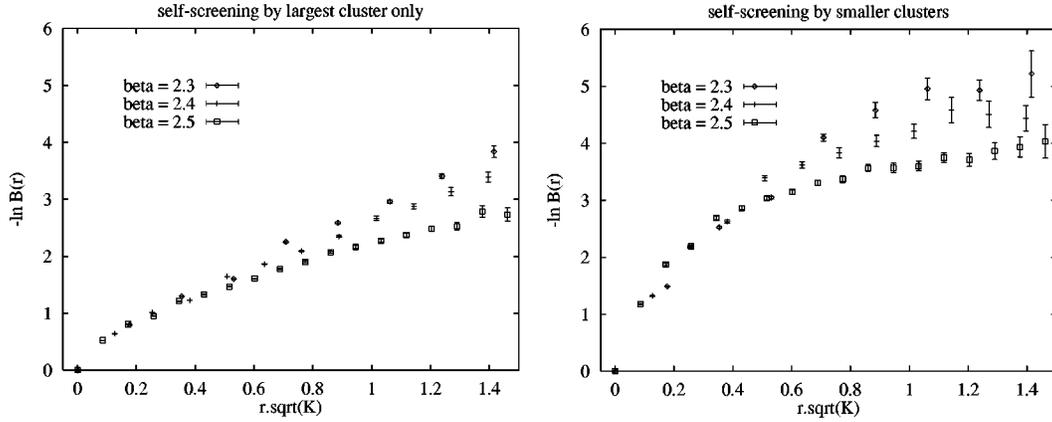


FIG. 3. Screening of flux as a function of distance in physical units for monopoles at  $\beta=2.3, 2.4$  and  $2.5$ . The left hand plot uses monopoles from the largest cluster only; the right hand plot from the remaining, smaller clusters.

cluster scales confirms that it is indeed the monopoles in this cluster that provide the confining monopole plasma.

Given the above discussion it would appear that the clearest way to reveal the screening of the confining monopoles would be to consider only the flux from those monopoles that are in the largest cluster, but to include all monopoles in the screening of that flux. This we do in Fig. 4. We now observe a flux that scales at all  $r$ . Moreover it can almost be described by a single exponential at all  $r$ . (We should use a lattice version of the Coulomb interaction at very small  $r$ , but we ignore this potential improvement here.) We extract a screening mass of  $m_s = 2.71(15)\sqrt{\sigma}$ , which is roughly consistent with our other values.

Before leaving this topic, it is interesting to ask if this screening mass has anything to do with the spectrum of the underlying SU(2) theory. Abelian dominance suggests that this is just the effective gluon mass in the maximally Abelian gauge. There have been speculations in the past that gauge-fixed quark mass calculations (typically performed in the Landau gauge) are telling us about the constituent quark

mass. So the analogous speculation here would be that our screening mass is related to a constituent gluon mass. It is therefore amusing to note that the lightest glueballs in the SU(2) theory are the scalar and the tensor, with continuum masses of  $m_{0^+} = 3.87(12)\sqrt{\sigma}$  and  $m_{2^+} = 5.63(11)\sqrt{\sigma}$  respectively [18,20]. In a simple constituent gluon picture of the low-lying glueball spectrum one would expect these states to arise from two gluons in an  $L=0$  state, with the spins aligned to give the tensor and anti-aligned to give the scalar. Thus to leading order in the spin-spin interaction the scalar and tensor masses will be equally split from the mass that they would possess if the spin-spin interaction were not present. The observed splitting from the average value is  $\sim \pm 20\%$  which is small enough for the leading order argument to be plausible. The average mass value will then equal that of two constituent gluons, neglecting binding energies (which have to be small if a constituent gluon picture is to have any chance of making sense). We observe that our screening mass is indeed in the right ballpark to be thought of as such a ‘‘constituent gluon’’ mass.

To sum up this section, we have established that the largest cluster is a quite different animal from all the other clusters. It permeates the entire volume, has a constant density and screening length in physical units, and drives confinement. It would seem natural to think of this largest cluster as being a simple example of naive percolation at work. But, as we have seen, this is not the case. If percolation is at work, it is at work within the SU(2) field configurations of which our monopoles are but a skeletal representation.

