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Nonperturbative evolution equation for quantum gravity

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A scale-dependent effective action for gravity is introduced and an exact nonperturbative evolution equation is derived which governs its renormalization group flow. It is invariant under general coordinate transformations and satisfies modified Becchi-Rouet-Stora Ward identities. The evolution equation is solved for a simple truncation of the space of actions. In $2+\varepsilon$ dimensions, nonperturbative corrections to the β function of Newton's constant are derived and its dependence on the cosmological constant is investigated. In 4 dimensions, Einstein gravity is found to be ''antiscreening;'' i.e., Newton's constant increases at large distances. [S0556-2821(98)02302-9]

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I. INTRODUCTION

In many of the traditional approaches to quantum gravity the Einstein-Hilbert term has been regarded as a fundamental action which should be quantized along the same lines as the familiar renormalizable field theories in flat space, such as QED, for example [1]. It was soon realized that this program is not only technically rather involved but also leads to severe conceptual difficulties. In particular, the nonrenormalizability of the theory hampers a meaningful perturbative analysis. While this does not rule out the possibility that the theory exists nonperturbatively, not much is known in this direction. However, it could also be argued that gravity, as we know it, should not be quantized at all, because Einstein gravity is an effective theory [2] which results from quantizing some yet unknown fundamental theory. If so, the Einstein-Hilbert term is an effective action analogous to the Heisenberg-Euler action in QED and it should not be compared to the ''microscopic'' action of electrodynamics.

It seems not unreasonable to assume that the truth lies somewhere between these two extreme points of view, i.e., that Einstein gravity is an effective theory which is valid near a certain nonzero momentum scale k. This means that it arises from the fundamental theory by a ''partial quantization'' in which only excitations with momenta larger than k are integrated out, while those with momenta smaller than k are not included. (The interpretation of the Einstein-Hilbert term as a fundamental or an ordinary effective action is recovered in the limits $k\to\infty$ and $k\to 0$, respectively.) An ''effective theory at scale k,'' when evaluated at the tree level, should correctly describe all gravitational phenomena which involve a typical momentum scale k acting as a physical infrared cutoff. Only if one is interested in processes with momenta $k'\ll k$, do loop calculations become necessary; they amount to integrating out the missing field modes in the momentum interval $[k',k]$.

We shall regard the scale-dependent action for gravity, henceforth denoted Γ_k, as a Wilsonian effective action which is obtained from the fundamental (''microscopic'') action S by a kind of coarse-graining analogous to the iterated block-spin transformations which are familiar from lattice systems [3]. In the continuum, Γ_k will be defined in terms of a modified functional integral over e^{-S} in which the contributions of all field modes with momenta smaller than k are suppressed. In this manner Γ_k interpolates between S (for $k\to\infty$) and the effective action Γ (for $k\to 0$). The trajectory in the space of all action functionals can be obtained as the solution of a certain functional evolution equation, the exact renormalization group equation. Its form is independent of the action S under consideration. The latter enters via the initial conditions for the renormalization group trajectory; it is specified at some UV cutoff scale Λ: $\Gamma_\Lambda = S$. If S is a truly fundamental action, Λ is sent to infinity at the end.

The renormalization group equation can also be used to evolve effective actions, known at some point Λ, towards smaller scales $k<\Lambda$. In this case Λ is a fixed, finite scale.

In this framework, the (non)renormalizability of a theory is seen as a global property of the renormalization group flow for $\Lambda\to\infty$. The evolution equation by itself is perfectly finite and well behaved in either case, because it describes only infinitesimal changes of the cutoff.

In this paper we shall give a precise meaning to the notion of a scale-dependent gravitational action $\Gamma_k[g_{\mu\nu}]$ and we shall derive the associated evolution equation. We employ a formulation in which the metric is the fundamental dynamical variable. Alternative approaches based upon the spin connection and the vielbeins are also possible, but they will not be considered here. By using a variant of the background gauge technique we are able to make $\Gamma_k[g_{\mu\nu}]$ invariant under general coordinate transformations. This property is very important if one wants to find nonperturbative solutions of the evolution equations in terms of simple truncations of the space of actions. Our construction of $\Gamma_k[g_{\mu\nu}]$ parallels the definition of the ''effective average action'' [4,5] which was widely used recently [6–9].[1] The remarkable successes of this method in flat space are partly due to the fact that it allows for nonperturbative solutions when no small expansion parameter is available, and that Γ_k has a built-in infrared cutoff. Therefore the low-momentum behavior of (almost) massless theories can be investigated even in cases where IR divergences render standard perturbation theory inapplicable.

