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# Radiation and string atmosphere for relativistic stars

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We extend the Vaidya radiating metric to include both a radiation field and a string fluid. Assuming diffusive transport for the string fluid, we find new analytic solutions of Einstein's field equations. Our new solutions represent an extension of the Xanthopoulos superposition. [S0556-2821(98)50210-X]

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Vacuum solutions of Einstein's field equations have played an important role in our understanding of curvature effects and relativistic behavior even though it is obvious that real stars do not sit in a vacuum but have particle and radiation atmospheres. Quantum effects allow atmospheres to be added to classical vacuums; for example black holes are associated with atmospheres of Hawking radiation [1,2]. In addition to their intrinsic value as exact solutions, vacuum solutions in general relativity are approximate string theory solutions for curvature, small compared to the Planck scale [3]. The intense level of activity in string theory has lead to the idea that many of the classic vacuum scenarios, such as the static Schwarzschild point mass or black hole, may have atmospheres composed of a fluid or field of strings [4]. One of the well known examples of radiation atmospheres is the Vaidya metric [5], generated from the vacuum Schwarzschild solution by allowing the Schwarzschild mass to be a function of retarded time. The resulting stress-energy content describes outgoing short-wavelength photons.

In this Rapid Communication we extend the Vaidya metric by allowing the mass to be a function of both retarded time and distance along the outgoing null geodesics. The effect of this extension is to create two fluids outside the star, the original null fluid and a new fluid composed of strings. Given the recent links [6,7] between black holes and string theories, this result is of interest by itself. It is additionally interesting since our new analytic solutions for the mass function allow the metric to be written as a superposition of a string fluid and vacuum Schwarzschild. We have thereby extended the Xanthopolous superposition [8].

The string fluid tension and density depend on spatial derivatives of the mass function. Assuming a specific model for propagation of the density allows the generation of new densities and hence new mass functions. We choose to propagate the density diffusively as a particular example of mass transport.

Our sign conventions are $2A_{\nu;[\alpha\beta]}=A_\mu R^\mu{}_{\nu\alpha\beta}$, $R_{\mu\nu}=R^\alpha{}_{\mu\nu\alpha}$, and metric signature (+,-,-,-). Greek indices range over (0,1,2,3) = $(u,r,\vartheta,\varphi)$. $\dot{m}$ abbreviates $\partial m/\partial u$, $m'$ abbreviates $\partial m/\partial r$, and $m''$ represents $\partial^2 m/\partial r^2$. Overhead carets denote unit vectors.

The spacetime metric covering the region exterior to a spherical star is given by

$$ds^2=Adu^2+2dudr-r^2(d\vartheta^2+\sin^2\vartheta d\varphi^2) \tag{1}$$

where $A=1-2m(u,r)/r$. Initially $m(u,r)=m_0$ provides the vacuum Schwarzschild solution in the region $r>2m_0$. At later times $m(u,r)$ admits a two-fluid description of diffusing matter and outward flowing short-wavelength photons (sometimes called a "null fluid"). Metric (1) is spherically symmetric and given in retarded time coordinate $u$. With the use of a Newman-Penrose null tetrad the Einstein tensor is computed from Eq. (1) and given by

$$G_{\mu\nu}=-2\Phi_{11}(l_\mu n_\nu+n_\mu l_\nu+m_\mu\bar{m}_\nu+\bar{m}_\mu m_\nu) -2\Phi_{22}l_\mu l_\nu-6\Lambda g_{\mu\nu}. \tag{2}$$

Here the null tetrad components of the Ricci tensor are

$$\Phi_{11}=-(rm''-2m')/(4r^2), \tag{3a}$$

$$\Phi_{22}=-\dot{m}/r^2, \tag{3b}$$

$$\Lambda=R/24=(rm''+2m')/(12r^2). \tag{3c}$$

The metric is Petrov type $D$ with $l_\mu$ and $n_\mu$ principal null vectors, $l_\mu$ geodesic, and

$$l_\mu dx^\mu=du, \tag{4a}$$

$$n_\mu dx^\mu=(A/2)du+dr, \tag{4b}$$

$$m_\mu dx^\mu=-(r/\sqrt{2})(d\vartheta+i\sin\vartheta d\varphi). \tag{4c}$$

In order to clearly see the two-fluid description we introduce a timelike unit velocity vector $\hat{v}^\mu$ and three unit spacelike vectors $\hat{r}^\mu$, $\hat{\vartheta}^\mu$, $\hat{\varphi}^\mu$ such that

$$g_{\mu\nu}=\hat{v}_\mu\hat{v}_\nu-\hat{r}_\mu\hat{r}_\nu-\hat{\vartheta}_\mu\hat{\vartheta}_\nu-\hat{\varphi}_\mu\hat{\varphi}_\nu.$$