## V. THE SMALLER CLUSTERS

### A. The cluster spectrum

We have frequently stated that the number of secondary clusters falls off approximately as a cubic power of the cluster length  $l$ . Some evidence for this is shown in Fig. 5 where we display the spectra for three different volumes at  $\beta = 2.3$ . These spectra all fall roughly as  $1/l^3$  for the range of  $l$  where our calculations are accurate. There is also some evidence that at very large  $l$  there is a change in the functional form. There seem to be finite size effects there, and the in-

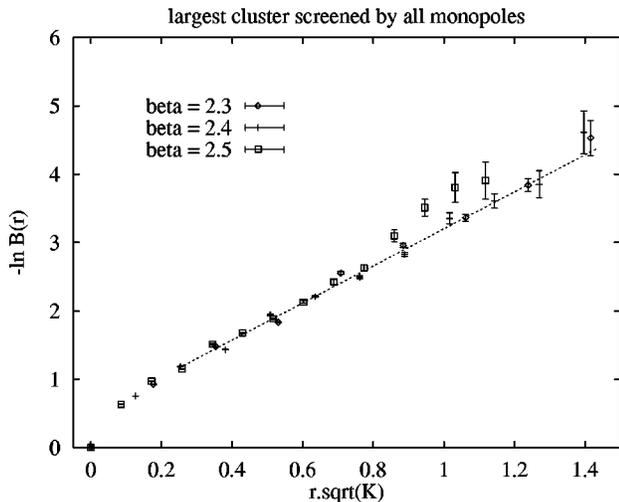


FIG. 4. Screening of flux from monopoles of the largest cluster by all other monopoles as a function of distance in physical units for monopoles at  $\beta=2.3, 2.4$  and  $2.5$ ; with a linear fit shown.

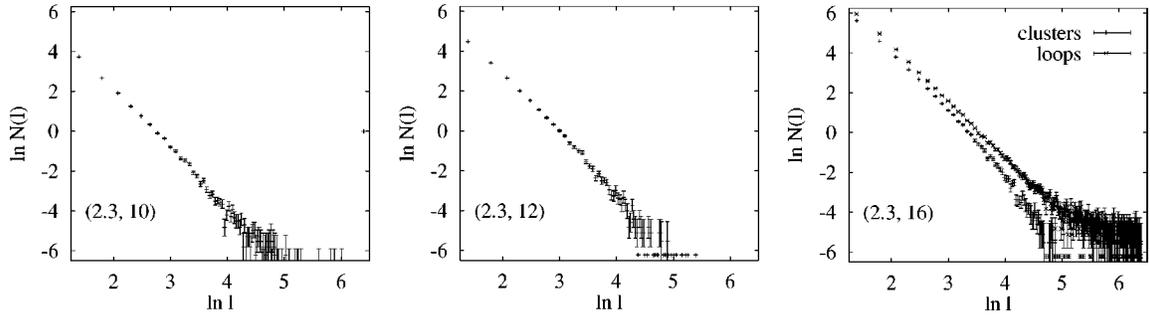


FIG. 5. Monopole cluster spectra at  $\beta=2.3$  on  $L=10, 12$  and  $16$ . The loop spectrum is shown for comparison on  $L=16$ .

dication is that on large enough volumes, the spectrum for very large values of  $l$  might fall off more steeply. More accurate calculations than ours are needed to determine whether this is indeed so. In Figs. 5 and 6 we also display the spectra obtained on  $16^4$  lattices at  $\beta=2.3, 2.4$ , and  $2.5$ . This shows that the  $\sim 1/l^3$  behavior does not depend on  $a$ . We note that as  $a$  decreases, the very large  $l$  end of the spectrum on the  $16^4$  lattices appears to show finite volume effects; perhaps not surprising since  $16a(\beta=2.5) \approx 8a(\beta=2.3)$ . The slight curvature of the spectrum leads to the fit parameters of the power law depending weakly on the range of  $l$  that we choose to fit. Nevertheless, we are able to conclude that all the fits to our data have an exponent in the range  $\gamma \in [2.85, 3.15]$ .

In Fig. 5 we also show for comparison the spectrum obtained when the monopole currents are divided into loops. The main difference is in the normalization; there are more loops of a given size than clusters. Some proportion of the small loops of a given size will be part of larger clusters, and in particular the largest cluster on the lattice. It is interesting, nonetheless, to note that the exponent of the power law for the loop spectrum is in general slightly smaller than that for the more fundamental (we believe) cluster spectrum.

