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Analytic study of the null singularity inside spherical charged black holes

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We study analytically the features of the Cauchy horizon (CH) singularity inside a spherically symmetric charged black hole, nonlinearly perturbed by a self-gravitating massless scalar field. We derive exact expressions for the divergence rate of the blue-shift factors, namely the derivatives in the outgoing direction of the scalar field Φ and the area coordinate r. Both derivatives are found to grow along the contracting CH exactly like $1/r$. Our results are valid everywhere along the CH singularity, down to the point of full focusing. These exact analytic expressions are verified numerically.

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I. INTRODUCTION

In the last few years, the investigation of spinning and charged black holes led to a new picture of the spacetime singularity inside such black holes. According to this new picture, the Cauchy horizon (CH) evolves into a curvature singularity, which has the following two remarkable features: (i) It is *null* (rather than spacelike). (ii) It is *weak* (in Tipler's terminology [1]); namely, the tidal deformation experienced by extended physical objects is finite at the null singularity. In the case of a spinning black hole, the evidence for the occurrence of the null weak singularity has emerged from a systematic linear and nonlinear perturbation analysis [2]. For the toy model of a spherical charged black hole, the main features of the singularity at the inner horizon were first deduced analytically from simplified models based on null fluids [3–5], and later confirmed numerically for a model with a self-gravitating scalar field [6,7]. (See also the approximate leading-order analysis in [8].) In addition, the *local* existence and genericity of the null weak singularity were shown mathematically in Ref. [9], and more recently (in the framework of plane-symmetric spacetimes) in Ref. [10]. The compatibility of a null weak singularity with the constraint equations was shown in Ref. [11].

We shall consider here the spherically symmetric model of a charged black hole nonlinearly perturbed by a self-gravitating, minimally coupled, massless scalar field. Despite its relative simplicity (compared to the analogous model of a spinning black hole), no systematic analytic study of this model has been carried out so far. The goal of this paper is to present a simple analytic calculation, which may be the first step towards such a thorough analytic study: We quantitatively analyze the evolution of the divergent blue-shift factors along the contracting CH. It is well known (from both theoretical considerations and numerical simulations) that the singularity at the CH is characterized by finite values of the scalar field Φ and the area coordinate r. (The latter is also known to decrease monotonically with increasing affine parameter along the CH, due to the outflux of energy-momentum carried by the scalar field.) However, the *gradients* of Φ and r diverge at the CH. More specifically, let V be a ''Kruskal-like'' ingoing null coordinate (i.e. an ingoing null coordinate for which the double-null metric function g_{UV} is finite and nonvanishing at the Cauchy horizon—see below). Then, $r_{,V}$ and $\Phi_{,V}$ diverge at the CH. In this paper we shall calculate the evolution of $r_{,V}$ and $\Phi_{,V}$ along the contracting CH. We shall show, analytically, that the divergence rate of both entities is exactly proportional to $1/r$. Our method of calculation is non-perturbative, and is therefore valid also in the region of strong focusing; however, we shall use the perturbative results (applicable at the early section of the CH) to determine the two overall coefficients characterizing the blue-shift divergence.

The paper is organized as follows. In Sec. II we describe the physical model of the self-gravitating massless scalar field on a spherical charged black hole, and present the field equations. In Sec. III we carry out a leading-order perturbation analysis of r (and use previous perturbative results for Φ) and calculate the v-derivatives of Φ and r at the very early part of the CH (where the focusing effect is still negligible). Then, in Sec. IV we perform a fully nonlinear (and non-perturbative) calculation of these v-derivatives, which is valid everywhere along the contracting CH, down to the point of full focusing, where $r=0$ and the singularity becomes spacelike [6]. This nonlinear analysis leaves two coefficients undetermined—one for each field—and we determine these two coefficients by matching the nonlinear results to the linear results applicable at the asymptotically early part of the CH. Our results are in excellent agreement with the numerically obtained results [7].

II. PHYSICAL MODEL AND FIELD EQUATIONS

We consider here the model of a spherically symmetric charged black hole, nonlinearly perturbed by a self-gravitating, spherically symmetric, minimally coupled, massless scalar field Φ (the same model as that analyzed numerically in [6,7]). This model allows us to obtain nontrivial radiative dynamics, while retaining the simplicity of the spherical symmetry.