For the purposes of quantum gravity, both of these fea-

[1]For related work using similar techniques see Refs. [10, 11, 12].

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tures are very welcome, of course. In fact, in quantum cosmology one of the most intriguing questions is how quantized Einstein gravity behaves at extremely large distances. It has been argued [13,14] that in the presence of a nonzero cosmological constant there should be very strong renormalization effects in the infrared which might even provide a mechanism for a dynamical relaxation of the cosmological constant. The method which we are going to develop would be ideally suited to study problems of this type. Since only long distance physics is involved here, there are good chances that this can be done without knowing the microscopic theory of quantum gravity. (See Ref. [2] for a related discussion.)

The ''effective average action'' used in this paper should not be confused with the closely related ''average action'' which was introduced earlier [15]. The former obeys a more convenient evolution equation while the latter has a simple interpretation in terms of field averages. Their precise relation is explained in Ref. [16]. The average action has been used in a gravitational context in Refs. [17, 18], but no exact evolution equation was formulated.[2] The evolution of the effective average action in a gravitational background was studied in Ref. [22] in the context of Liouville field theory. For a review of the effective average action and its applicaton to Yang-Mills theory we refer to [23].

The remaining sections of this paper are organized as follows. In Sec. II we give the definition of Γ_k and derive the exact, nonperturbative renormalization group equation. In Sec. III we establish the modified Ward identities satisfied by Γ_k, and we show that the conventional diffeomorphism Ward identities are recovered in the limit $k\rightarrow 0$. In its general form, the evolution equation describes a flow on the infinite dimensional space of all action functionals. Approximate nonperturbative solutions can be found by truncating the space of actions, i.e., by projecting the flow on a finite-dimensional subspace. In Sec. IV we investigate the ''Einstein-Hilbert truncation'' where only the operators $\int\sqrt{g}$ and $\int\sqrt{g}R$ are retained. In Sec. V we determine the resulting scale dependence of Newton's constant and of the cosmological constant. As an example, gravity in $2+\varepsilon$ and in 4 dimensions is discussed in detail.

II. THE RENORMALIZATION GROUP EQUATION

In this section we introduce the effective average action for Euclidean quantum gravity in d dimensions and we derive the exact renormalization group equation which governs its scale dependence.

We are going to employ the background gauge fixing technique [24,25], which means that we decompose the integration variable $\gamma_{\mu\nu}(x)$ in the functional integral over all metrics according to

$$\gamma_{\mu\nu}(x)=\bar{g}_{\mu\nu}(x)+h_{\mu\nu}(x). \tag{2.1}$$

Here $\bar{g}_{\mu\nu}$ is a fixed background metric so that the integration over $\gamma_{\mu\nu}$ may be replaced by an integration over $h_{\mu\nu}$. We consider the following scale-dependent modification of the generating functional for the connected Green's functions

$$\begin{aligned}\exp\{W_k[t^{\mu\nu},\sigma^{\mu},\bar{\sigma}_{\mu};\beta^{\mu\nu},\tau_{\mu};\bar{g}_{\mu\nu}]\} &= \int \mathcal{D}h_{\mu\nu}\mathcal{D}C^{\mu}\mathcal{D}\bar{C}_{\mu}\,\exp\{-S[\bar{g}+h]-S_{\mathrm{gf}}[h;\bar{g}]\\ &\quad -S_{\mathrm{gh}}[h,C\bar{C};\bar{g}]-\Delta_k S[h,C,\bar{C};\bar{g}]-S_{\mathrm{source}}\}.\end{aligned} \tag{2.2}$$

Here $S[\gamma]=S[\bar{g}+h]$ is the classical action which is assumed to be invariant under the general coordinate transformations

$$\delta\gamma_{\mu\nu}=\mathcal{L}_v\gamma_{\mu\nu}\equiv v^{\rho}\partial_{\rho}\gamma_{\mu\nu}+\partial_{\mu}v^{\rho}\gamma_{\rho\nu}+\partial_{\nu}v^{\rho}\gamma_{\mu\rho}, \tag{2.3}$$

where $\mathcal{L}_v$ denotes the Lie derivative with respect to the vector field v^{μ}. For the time being let us also assume that S is positive definite.