The simplest way to understand this cluster spectrum would be if there were, in essence, only the one current cluster in each field configuration (the very large cluster that we described in the previous section) and that the secondary clusters then arose when small portions of this largest cluster were randomly “pinched” off. The power law spectrum would then have to arise from the relative probability of pinching off portions of the largest cluster of different lengths. Were this the case, the number of clusters of a given length on a configuration (particularly the smallest and most numerous) would be expected to be proportional to the (re-

maining) length of the largest cluster from which they were formed. Unfortunately this turns out not to be even approximately the case, there being no correlation, either positive or negative. (We might also expect the smaller clusters to be preferentially located near current links of the largest cluster, although we did not test this.)

### B. Cluster sizes

What do we know about the sizes of these secondary clusters? We can estimate the cluster radius using the first moment of the current links about the centroid of the cluster. If the cluster were composed of  $n$  current links of charge  $\{j^i: i=1, n\}$  with centers at  $\{x_\mu^i\}$ , then the centroid is

$$\bar{x}_\mu = \frac{1}{l} \sum_{i=1}^n x_\mu^i |j^i| \quad (27)$$

where the length is

$$l = \sum_{i=1}^n |j^i|. \quad (28)$$

The distance of the center of a link from the centroid is  $d^i$ , and the effective radius of the cluster is

$$r_{\text{eff}} = \frac{1}{l} \sum_{i=1}^n d^i |j^i|. \quad (29)$$

We plot this as a function of length in Fig. 7, and find that it is well fitted by the functional form  $r_{\text{eff}}(l) = s + t\sqrt{l}$ . This suggests that the monopole is essentially performing a random walk. Is the step size of this walk fixed in lattice or in physical units? If it were fixed in lattice units we would expect  $t$  to be independent of  $a$ . If the step size were fixed in physical units then we would have  $r\sqrt{K} \propto \sqrt{l\sqrt{K}}$ , and so would expect  $t^2 \propto 1/\sqrt{K}$ . Our calculations, examples of which are presented in Table V, show us that the coefficient  $t$  varies very weakly with  $a$  if at all. There is some variation in our fitted value of  $t$  depending on the range of  $l$  used. But our overall conclusion is that if we insist on parametrizing  $t$  by a power of  $a$  then that power is small:  $t^2 \propto (\sqrt{K})^{-1/8 \pm 1/8}$ . So, although there is some room for a residual weak dependence on  $a$ , the evidence is that the step size in the cluster random walk does not know about physical units.

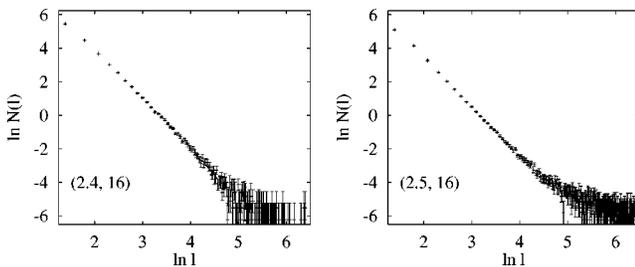


FIG. 6. Monopole cluster spectra on  $L=16$  at  $\beta=2.4$  and  $2.5$ . The equivalent for  $\beta=2.3$  is given in Fig. 5.

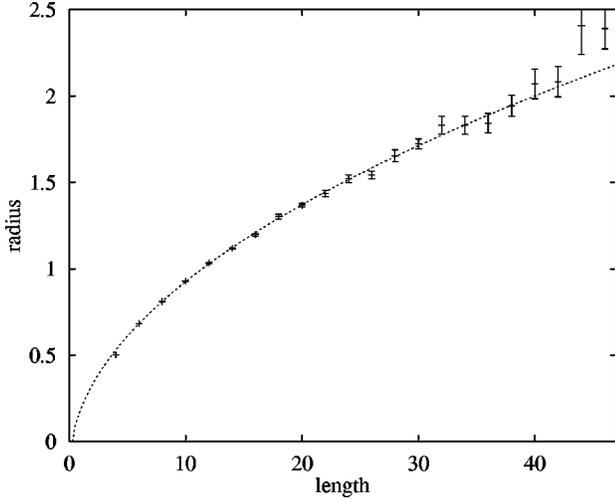


FIG. 7. The cluster radii and a fit  $r_{\text{eff}} = s + t\sqrt{l}$  at  $\beta = 2.3$  on  $L = 12$ .