We write the general spherically symmetric line element in double-null coordinates,

$$ds^2=-f(u,v)du\,dv+r^2(u,v)d\Omega^2, \tag{1}$$

where $d\Omega^2$ is the line element on the unit two-sphere. The energy-momentum tensor of a massless scalar field is

$$T^{\rm s}_{\mu\nu}=\frac{1}{4\pi}\left(\Phi_{,\mu}\Phi_{,\nu}-\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta}\right). \tag{2}$$

The energy-momentum associated with the general spherically symmetric free electromagnetic field is

$$T^{\rm em}_{\mu\nu}=\frac{Q^2}{8\pi r^4}\begin{pmatrix} 0 & f/2 & 0 & 0\\ f/2 & 0 & 0 & 0\\ 0 & 0 & r^2 & 0\\ 0 & 0 & 0 & r^2\sin^2\theta\end{pmatrix}, \tag{3}$$

where Q is the electric charge.

For a spherically symmetric scalar field, the Klein-Gordon (KG) equation reduces to

$$\Phi_{,uv}+\frac{1}{r}(r_{,u}\Phi_{,v}+r_{,v}\Phi_{,u})=0. \tag{4}$$

The Einstein field equations, $G_{\mu\nu}=8\pi(T^{\rm s}_{\mu\nu}+T^{\rm em}_{\mu\nu})$, include two evolution equations

$$r_{,uv}=-\frac{r_{,u}r_{,v}}{r}-\frac{f}{4r}\left(1-\frac{Q^2}{r^2}\right) \tag{5}$$

$$f_{,uv}=\frac{f_{,u}f_{,v}}{f}+f\left\{\frac{1}{2r^2}\left[4r_{,u}r_{,v}+f\left(1-2\frac{Q^2}{r^2}\right)\right]-2\Phi_{,u}\Phi_{,v}\right\}, \tag{6}$$

and two constraint equations:

$$r_{,uu}-(\ln f\,)_{,u}r_{,u}+r(\Phi_{,u})^2=0 \tag{7}$$

$$r_{,vv}-(\ln f\,)_{,v}r_{,v}+r(\Phi_{,v})^2=0. \tag{8}$$

The form of the above line element and field equations is invariant to a coordinate transformation of the form $v\rightarrow\overline{v}(v)$, $u\rightarrow\overline{u}(u)$. In what follows u and v will denote *generic*, unspecified, double-null coordinates. Below we shall often use specific types of null coordinates for specific calculations, and in order to avoid confusion, we shall assign a special symbol to each of these specific coordinates. Thus, we shall denote the standard Eddington-Finkelstein null coordinates of Reissner-Nordström (RN) by u_e and v_e. We shall also use U and V to denote *Kruskal-like* coordinates, i.e. double-null coordinates which regularize the line element (1) at the inner horizon. In addition, in Sec. IV we shall define two other types of ingoing null coordinates, V_r and V_Φ.

III. LINEAR REGIME

Previous analytic and numerical studies have indicated that the geometry at (and near) the early part of the CH may well be described by the background metric functions of the static (or stationary) black-hole solution plus a small metric perturbation. This is found to be the situation both in vacuum spinning black holes (analytically) [2] and in the present model of a spherical charged black hole (numerically) [7]. Moreover, the perturbations become arbitrarily small as one approaches the asymptotic past of the CH. In the very early part of the CH, the perturbations are dominated by their linear part, and the singularity is well described by the linear metric perturbation. In the later part of the CH, however, nonlinear effects become exceedingly important, as demonstrated, e.g., by the contraction of the CH.

Accordingly, we shall schematically divide the CH into two parts:

(1) The *linear regime*, i.e. the asymptotically early part of the CH, where the metric perturbations (and the scalar field) are still very small, and a leading-order analysis is adequate;

(2) The *nonlinear regime*, i.e. the later part of the CH, where the focusing (and possibly other nonlinear effects) become important. At the future end of the nonlinear regime the area of the CH shrinks to zero, and the singularity becomes spacelike.