Furthermore, S_{gf} denotes the gauge-fixing term for the gauge condition $F_{\mu}(\bar{g},h)=0$,

$$S_{\mathrm{gf}}[h;\bar{g}]=\frac{1}{2\alpha}\int d^dx\,\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}F_{\mu}F_{\nu} \tag{2.4}$$

and S_{gh} is the action for the corresponding Faddeev-Popov ghosts C^{μ} and $\bar{C}_{\mu}$:

$$\begin{aligned}S_{\mathrm{gh}}[h,C,\bar{C};\bar{g}]&=-\kappa^{-1}\int d^dx\,\bar{C}_{\mu}\bar{g}^{\mu\nu}\,\frac{\partial F_{\nu}}{\partial h_{\alpha\beta}}\\ &\quad\times\mathcal{L}_C(\bar{g}_{\alpha\beta}+h_{\alpha\beta}).\end{aligned} \tag{2.5}$$

The Faddeev-Popov action S_{gh} is obtained along the same lines as in Yang-Mills theory: one applies a gauge transformation

$$\begin{aligned}\delta h_{\mu\nu}&=\mathcal{L}_v\gamma_{\mu\nu}=\mathcal{L}_v(\bar{g}_{\mu\nu}+h_{\mu\nu})\\ \delta\bar{g}_{\mu\nu}&=0\end{aligned} \tag{2.6}$$

to F_{μ} and replaces the parameters v^{μ} by the ghost field C^{μ}. The integral over C^{μ} and $\bar{C}_{\mu}$ provides a representation of the Faddeev-Popov determinant $\det[\delta F_{\mu}/\delta v^{\nu}]$ then. In Eq. (2.5) we introduced the constant (proportional to the Planck mass)

$$\kappa\equiv(32\pi\bar{G})^{-1/2}, \tag{2.7}$$

where $\bar{G}$ denotes the bare Newtonian constant. In principle our construction works for an arbitrary background gauge fixing. It is particularly convenient to use a F_{μ} which is linear in the quantum field $h_{\mu\nu}$:

$$F_{\mu}=\sqrt{2}\kappa\mathcal{F}_{\mu}^{\alpha\beta}[\bar{g}]h_{\alpha\beta}. \tag{2.8}$$

We shall mostly employ the harmonic coordinate condition for which $\mathcal{F}_{\mu}^{\alpha\beta}$ is the following first order differential operator constructed from $\bar{g}_{\mu\nu}$:

$$\mathcal{F}_{\mu}^{\alpha\beta}=\delta_{\mu}^{\beta}\bar{g}^{\alpha\gamma}\bar{D}_{\gamma}-\frac{1}{2}\,\bar{g}^{\alpha\beta}\bar{D}_{\mu}. \tag{2.9}$$

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[2] For recent work on related renormalization group flows see also Refs. [19, 20, 21].

The covariant derivative $\bar{D}_\mu$ involves the Christoffel symbols $\bar{\Gamma}^\rho_{\mu\nu}$ of the background metric $\bar{g}_{\mu\nu}$. For the gauge fixing (2.8) with (2.9) the ghost action reads

$$S_{\rm gh}[h,C,\bar{C};\bar{g}]=-\sqrt{2}\int d^dx\,\sqrt{\bar{g}}\,\bar{C}_\mu\mathcal{M}[g,\bar{g}]^\mu{}_\nu C^\nu, \tag{2.10}$$

with the Faddeev-Popov operator

$$\mathcal{M}[g,\bar{g}]^\mu{}_\nu=\bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda(g_{\rho\nu}D_\sigma+g_{\sigma\nu}D_\rho)-\bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda g_{\sigma\nu}D_\rho. \tag{2.11}$$

The essential piece in Eq. (2.2) is the IR cutoff for the gravitational field $h_{\mu\nu}$ and for the ghosts:

$$\Delta_kS[h,C,\bar{C};\bar{g}]=\frac{1}{2}\kappa^2\int d^dx\,\sqrt{\bar{g}}\,h_{\mu\nu}R_k^{\rm grav}[\bar{g}]^{\mu\nu\rho\sigma}h_{\rho\sigma}+\sqrt{2}\int d^dx\,\sqrt{\bar{g}}\,\bar{C}_\mu R_k^{\rm gh}[\bar{g}]C^\mu. \tag{2.12}$$