We note that the values of  $l$  where we saw, in Figs. 5 and 6, evidence of finite size effects in  $N(l)$ , do indeed correspond to cluster sizes,  $r_{\text{eff}}$ , that might plausibly feel the boundaries of our periodic lattices.

### C. Scaling properties

We now turn to the normalization of the spectrum of these secondary clusters and ask what scaling properties it possesses. We have already seen that the total density does not scale: that is, the total length of the secondary clusters is not proportional to the volume when both are expressed in physical units. This in itself is no surprise, however. When we decrease  $a$  by a factor of say 2, then the total current length acquires an additional contribution that is  $\sim \int_4^8 lN(l)dl$  in units of the smaller lattice spacing. (Since the smallest cluster has length  $4a$ .) This will be a significant contribution because the spectrum grows rapidly at small  $l$ . So if nothing else, we expect a significant scaling violation from the growing tail of ultraviolet clusters and any test of scaling must take this into account. The simplest form of physical scaling would be to consider only those clusters whose length is larger than some fixed physical length  $l_p$ , i.e.  $l \geq l_p/\sqrt{K}$ , and then to demand that the total length of these clusters is proportional to the volume when both are expressed in physical units. We now see what this implies for the observed spectrum

$$N(l) = \frac{C(L,a)}{l^\gamma} \quad (30)$$

TABLE VI. Power law fits and scaling behavior of the smaller clusters, including the assumption that scaling is controlled by  $\gamma = 3$ .

$L = 16$ :	$\ln C$	$\gamma$	$c_p^1(\gamma)$	$c_p^1(\gamma = 3)$	$c_p^2(\gamma)$	$c_p^2(\gamma = 3)$
$\beta = 2.3$ :	10.38 (21)	3.11 (8)	3.52 (80)	2.97 (4)	1.24 (28)	1.05 (2)
$\beta = 2.4$ :	9.72 (11)	2.90 (4)	4.45 (60)	5.20 (25)	1.14 (15)	1.34 (4)
$\beta = 2.5$ :	9.30 (12)	2.94 (4)	6.30 (90)	6.75 (24)	1.10 (15)	1.16 (9)

TABLE V. Fitting  $r_{\text{eff}}(l) = s + t\sqrt{l}$  to the clusters.

	$s$	$t$
$(\beta = 2.3, L = 12)$ :	-0.150 (3)	0.340 (2)
$(\beta = 2.4, L = 14)$ :	-0.175 (3)	0.350 (1)
$(\beta = 2.5, L = 16)$ :	-0.191 (2)	0.355 (1)

with  $\gamma \sim 3$ . (For these purposes any deviation at very large  $l$  is negligible, and the deviations at the small  $l$  ultraviolet scale are irrelevant.) Scaling would imply

$$\sqrt{K} \int_{l_p/\sqrt{K}} lN(l)dl = \sqrt{K} \int_{l_p/\sqrt{K}} l \frac{C(L,a)}{l^\gamma} dl \propto (L\sqrt{K})^4 \quad (31)$$

which requires

$$C(L,a) \propto L^4 (\sqrt{K})^{5-\gamma}. \quad (32)$$

This is to be contrasted with what we should expect if these secondary clusters only knew about the ultraviolet length scale,  $a$ :  $C(L,a) \propto L^4$ . As we have already seen, in Table II, the factor of  $L^4$  is certainly there. What is at issue is the dependence on  $\sqrt{K}$ . Scaling requires that the quantity

$$c_p^1 = \frac{C}{L^4 \cdot (\sqrt{K})^{5-\gamma}} \quad (33)$$

should be independent of  $\beta$ . In Table VI we show the values of  $c_p^1$  that we have obtained on our  $L = 16$  lattices both when we use the value of  $\gamma$  obtained from the power law fit, and when we impose a fixed value  $\gamma = 3$  at all values of  $\beta$  (as our above analysis assumes). As we see, rather than being constant  $c_p^1$  increases roughly as  $1/K$ . This is what one expects (with  $\gamma \approx 3$ ) if the clusters know only of the ultraviolet scale.