The unit vectors are defined by

$$\hat{v}_\mu dx^\mu=A^{1/2}du+A^{-1/2}dr, \tag{5a}$$

$$\hat{r}_\mu dx^\mu=A^{-1/2}dr, \tag{5b}$$

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$$\hat{\vartheta}_\mu dx^\mu = r d\vartheta, \tag{5c}$$

$$\hat{\varphi}_\mu dx^\mu = r \sin\vartheta d\varphi. \tag{5d}$$

The Einstein tensor (2) can be written as a two-fluid system:

$$G_{\mu\nu} = (2\dot{m}/r^2) l_\mu l_\nu - (2m'/r^2)(\hat{v}_\mu \hat{v}_\nu - \hat{r}_\mu \hat{r}_\nu) + (m''/r)(\hat{\vartheta}_\mu \hat{\vartheta}_\nu + \hat{\varphi}_\mu \hat{\varphi}_\nu). \tag{6}$$

Spherical symmetry allows the function $m(u,r)$ to be identified as the mass within two-surfaces of constant $u$ and $r$, and invariantly defined from the sectional curvature of those surfaces:

$$-2m/r^3 = R_{\alpha\beta\mu\nu} \hat{\vartheta}^\alpha \hat{\varphi}^\beta \hat{\vartheta}^\mu \hat{\varphi}^\nu. \tag{7}$$

The string bivector is defined by

$$\Sigma^{\mu\nu} = \epsilon^{BC} \frac{\partial x^\mu}{\partial x^B} \frac{\partial x^\nu}{\partial x^C}, \quad (B,C) = (0,1) \;\; or \;\; (2,3).$$

Spherical symmetry requires that the string bivector have a world-sheet in either the $(u,r)$ or $(\vartheta,\varphi)$ plane. We require that the world-sheets be timelike, i.e. $\gamma := \frac{1}{2}\Sigma^{\mu\nu}\Sigma_{\mu\nu} < 0$, and so only the $\Sigma_{ur}$ component is non-zero. Here $\gamma = -1$. It is useful to write $\Sigma^{\mu\nu}$ in terms of unit vectors,

$$\Sigma^{\mu\nu} = \hat{r}^\mu \hat{v}^\nu - \hat{v}^\mu \hat{r}^\nu, \tag{8}$$

and so $\Sigma^{\mu\alpha}\Sigma_\alpha{}^\nu = \hat{v}^\mu \hat{v}^\nu - \hat{r}^\mu \hat{r}^\nu$. We follow Letelier [9,10] and write a string energy-momentum tensor by analogy with one for a perfect fluid. The string energy-momentum is given by

$$T^{string}_{\mu\nu} = \rho(-\gamma)^{1/2} \hat{\Sigma}_\mu{}^\alpha \hat{\Sigma}_{\alpha\nu} - p_\perp H_{\mu\nu},$$

where $H^\mu_\nu = \delta^\mu_\nu - \hat{\Sigma}^{\mu\alpha}\hat{\Sigma}_{\alpha\nu}$, $H^\mu_\nu \hat{\Sigma}^{\nu\beta} = 0$. We have written $\hat{\Sigma}^{\mu\nu} := (-\gamma)^{-1/2}\Sigma^{\mu\nu}$, so that $\hat{\Sigma}^{\mu\nu}$ is invariant under reparametrizations of the world-sheets [9]. Einstein's field equations, $G_{\mu\nu} = -8\pi T_{\mu\nu}$, allow the matter portion of Eq. (6) to be identified as a string fluid:

$$T_{\mu\nu} = \psi l_\mu l_\nu + \rho \hat{v}_\mu \hat{v}_\nu + p_r \hat{r}_\mu \hat{r}_\nu + p_\perp(\hat{\vartheta}_\mu \hat{\vartheta}_\nu + \hat{\varphi}_\mu \hat{\varphi}_\nu). \tag{9}$$

Thus

$$4\pi\psi = -\dot{m}/r^2, \tag{10a}$$

$$4\pi\rho = -4\pi p_r = m'/r^2, \tag{10b}$$

$$8\pi p_\perp = -m''/r. \tag{10c}$$

The equation of motion $T^{\mu\nu}{}_{;\nu} = 0$ is identically satisfied for the components of $T_{\mu\nu}$ given in Eq. (10).

As an example of mass transport we assume the strings diffuse and that string diffusion is like point particle diffusion where the number density diffuses from higher numbers to lower according to

$$\partial_u n = \mathcal{D}\nabla^2 n. \tag{11}$$

$\nabla^2 = r^{-2}(\partial/\partial r) r^2 (\partial/\partial r)$, and $\mathcal{D}$ is the positive coefficient of self-diffusion. Classical transport theory derives the diffusion equation by starting with Fick's law

$$\vec{J}_{(n)} = -\mathcal{D}\vec{\nabla} n \tag{12}$$

where $\vec{\nabla}$ is a purely spatial gradient. Then 4-current conservation $J^\mu_{(n);\mu} = 0$, where

$$J^\mu_{(n)}\partial_\mu = (n, \vec{J}_{(n)}) = n\partial_u - \mathcal{D}(\partial n/\partial r)\partial_r, \tag{13}$$

yields the diffusion equation (11). We label the 4-current $J_{(n)}$ to indicate $n$ diffusion but we could have also written $J_{(\rho)}$ since the string number density and string fluid density must be related by $\rho = M_s n$ where $M_s$ is the constant mass of the string species. $M_s$ must be a multiple of the Planck mass since it is only over Planck length scales that point particles resolve into strings.