The cutoff operators $R_k^{\rm grav}$ and $R_k^{\rm gh}$ serve the purpose of discriminating between high-momentum and low-momentum modes. Eigenmodes of $-\bar{D}^2$ with eigenvalues $p^2\gg k^2$ are integrated out in Eq. (2.2) without any suppression whereas modes with small eigenvalues $p^2\ll k^2$ are suppressed by a kind of momentum-dependent mass term. The operators $R_k^{\rm grav}$ and $R_k^{\rm gh}$ describe the transition from the high-momentum to the low-momentum regime. Either of them has the structure

$$R_k[\bar{g}]=\mathcal{Z}_kk^2R^{(0)}(-\bar{D}^2/k^2), \tag{2.13}$$

where the dimensionless function $R^{(0)}$ interpolates smoothly between $R^{(0)}(0)=1$ and $\lim_{u\to\infty}R^{(0)}(u)=0$. A convenient choice is, for example,

$$R^{(0)}(u)=u[\exp(u)-1]^{-1}. \tag{2.14}$$

The factors $\mathcal{Z}_k$ are different for the graviton and the ghost cutoff. For the ghost $\mathcal{Z}_k\equiv Z_k^{\rm gh}$ is a pure number, whereas for the metric fluctuation $\mathcal{Z}_k\equiv\mathcal{Z}_k^{\rm grav}$ is a tensor constructed from the background metric $\bar{g}_{\mu\nu}$. In the simplest case one would take

$$(\mathcal{Z}_k^{\rm grav})^{\mu\nu\rho\sigma}=\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}Z_k^{\rm grav}. \tag{2.15}$$

In Sec. IV we shall employ a slightly more refined choice. There we shall also explain how the factors $Z_k^{\rm gh}$ and $Z_k^{\rm grav}$ should be chosen. Note that the cutoff action (2.12) is quadratic in the quantum fields $h_{\mu\nu}$, C^μ and $\bar{C}_\mu$. This is an important prerequisite for obtaining a tractable evolution equation later on. The requirement of a quadratic Δ_kS forces us to use the covariant Laplacian $\bar{D}^2\equiv\bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$ in the *background* metric as the operator which discriminates between high-momentum and low-momentum modes.

In Eq. (2.2) we coupled $h_{\mu\nu}$, C^μ and $\bar{C}_\mu$ to the sources $t^{\mu\nu}$, $\bar{\sigma}_\mu$ and σ^μ, respectively:

$$S_{\rm source}=-\int d^dx\,\sqrt{\bar{g}}\{t^{\mu\nu}h_{\mu\nu}+\bar{\sigma}_\mu C^\mu+\sigma^\mu\bar{C}_\mu+\beta^{\mu\nu}\mathcal{L}_C(\bar{g}_{\mu\nu}+h_{\mu\nu})+\tau_\mu C^\nu\partial_\nu C^\mu\}. \tag{2.16}$$

The sources $\beta^{\mu\nu}$ and τ_μ couple to the Becchi-Rouet-Stora (BRS) variations of $h_{\mu\nu}$ and C^μ, respectively. In fact, it is not difficult verify that $S+S_{\rm gf}+S_{\rm gh}$ is invariant under the BRS transformations (ε is an anticommuting parameter)

$$\delta_\varepsilon h_{\mu\nu}=\varepsilon\kappa^{-2}\mathcal{L}_C\gamma_{\mu\nu}=\varepsilon\kappa^{-2}\mathcal{L}_C(\bar{g}_{\mu\nu}+h_{\mu\nu}) \tag{2.17}$$

$$\delta_\varepsilon\bar{g}_{\mu\nu}=0$$

$$\delta_\varepsilon C^\mu=\varepsilon\kappa^{-2}C^\nu\partial_\nu C^\mu$$

$$\delta_\varepsilon\bar{C}_\mu=\varepsilon(\alpha\kappa)^{-1}F_\mu.$$

Given the functional W_k, we introduce k-dependent classical fields

$$\bar{h}_{\mu\nu}=\frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta t^{\mu\nu}},\quad \xi^\mu=\frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta\bar{\sigma}_\mu},\quad \bar{\xi}_\mu=\frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta\sigma^\mu}, \tag{2.18}$$