We have tested a particular formulation of scaling which, naively, would seem to be the most reasonable. It is not unique, however. A plausible alternative would be to focus on the total number of clusters instead of their total length. If we consider the total number of clusters whose length is greater than some constant in physical units, then in fact we find the same criterion as above. To get something different we might, for example, ask (as in [7]) whether perhaps it is  $N(l)$  itself that scales with the physical volume, when  $l$  is chosen fixed in physical units. This would require  $c_p^2 = C/L^4 \cdot (\sqrt{K})^{4-\gamma}$  to be independent of  $\beta$ . In Table VI we show this, again for  $\gamma$  from the power law fits at different  $\beta$  and with a single, imposed value of  $\gamma = 3$ . Using the fitted  $\gamma$ ,

this quantity appears to scale much better. This result is not robust; imposing a fixed  $\gamma$ , however, where the statistical errors are less, this scaling appears less good. Without some argument for keeping the measure  $dl$  in lattice units (which is what we have just done), however, it does not really make sense as a scaling criterion. It seems that if we take the secondary clusters at face value, they certainly do not have the right scaling properties to be physical objects.

Given that the secondary clusters do not scale as ‘‘physical objects,’’ we can ask whether they scale as purely lattice artifacts. If so we would expect the total current length to be  $\propto L^4$  but to be independent of  $a$ . So if we focus on the  $L=16$  lattices in Table II, we would expect  $(l_{\text{tot}} - l_{\text{max}})$  to be independent of  $\beta$ . In fact the values are 3226, 2997, and 2244 at  $\beta=2.3, 2.4$  and  $2.5$  respectively. We know that the  $\beta=2.5$  value is suppressed below its true value because there is some overlap between the largest cluster and the secondary spectrum: so  $l_{\text{max}}$  is certainly overestimated. Nonetheless, even allowing for that, there does appear to be some significant  $a$  dependence,  $C(L,a) \propto L^4(\sqrt{K})^{0.2-1.0}$ , but it is quite weak suggesting that the spectrum is influenced more by the ultraviolet than by the physical length scale.

#### D. Clusters as 4-balls

Of course the monopole currents are only images, through gauge-fixing, of some unknown structures in the SU(2) gauge fields. It is the latter that one would hope to be physical. In fact our observed cluster spectrum does provide some intriguing hints as to what these structures might be. As we have seen, a monopole cluster of length  $l$  is localized within a region in space-time of size  $r \approx t\sqrt{l}$ . We call such an object a ‘‘4-ball’’ for obvious reasons. What is the spectrum,  $N_B(r)dr$ , of these 4-balls? It is easy to see that if the radius is related to the length by  $r \propto \sqrt{l}$ , the cluster spectrum  $N(l)dl = C/l^3$  translates into the following 4-ball spectrum:

$$N_B(r)dr = C_B \frac{dr}{r} \times \frac{1}{r^4}. \quad (34)$$

We recognize this to be simply the general scale invariant distribution of objects of radius  $r$  in four dimensions. (Such an object takes up a volume  $\sim r^4$  and hence there are  $\sim 1/r^4$  ways of placing it in a unit volume. And  $dr/r$  is a scale-invariant measure.) This is precisely the formula one has for the density of instantons, before one includes the effects of the scale anomaly through the running of the coupling. Since we know that an isolated instanton, when projected to the maximally Abelian gauge, generates a monopole current loop of size comparable to its core size [8], it would be tempting to put forward the elegant hypothesis that these 4-balls are just SU(2) instantons and that the secondary clusters are simply the associated monopole loops. Unfortunately things cannot be so simple. Although we do not know how the scale-breaking affects the instanton density at large sizes, we do know what it does to the distribution at small sizes: the  $dr/r^5$  is transformed into  $r^{7/3}dr$ . This is nothing at all like our 4-ball number density.