In this section we shall consider the linear regime, and obtain expressions for the blue-shift factors, namely, the v-derivatives of Φ and r. The nonlinear regime will be the subject of Sec. IV.

In the linear regime we may treat Φ as a linear KG field over a fixed RN background. The evolution of such a field was analyzed in [12] and more recently in [13,14] (using a different method). For a spherically symmetric scalar field satisfying an inverse power-law

$$\Phi\cong v_e{}^{-n}\quad(\text{EH}) \tag{9}$$

at the event horizon (EH), the asymptotic behavior at the early part of the CH was found to be [12,13]

$$\Phi\cong Av_e{}^{-n}+Bu_e{}^{-n}\quad(\text{CH}), \tag{10}$$

where A and B are constants which only depend on the ratio Q/M, M being the black-hole mass. Since we are primarily interested here in the v-derivative of Φ, we only need the value of A, which was found in Refs. [12, 14] to be

$$A=\frac{1}{2}\frac{r_+}{r_-}\left(\frac{r_+}{r_-}+\frac{r_-}{r_+}\right), \tag{11}$$

$r_\pm$ being the value of r at the outer and inner horizons of RN.

One finds that both at the EH and at the CH

$$\Phi_{,v_e}\propto v_e{}^{-p}, \tag{12}$$

where $p\equiv n+1$, and

$$\Phi^{\rm CH}_{,v}/\Phi^{\rm EH}_{,v}\rightarrow A. \tag{13}$$

Here and below the arrow denotes the limit of large advanced time (corresponding to $v_e\rightarrow\infty$). Note that the last relation is explicitly gauge invariant, so it holds for any type of ingoing null coordinate v, and not only for $v=v_e$.

Next, we consider the v-derivative of r at the CH. In the pure RN geometry, $r_{,v_e}$ dies off exponentially (in v_e) at the CH. In the presence of the self-gravitating scalar field, however, $r_{,v_e}$ decays as a *power-law* of v_e (see below). In the asymptotically early portion of the CH (the ‘‘linear regime’’) which concerns us here, the effect of the scalar field is dominated by the second-order term (i.e., the term quadratic in derivatives of the scalar field), and higher-order corrections are negligible. We shall now calculate this leading-order term of $r_{,v_e}$.

Viewing Eq. (8) as a linear first-order differential equation for $r_{,v}$, we formally integrate it and obtain

$$r_{,v}(v)=-f(v)\int^{v}\frac{r(v')}{f(v')}[\Phi_{,v}(v')]^2dv'. \tag{14}$$

Here [and also in Eq. (15) below] the integration is done along lines of constant u, and we omit the dependence on u for brevity. From this exact expression we now extract the term quadratic in (derivatives of) Φ. Since $\Phi_{,v}{}^2$ appears explicitly in the integrand, we simply need to replace f and r in the right-hand side (RHS) by the corresponding unperturbed metric functions of RN, which we denote $f_{\rm RN}$ and $r_{\rm RN}$:

$$r_{,v}(v)=-f_{\rm RN}(v)\int^{v}\frac{r_{\rm RN}(v')}{f_{\rm RN}(v')}[\Phi_{,v}(v')]^2dv'. \tag{15}$$

Note that this equation is invariant to the choice of gauge for the coordinate v. We shall now evaluate the integral at the RHS, using the null coordinate v_e. In this gauge, $f_{\rm RN}$ decays exponentially at the CH:

$$f_{\rm RN}\propto e^{-\kappa_- v_e}, \tag{16}$$

where κ_- is the surface gravity of the inner horizon. On the other hand, $\Phi_{,v_e}$ decays as an inverse power of v_e, and $r_{\rm RN}$ approaches a nonzero constant, r_-, at the CH. Therefore, since the relative change of $\Phi_{,v}$ (and $r_{\rm RN}$) is exponentially slower than that of $f_{\rm RN}$, to the leading order in $1/v_e$ we can take $\Phi_{,v}$ outside the integral, and substitute $r_{\rm RN}\cong r_-$ [as well as Eq. (16) for $f_{\rm RN}$]. Doing so, we obtain (to the leading order in $1/v_e$)[1]

$$r_{,v_e}\cong-\frac{r_-}{\kappa_-}(\Phi^{\rm CH}_{,v_e})^2. \tag{17}$$

Finally, using Eq. (13), we find

$$\frac{r_{,v_e}}{(\Phi^{\rm EH}_{,v_e})^2}\to-\frac{r_-}{\kappa_-}A^2. \tag{18}$$