and we formally solve for the sources $(t^{\mu\nu},\sigma^\mu,\bar{\sigma}_\mu)$ as functionals of the fields $(\bar{h}_{\mu\nu},\xi^\mu,\bar{\xi}_\mu)$ and of $(\beta^{\mu\nu},\tau_\mu;\bar{g}_{\mu\nu})$. Then the Legendre transform $\tilde{\Gamma}_k$ of W_k depends on the classical fields and parametrically on β, τ and $\bar{g}$:

$$\tilde{\Gamma}_k[\bar{h},\xi,\bar{\xi};\beta,\tau;\bar{g}]=\int d^dx\,\sqrt{\bar{g}}\{t^{\mu\nu}\bar{h}_{\mu\nu}+\bar{\sigma}_\mu\xi^\mu+\sigma^\mu\bar{\xi}_\mu\}-W_k[t,\sigma,\bar{\sigma};\beta,\tau;\bar{g}]. \tag{2.19}$$

By definition, the effective average action Γ_k obtains from $\tilde{\Gamma}_k$ by subtracting the cutoff action Δ_kS with the classical fields inserted:

$$\Gamma_k[\bar{h},\xi,\bar{\xi};\beta,\tau;\bar{g}]=\tilde{\Gamma}_k[\bar{h},\xi,\bar{\xi};\beta,\tau;\bar{g}]-\Delta_kS[\bar{h},\xi,\bar{\xi};\bar{g}], \tag{2.20}$$

It is convenient to define the metric

$$g_{\mu\nu}(x)\equiv\bar{g}_{\mu\nu}(x)+\bar{h}_{\mu\nu}(x) \tag{2.21}$$

as the classical analogue of the quantum metric $\gamma_{\mu\nu}\equiv\bar{g}_{\mu\nu}+h_{\mu\nu}$ and to consider Γ_k as a functional of $g_{\mu\nu}$ rather than $\bar{h}_{\mu\nu}$:

$$\Gamma_k[g_{\mu\nu},\bar{g}_{\mu\nu},\xi^\mu,\bar{\xi}_\mu;\beta,\tau]\equiv\Gamma_k[g_{\mu\nu}-\bar{g}_{\mu\nu},\xi^\mu,\bar{\xi}_\mu;\beta,\tau;\bar{g}_{\mu\nu}]. \tag{2.22}$$

The main virtue of the background technique employed here is that the functional Γ_k is invariant under general coordinate transformations where all its arguments transform as tensors of the corresponding rank:

$$\Gamma_k[\Phi+\mathcal{L}_v\Phi]=\Gamma_k[\Phi],\quad \Phi\equiv\{g_{\mu\nu},\bar{g}_{\mu\nu},\xi^\mu,\bar{\xi}_\mu;\beta^{\mu\nu},\tau_\mu\}. \tag{2.23}$$

Note that in Eq. (2.23), contrary to the ‘‘gauge transformation’’ (2.6), the background metric also transforms as an ordinary tensor field: $\delta\bar{g}_{\mu\nu}=\mathcal{L}_v\bar{g}_{\mu\nu}$. Equation (2.23) is a consequence of

$$W_k[\mathcal{J}+\mathcal{L}_v\mathcal{J}]=W_k[\mathcal{J}],\quad \mathcal{J}\equiv\{t^{\mu\nu},\sigma^\mu,\bar{\sigma}_\mu;\beta^{\mu\nu},\tau_\mu;\bar{g}_{\mu\nu}\}. \tag{2.24}$$

This invariance property follows from Eq. (2.2) if one performs a compensating transformation on the integration variables $h_{\mu\nu}$, C^μ and $\bar{C}_\mu$. At this point we assume that the measure is diffeomorphism invariant.

The general coordinate invariance of Γ_k is of major practical importance because if we know *a priori* that no symmetry-violating terms are generated during the evolution it is sufficient to use truncations which consist of invariant combinations of the fields only. The conventionally defined effective action of the metric, $\Gamma[g_{\mu\nu}]$, obtains in the limit of a vanishing IR cutoff by setting the ghosts β and τ to zero and by identifying $\bar{g}_{\mu\nu}$ with $g_{\mu\nu}$:

$$\Gamma[g_{\mu\nu}]=\lim_{k\to 0}\Gamma_k[g_{\mu\nu},g_{\mu\nu},0,0;0,0]. \tag{2.25}$$

As a consequence, $\Gamma[g_{\mu\nu}]$ is invariant under $\delta g_{\mu\nu}=\mathcal{L}_v g_{\mu\nu}$. Even though we are mostly interested in the functional

$$\bar{\Gamma}_k[g_{\mu\nu}]\equiv\Gamma_k[g_{\mu\nu},g_{\mu\nu},0,0;0,0] \tag{2.26}$$

which depends on $g_{\mu\nu}$ only, an exact renormalization group equation can be formulated only if one keeps track of the dependence on ξ, $\bar{\xi}$ and $\bar{g}$ as well. For the derivation of the (modified) BRS Ward identities satisfied by Γ_k the dependence on β and τ must be retained in addition.