It is interesting to repeat our previous scaling analysis, but this time assuming that it is the 4-balls that are physical rather than the clusters themselves. That is to say, we impose that the number of 4-balls of radius larger than some fixed length in physical units, should be proportional to the physical volume. It is easy to see that this implies that  $C_B(L,a) \propto L^4$ . One obtains the same result, however, if one constrains the density to be constant in lattice units, or any other units, because the 4-ball density just reflects naive dimensional counting. Thus we expect rather generally that

$$N(l)dl = \frac{C(L,a)dl}{l^3} \propto \frac{L^4}{t^4} \frac{dl}{l^3} \quad (35)$$

where we have gone from the 4-ball density to the cluster spectrum using  $r \sim t\sqrt{l}$ . This implies that

$$C(L,a) \propto \frac{L^4}{t^4} \propto L^4(\sqrt{K})^{0.25 \pm 0.25} \quad (36)$$

using our results for the  $a$ -dependence of  $t$ . This weak  $a$ -dependence is entirely consistent with what we observe for the cluster spectrum:  $C(L,a) \propto L^4(\sqrt{K})^{0.2-1.0}$ . Thus the spectrum of secondary clusters is consistent, in every respect, with arising from a scale-invariant density of 4-balls.

As an aside, we note that the largest cluster from the distribution  $N(l)$  has a length  $l_{2\text{nd}} \propto C(L,a)^{1/(\gamma-1)}$ . Putting in  $\gamma=3$  and the form for  $C(L,a)$  as in the previous paragraph, we see that  $l_{2\text{nd}} \propto L^2(\sqrt{K})^{0.1-0.25}$ . By contrast the length of the largest cluster varies as:  $l_{\text{max}} \propto L^4(\sqrt{K})^3$ . From this we see that if we wish to maintain  $l_{\text{max}} \gg l_{2\text{nd}}$  as  $a \rightarrow 0$  then the lattice size in physical units must grow roughly as  $(1/\sqrt{K})^{1/2}$ . Thus, for example, the two types of clusters begin to overlap on our  $16^4$  lattice at  $\beta=2.5$  (rendering some of the calculations there ambiguous) despite the fact that they did not do so on the  $8^4$  lattice at  $\beta=2.3$ . This is something we did not, of course, anticipate when originally choosing our lattice sizes.

## VI. SUMMARY

In this paper we have shown that the magnetic monopole currents that we obtain, when gauge fixing SU(2) fields to the maximally Abelian gauge, divide into two quite distinct classes (on large enough volumes): a single very large cluster and a distribution of very much smaller clusters.

The very large cluster has a length that is proportional to the space-time volume when both are expressed in physical units. Moreover we have shown that it is this cluster that generates the string tension. We have also seen that, within this largest cluster, the effective screening length relevant to confinement is constant in physical units. Thus it is this cluster that represents all the interesting infrared physics of the SU(2) fields.

That there is always just one very large cluster is a significant fact since, as we saw, even under a minimal amount of smoothing this cluster readily hives off secondary clusters that are much larger than those that we observe in the unsmoothed fields.

The secondary clusters are localized compact objects obtained by the monopole performing a random walk on the length of the lattice spacing. This is in contrast to the largest cluster whose observed scaling demands that the step size be on the length of the physical length scale. These secondary clusters contribute nothing to the string tension even where they constitute a sizeable fraction of the total magnetic current.

One might be tempted to ignore these secondary clusters as being of no physical importance. They do seem quite remarkable in at least one respect, however. When one treats them as localized objects in space-time (“4-balls”), one finds that the number density is of the simplest scale-invariant form. This is reminiscent of classical instantons, but unfortunately incompatible with the real instanton density at small distances.

The calculations of this paper can be improved upon in many ways. In particular better calculations could clarify what happens to the distribution of secondary clusters at very large  $l$  and it would be useful to calculate the 4-ball number density directly (as we would have done if we had not deduced their relevance after completing the simulations).

The monopole content of the vacuum thus seems to split

up into two types of cluster. First there is the confining cluster that knows about the physical length scale (ultimately due to the breaking of scale invariance) but does not seem to know anything at all about the lattice length scale. Secondly there are the other, smaller clusters. These can be thought of as compact objects that satisfy a scale invariant distribution: while they know about the lattice spacing, they apparently know little about the breaking of scale invariance. This is unexpected and puzzling, because these clusters should somehow reflect fluctuations in the SU(2) fields. Of course, because the gauge-fixing procedure is completely non-local, it is possible that the monopoles we observe only reflect an effective theory that possesses the same infrared physics as the non-Abelian theory. To resolve this puzzle would be of interest.

#### ACKNOWLEDGMENTS

The work of A.H. was supported in part by United States Department of Energy grant DE-FG05-91 ER 40617. M.T. was supported by United Kingdom PPARC grants GR/K55752 and GR/K95338.

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