In particular, we have in the linear regime

$$r_{,v_e}\propto v_e^{-2p}. \tag{19}$$

One clarification should be made here concerning the precise meaning of the parameters r_- and κ_-, and the coordinate v_e, in the perturbed spacetime. (Originally these entities are only defined in the pure RN geometry.) We know that outside the black hole, both the scalar field and the metric perturbations decay at late time, and the geometry approaches that of RN. In particular, the mass function approaches a limiting value M. We thus define r_- and κ_- according to the value of these parameters in the asymptotic RN geometry, i.e. according to their standard definition in terms of M and Q (with M being the above late-time limit of the mass function; note that the charge Q is a fixed parameter in our model). In a similar way, we also define the coordinate v_e with respect to this late-time asymptotic RN geometry. More specifically, we may define v_e according to the affine parameter λ along a line of constant $r>r_+$ (or along the EH), by taking $v_e(\lambda)$ to be the same function as in the pure RN geometry (with a mass parameter M defined as above). Note that once the entities M, r_-, κ_-, and v_e were defined in the linear regime, their extension to the nonlinear regime is trivial.

One might be puzzled by the relevance of the asymptotic external mass parameter to the internal dynamics near the perturbed CH (and particularly to the definition of the inner-horizon parameters r_- and κ_-), especially when the divergence of the mass function at the CH is recalled. The resolution of this puzzle relays on the basic features of the geometry inside perturbed charged (or spinning) black holes: On the one hand, the geometry is drastically different from that of RN (or Kerr), as expressed by the divergence of curvature at the CH. On the other hand, the geometry is very similar to RN (or Kerr) in terms of the metric functions: The metric perturbations are arbitrarily small at the asymptotic past of the CH. [Roughly speaking, the divergence of curvature indicates the divergence of *derivatives* of the metric functions (with respect to the regular background coordinate V) at the CH.] This smallness of metric perturbations is the necessary basis for the entire perturbative approach: As it turns out, the perturbation analysis (when properly formulated) respects the smallness of the metric perturbations, and not the divergence of curvature. That is, the typical ratio of two successive terms in the nonlinear perturbation expansion is comparable to the small metric perturbations, and not to the diverging curvature (this is fortunate, because otherwise the perturbative approach would render useless). This was demonstrated analytically for spinning black holes [2], and numerically for charged ones [7].

In the above analysis of the linear regime (based on the perturbative approach), r_- and κ_- appear as *parameters of the background RN geometry*, and their definition should therefore be based on the asymptotic mass function M. On the other hand, the divergence of the mass function (whose definition also involves the *derivatives* of r) at the perturbed CH merely reflects the divergence of $r_{,V}$ there, due to the *perturbation* (which undergoes infinite blue-shift). Obviously, this divergence has no relevance to the background parameters r_- and κ_-.

IV. NONLINEAR REGIME

We turn now to analyze the divergence rates of $r_{,v}$ and $\Phi_{,v}$ along the nonlinear, strong-focusing, portion of the CH. Here, it will be insufficient to calculate the leading-order

[1]In the transition from Eq. (14) to Eq. (15), we got rid of all terms of order higher than quadratic in Φ. Thus, in principle Eq. (15) should include both the zero-order and the second-order parts of $r_{,v}$. The zero-order term is represented by the (implicit) integration constant in Eq. (15). This zero-order term is exponentially small, however, and is thus negligible compared to the quadratic term in Eq. (17).

perturbations, so we must carry out a full nonlinear calculation.