The derivation of the evolution equation for Γ_k proceeds as follows. Taking a derivative of the functional integral (2.2) with respect to the renormalization group ‘‘time’’ $t\equiv\ln k$ one obtains, in matrix notation,

$$-\partial_t W_k=\frac{1}{2}\,\mathrm{Tr}[\langle h\otimes h\rangle(\partial_t\hat{R}_k)_{\bar{h}\bar{h}}]-\mathrm{Tr}[\langle\bar{C}\otimes C\rangle(\partial_t\hat{R}_k)_{\bar{\xi}\xi}]. \tag{2.27}$$

Here $\hat{R}_k$ is a matrix in field space whose non-zero entries are

$$(\hat{R}_k)^{\mu\nu\rho\sigma}_{\bar{h}\bar{h}}=\kappa^2(R_k^{\rm grav}[\bar{g}])^{\mu\nu\rho\sigma}$$
$$(\hat{R}_k)_{\bar{\xi}\xi}=\sqrt{2}R_k^{\rm gh}[\bar{g}]. \tag{2.28}$$

The right-hand side (RHS) of Eq. (2.27) can be expressed in terms of Γ_k by noting that the connected two-point function

$$G_{ij}(x,y)\equiv\langle\chi_i(x)\chi_j(y)\rangle-\varphi_i(x)\varphi_j(y)$$
$$=\frac{1}{\sqrt{\bar{g}(x)\bar{g}(y)}}\frac{\delta^2 W_k}{\delta J^i(x)\delta J^j(y)} \tag{2.29}$$

and

$$\tilde{\Gamma}_k^{(2)ij}(x,y)\equiv\frac{1}{\sqrt{\bar{g}(x)\bar{g}(y)}}\frac{\delta^2\tilde{\Gamma}_k}{\delta\varphi_i(x)\delta\varphi_j(y)} \tag{2.30}$$

are inverse matrices in the sense that

$$\int d^dy\sqrt{\bar{g}(y)}G_{ij}(x,y)\tilde{\Gamma}_k^{(2)jl}(y,z)=\delta_i^l\,\frac{\delta(x-z)}{\sqrt{\bar{g}(z)}}. \tag{2.31}$$

Here we used the shorthand notation $\chi_i\equiv\{h,C,\bar{C}\}$, $J^i\equiv\{t,\sigma,\bar{\sigma}\}$ and $\varphi_i\equiv\{\bar{h},\xi,\bar{\xi}\}$. Thus one obtains the evolution equation

$$\partial_t\Gamma_k[\bar{h},\xi,\bar{\xi};\beta,\tau;\bar{g}]=\frac{1}{2}\,\mathrm{Tr}[(\Gamma_k^{(2)}+\hat{R}_k)^{-1}_{\bar{h}\bar{h}}(\partial_t\hat{R}_k)_{\bar{h}\bar{h}}]-\frac{1}{2}\,\mathrm{Tr}[\{(\Gamma_k^{(2)}+\hat{R}_k)^{-1}_{\bar{\xi}\xi}-(\Gamma_k^{(2)}+\hat{R}_k)^{-1}_{\xi\bar{\xi}}\}(\partial_t\hat{R}_k)_{\bar{\xi}\xi}]. \tag{2.32}$$

If one evaluates the RHS of this equation in terms of position-space matrix elements then $\Gamma_k^{(2)}$ is defined by a formula similar to Eq. (2.30) and the integration implied by ‘‘Tr’’ has to be interpreted as $\int d^dx\sqrt{\bar{g}(x)}$. The matrix elements in the ghost sector are defined in terms of left derivatives, e.g.,

$$((\Gamma_k^{(2)})_{\bar{\xi}\xi})_{\mu x}{}^{\nu y}=\frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta}{\delta\xi^\mu(x)}\frac{1}{\sqrt{\bar{g}(y)}}\frac{\delta\Gamma_k}{\delta\bar{\xi}_\nu(y)}. \tag{2.33}$$

For any cutoff which is qualitatively similar to Eq. (2.14) the traces on the RHS of Eq. (2.32) are well convergent, both in the IR and the UV. By virtue of the factor $\partial_t\hat{R}_k$, the dominant contributions come from a narrow band of generalized momenta centered around k. Large momenta are exponentially suppressed.