By rewriting the $T_{\mu\nu}$ components (10a) and (10b) as $\dot{m} = -4\pi r^2 \psi$ and $m' = 4\pi r^2 \rho$, we can write the integrability condition for $m$ as

$$\dot{\rho} + r^{-2}\partial_r(r^2\psi) = 0. \tag{14}$$

If we compare the diffusion equation (11) ($n$ replaced by $\rho$)

$$\dot{\rho} = \mathcal{D} r^{-2} \partial/\partial r (r^2 \partial\rho/\partial r) \tag{15}$$

with $\dot{\rho}$ in Eq. (14) we obtain

$$\dot{m} = 4\pi\mathcal{D} r^2 \partial\rho/\partial r. \tag{16}$$

Thus solving the diffusion equation for $\rho$ and then integrating those solutions to obtain $m$ provides exact Einstein solutions for diffusing string fluids.

There are many analytic solutions of Eq. (15) and three of them are

$$\rho = \rho_0 + k_1/r, \tag{17a}$$

$$\rho = \rho_0 + k_3 u^{-3/2} \exp[-r^2/(4\mathcal{D}u)], \tag{17b}$$

$$\rho = \rho_0 + (k_4/r)\{1 + (\sqrt{\pi}/2)\mathrm{erf}[r(4\mathcal{D}u)^{-1/2}]\}. \tag{17c}$$

Upon integrating $m' = 4\pi r^2 \rho$ and $\dot{m} = 4\pi\mathcal{D} r^2 \partial\rho/\partial r$ we obtain the following masses, listed consecutively, for the densities above:

$$m(u,r) = m_0 + (4\pi/3) r^3 \rho_0 + 2\pi k_1 (r^2 - 2\mathcal{D}u), \tag{18a}$$

$$m(u,r) = m_0 + (4\pi/3) r^3 \rho_0 + 16\pi k_3 \mathcal{D}^{3/2}[-\eta\exp(-\eta^2) + (\sqrt{\pi}/2)\mathrm{erf}(\eta)], \tag{18b}$$

$$m(u,r)=m_0+(4\pi/3)r^3\rho_0+2\pi r^2k_4\left\{\left(1-\frac{1}{2\eta^2}\right)+\left[(\sqrt{\pi}/2)\left(1-\frac{1}{2\eta^2}\right)\mathrm{erf}(\eta)+(2\eta)^{-1}\exp(-\eta^2)\right]\right\}, \tag{18c}$$

where $\eta:=r\,(4\mathcal{D}u)^{-1/2}$.

It is clear that metric (1) can be written in Kerr-Schild form as $\eta_{\mu\nu}-[2m(u,r)/r]l_\mu l_\nu$. When $m(u,r)=m_0$ for the Schwarzschild solution, $(2m_0/r)l_\mu l_\nu$ solves the vacuum field equations linearized about flat space. This was Xanthopoulos' original superposition. His generalization [8] has $\hat{g}_{\mu\nu}=g_{\mu\nu}+H(x^\mu)l_\mu l_\nu$ with $H(x^\mu)l_\mu l_\nu$ required to solve the vacuum field equations linearized about $g_{\mu\nu}$. Referring to the mass solutions above, we can write $1-2m(u,r)/r=1-2m_0/r-2\tilde{m}/r$. Metric (1) then has the form $\eta_{\mu\nu}-(2m_0/r)l_\mu l_\nu-(2\tilde{m}/r)l_\mu l_\nu$ which is clearly

$$\hat{g}_{\mu\nu}=g^{Sch}_{\mu\nu}-(2\tilde{m}/r)l_\mu l_\nu. \tag{19}$$

For the Vaidya metric, with $\tilde{m}(u)$, the field equations linearized about $g^{Sch}_{\mu\nu}$ yield $G^{(1)}_{\mu\nu}=(d\tilde{m}/du)r^{-2}l_\mu l_\nu$ [11] which is not a vacuum solution and so $G^{(1)}_{\mu\nu}$ computed about Schwarzschild with $\tilde{m}(u,r)$ is *a fortiori* not zero. Since $\hat{g}_{\mu\nu}$ is an exact solution for a string fluid and $g^{Sch}_{\mu\nu}$ is an exact vacuum solution, we have extended Xanthopolous' generalization.

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