We shall base our calculation on two assumptions:

(1) For an appropriate choice of coordinates u,v, the line-element (1) is valid up to the singular CH, and both functions f and r are finite and nonvanishing along the entire CH. We shall denote such regular coordinates by U,V, and refer to them as *Kruskal-like* coordinates. (Of course, the choice of U and V is not unique.) We shall also set $V=0$ at the CH.

(2) There exists at least a single outgoing null geodesic, $u=u_0$, which intersects the CH and which satisfies the following two requirements:

(a) Along $u=u_0$, r and Φ are monotonic functions of v in a neighborhood of the CH,

(b) Along $u=u_0$, both $r_{,V}$ and $d\Phi/dr$ (i.e. $\Phi_{,V}/r_{,V}$) diverge at the CH.

The validity of assumption 1 is strongly supported by the perturbative approach, at least in the early part of the CH. Moreover, recent numerical simulations [7] confirm its validity down to contraction of the CH to less than 1% of its initial value.[2] Assumption 2 is justified, because at least in the asymptotically early part of the CH, Eqs. (12),(19) ensure the required monotonic behavior, and also imply $d\Phi/dr \propto {v_e}^p \to \infty$. In addition, in the linear regime the standard ingoing Kruskal-like coordinate, $V \equiv e^{-\kappa_- v_e}$, regularizes the line element at the CH, and satisfies $r_{,V} \propto {v_e}^{-2p} e^{\kappa_- v_e} \to \infty$. We can thus take u_0 to be in this asymptotically early section (in fact, the numerical simulations [7] confirm that the asymptotic relations of assumption 2 hold also in the nonlinear regime).

To analyze the evolution of $r_{,v}$, we shall use the evolution Eq. (5), viewing it as a first-order ordinary equation for $r_{,v}$. Our goal is to integrate this equation in the ingoing direction, along the CH. This integration would be trivial if the last term on the RHS (which couples this equation to the other evolution equation) were absent. Fortunately, on approaching the CH, this last term becomes arbitrarily small compared to the preceding one. For example, in a Kruskal-like V, the first term in the RHS diverges (at least at $u=u_0$), whereas the second one is finite. (Note that although each of these terms depends on the gauge, their ratio is gauge-invariant.) This suggests that, when integrating this equation along the CH, the last term could be dropped. In order to analyze this equation in a more systematic and elegant way, we define a new ingoing null coordinate V_r in the neighborhood of the CH, by

$$V_r(v) \equiv r(u=u_0,v), \tag{20}$$

and reexpress Eq. (5) in terms of V_r. To transform $f \equiv -2g_{uv}$ from v to V_r, we first calculate g_{UV_r}:

$$g_{UV_r} = g_{UV}/(dV_r/dV) = g_{UV}/(dr/dV)_{u_0}. \tag{21}$$

Since g_{UV} is finite (assumption 1), and $(dr/dV)_{u_0}$ diverges (assumption 2b), we find that g_{UV_r} vanishes everywhere along the CH, and so is g_{uV_r}. Defining $z(u) \equiv (r_{,V_r})_{\rm CH}$, Eq. (5) now reduces to the trivial equation $z_{,u} = -(r_{,u}/r)z$. Its general solution is

$$z = C/r, \tag{22}$$

where C is an integration constant. Note that this exact equality holds *everywhere* along the CH. Calibrating C at $u=u_0$, we find

$$r_{,V_r} = \frac{r_0}{r}(r_{,V_r})_{u_0} = \frac{r_0}{r} \ \ (\mathrm{CH}), \tag{23}$$

where r_0 is the r-value of the CH at $u=u_0$. The first of these two equalities has an explicit gauge-invariant form, so we can immediately transform it to a generic gauge and write it as

$$\frac{r_{,v}}{(r_{,v})_{u_0}} \to \frac{r_0}{r} \tag{24}$$

(later we shall use this result for $v=v_e$).