Solving the evolution equation (2.32) with the appropriate initial condition at the UV cutoff scale $\Lambda\to\infty$ is tantamount to computing the original functional integral (2.2). In order to determine the correct initial value Γ_Λ we consider the following integral equation satisfied by Γ_k:

$$\exp\{-\Gamma_k[\bar{h},\xi,\bar{\xi};\beta,\tau;\bar{g}]\}=\int \mathcal{D}h\mathcal{D}C\mathcal{D}\bar{C}\exp\left[-\tilde{S}[h,C,\bar{C};\beta,\tau;\bar{g}]+\int d^dx\left\{(h_{\mu\nu}-\bar{h}_{\mu\nu})\frac{\delta\Gamma_k}{\delta\bar{h}_{\mu\nu}}+(C^\mu-\xi^\mu)\frac{\delta\Gamma_k}{\delta\xi^\mu}+(\bar{C}_\mu-\bar{\xi}_\mu)\frac{\delta\Gamma_k}{\delta\bar{\xi}_\mu}\right\}\right]\exp\{-\Delta_k S[h-\bar{h},C-\xi,\bar{C}-\bar{\xi};\bar{g}]\}. \tag{2.34}$$

Here

$$\tilde{S}\equiv S+S_{\rm gf}+S_{\rm gh}-\int d^dx\sqrt{\bar{g}}\{\beta^{\mu\nu}\mathcal{L}_C(\bar{g}_{\mu\nu}+h_{\mu\nu})+\tau_\mu C^\nu\partial_\nu C^\mu\} \tag{2.35}$$

is expressed in terms of the ''microscopic'' fields $(h,C,\bar{C})$. Equation (2.34) was obtained by inserting the definition of Γ_k into Eq. (2.2) and using

$$\frac{\delta\tilde{\Gamma}_k}{\delta\bar{h}_{\mu\nu}}=\sqrt{\bar{g}}\,t^{\mu\nu},\quad \frac{\delta\tilde{\Gamma}_k}{\delta\bar{\xi}_\mu}=-\sqrt{\bar{g}}\,\sigma^\mu,\quad \frac{\delta\tilde{\Gamma}_k}{\delta\xi^\mu}=-\sqrt{\bar{g}}\,\bar{\sigma}_\mu. \tag{2.36}$$

The crucial observation is that for $k\to\infty$ the last exponential in Eq. (2.34) becomes proportional to a δ-functional which equates the quantum fields $(h,C,\bar{C})$ to their classical counterparts:

$$e^{-\Delta_k S}\underset{k\to\infty}{\sim}\delta[h-\bar{h}]\delta[C-\xi]\delta[\bar{C}-\bar{\xi}]. \tag{2.37}$$

As a consequence, the effective average action at the UV cutoff reads[3]

$$\Gamma_\Lambda[\bar{h},\xi,\bar{\xi};\beta,\tau;\bar{g}]=S[\bar{g}+\bar{h}]+S_{\rm gf}[\bar{h};\bar{g}]+S_{\rm gh}[\bar{h},\xi,\bar{\xi};\bar{g}]-\int d^dx\sqrt{\bar{g}}\{\beta^{\mu\nu}\mathcal{L}_\xi(\bar{g}_{\mu\nu}+\bar{h}_{\mu\nu})+\tau_\mu\xi^\nu\partial_\nu\xi^\mu\}. \tag{2.38}$$

It is this action Γ_Λ which has to be used as the initial condition for the evolution equation. We note that at the level of the functional $\bar{\Gamma}_k[g]$ Eq. (2.38) boils down to

$$\bar{\Gamma}_\Lambda[g_{\mu\nu}]=S[g_{\mu\nu}]. \tag{2.39}$$

As $\Gamma_k^{(2)}$ involves derivatives with respect to $g_{\mu\nu}$ at fixed $\bar{g}_{\mu\nu}$ it is clear that the evolution equation cannot be formulated in terms of $\bar{\Gamma}_k$ alone, however.