The analysis of the evolution of $\Phi_{,v}$ proceeds in a similar way. This time we use the KG Eq. (4), viewed as an ordinary differential equation for $\Phi_{,v}$, and integrate it along the CH. By virtue of assumption 2b, the second term in the parentheses in Eq. (4) is negligible at the CH (at least at $u=u_0$) compared to the preceding one. To make an optimal use of this fact, we transform Eq. (4) from v to the new ingoing null coordinate

$$V_\Phi(v) \equiv \Phi(u=u_0,v), \tag{25}$$

defined in a neighborhood of the CH. The last term in the transformed equation is proportional to $r_{,V_\Phi}$. But

$$r_{,V_\Phi} = r_{,V_r}(dV_r/dV_\Phi) = r_{,V_r}(dr/d\Phi)_{u_0}. \tag{26}$$

At the CH, $r_{,V_r} = C/r$ and $(dr/d\Phi)_{u_0} \to 0$ (assumption 2b), so the last term in the transformed Eq. (4) vanishes. Defining $y(u) \equiv (\Phi_{,V_\Phi})_{\rm CH}$, Eq. (4) becomes $y_{,u} = -(r_{,u}/r)y$, whose general solution is

$$y = K/r, \tag{27}$$

where K is an integration constant. Calibrating K at $u=u_0$, we find

$$\Phi_{,V_\Phi} = \frac{r_0}{r}(\Phi_{,V_\Phi})_{u_0} = \frac{r_0}{r} \ \ (\mathrm{CH}), \tag{28}$$

and again, the first equality may be immediately transformed to a generic gauge:

$$\frac{\Phi_{,v}}{(\Phi_{,v})_{u_0}} \to \frac{r_0}{r}. \tag{29}$$

We shall now match the non-linear results (24) and (29)

[2]If assumption 1 were valid only in a portion of the CH, then the analysis below would nevertheless be applicable to this portion (provided that the outgoing null ray considered in assumption 2 intersects this portion).

to the leading-order results at the linear regime. To that end, we take our reference outgoing ray $u=u_0$ to be in the asymptotically early section of the CH. We can then use the results of the previous section [e.g. Eqs. (13) and (18)] for $(r_{,v})_{u_0}$ and $(\Phi_{,v})_{u_0}$, and also substitute $r_0=r_-$. Combining Eq. (24) (with $v=v_e$) and Eq. (29) with Eqs. (18) and (13), respectively, we obtain

$$\frac{\Phi_{,v}}{\Phi_{,v}^{\mathrm{EH}}}\to\frac{r_-}{r}A \tag{30}$$

and

$$\frac{r_{,v_e}}{(\Phi_{,v_e}^{\mathrm{EH}})^2}\to-\frac{r_-{}^2}{r\kappa_-}A^2. \tag{31}$$

These exact relations hold everywhere along the CH.

More explicitly, for initial data $\Phi\cong v_e{}^{-n}$ at the EH, the asymptotic behavior at the CH is (to leading order in $1/v_e$)

$$\Phi_{,v_e}\cong-n\frac{r_-}{r}Av_e{}^{-(n+1)},\ \ r_{,v_e}\cong-n^2\frac{r_-{}^2}{r\kappa_-}A^2v_e{}^{-2(n+1)}. \tag{32}$$

These results take an especially simple form when expressed in terms of $\Psi\equiv r\Phi$ and r^2:

$$\Psi_{,v_e}\cong-nr_-Av_e{}^{-(n+1)},\ \ (r^2)_{,v_e}\cong-2n^2\frac{r_-{}^2}{\kappa_-}A^2v_e{}^{-2(n+1)}. \tag{33}$$

(Note that to the leading order in $1/v_e$, which concerns us here, the contribution of $r_{,v_e}$ to $\Psi_{,v_e}$ is negligible.) That is, to the leading order in $1/v_e$, the v-derivatives of Ψ and r^2 at the CH are *independent of* r (and u). The translation of the above results from v_e to any other type of ingoing null coordinate (e.g. V) is straightforward.

The above results are verified numerically in Ref. [7]. The terms at the two sides of Eq. (30) and (31) are evaluated numerically along an outgoing null ray that intersects the strong-focusing portion of the CH. We have checked the analytic results numerically up to a stage of 90% focusing of the CH, and found excellent agreement. We believe that these results hold along the entire CH.

It would be an interesting challenge to try generalizing these results to the CH singularity of a generic spinning vacuum black hole.

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