Up to now we assumed that the fundamental action S is positive definite and the Euclidean functional integral (2.2) makes sense as it stands. It is well known that this is not the case for the Einstein-Hilbert action, for example, because the conformal factor has a ''wrong sign'' kinetic term. Clearly it would be desirable to have an evolution equation which can be applied in such cases as well. It is quite remarkable therefore that the renormalization group equation (2.32), with a properly chosen cutoff, is well defined even if S and Γ_k are not positive definite. To see this, let us look at the first trace on the RHS of Eq. (2.32) and let us concentrate on the contribution of a fixed mode ϕ contained in the metric. We assume that ϕ is an eigenfunction of $\Gamma_k^{(2)}$ with eigenvalue $z_k p^2$ where p^2 is a positive eigenvalue of some covariant kinetic operator, typically of the form $-\bar{D}^2+R$ *terms*. For theories with $S>0$, the wave function renormalization z_k is positive (at least for large k). In this case the general rule [5,6] is to define the constant $\mathcal{Z}_k$ in the cutoff R_k, Eq. (2.13), as $\mathcal{Z}_k=z_k$ because this guarantees that for the low-momentum modes the effective inverse propagator $\Gamma^{(2)}+R_k$ becomes $z_k(p^2+k^2)$, as it should be.

The important question is how $\mathcal{Z}_k$ should be chosen if z_k is negative. If we continue to use $\mathcal{Z}_k=z_k$, the evolution equation is perfectly well defined because the inverse propagator $-|z_k|(p^2+k^2)$ never vanishes, and the traces of Eq. (2.32) are not suffering from any IR problems. In fact, if we write down the perturbative expansion for the functional trace, for instance, it is clear that all propagators are correctly cut off in the IR, and that loop momenta smaller than k are suppressed.

On the other hand, if we set $\mathcal{Z}_k=-z_k$, then $-|z_k|(p^2-k^2)$ introduces a spurious singularity at $p^2=k^2$, and the cutoff fails to make the theory IR finite in this case.

At first sight the choice $\mathcal{Z}_k=-z_k$ might have appeared more natural because only if $\mathcal{Z}_k>0$ is the factor $\exp(-\Delta_k S)\sim\exp(-\int R_k\phi^2)$ a damped exponential which suppresses the low momentum modes in the usual way. In this paper we shall nevertheless adopt the rule $\mathcal{Z}=z_k$ for either sign of z_k. We shall see that at least for the Einstein-Hilbert truncation of Sec. IV the evolution equations are well defined and consistent even though it is difficult to give a meaning to the functional integral itself. In the case $\mathcal{Z}_k=z_k<0$ the factor $\exp(+\int|R_k|\phi^2)$ unavoidably becomes a grow-

[3]Strictly speaking Eq. (2.38) is correct only up to local terms which at most change the bare parameters in S. Because the value of the bare parameters has no physical significance anyhow, we ignore these terms here.

ing exponential and it might seem that this enhances rather than suppresses the low-momentum modes. However, as suggested by the perturbative argument above, this conclusion is probably too naive. Moreover, if one invokes the usual prescription of rotating the contour of integration over ϕ so that it is parallel to the imaginary axis, both the kinetic term and the cutoff lead to damped exponentials.

Furthermore, it is important to note that the constructions in this section can be repeated for metrics on Lorentzian spacetimes. Then one deals with oscillating exponentials e^{iS}, and for arguments like the one leading to Eq. (2.37) one has to employ the Riemann-Lebesgue lemma. Apart from the obvious substitutions $\Gamma_k \to -i\Gamma_k$, $R_k \to -iR_k$, the evolution equation remains unaltered. For $\mathcal{Z}_k = z_k$ it has all the desired features, and $z_k < 0$ seems not to pose any special problem.

III. MODIFIED WARD IDENTITIES AND CONSISTENT TRUNCATIONS

We mentioned already that the classical action plus the gauge-fixing and ghost terms are invariant under the BRS transformations (2.17). Therefore the BRS variation of the total action $S_{\mathrm{tot}} \equiv S + S_{\mathrm{gf}} + S_{\mathrm{gh}} + \Delta_k S + S_{\mathrm{sources}}$ receives contributions only from the cutoff and the source terms. If we apply a BRS transformation to the integral defining W_k and assume that the measure is invariant we obtain

$$\langle \delta_\varepsilon S_{\mathrm{sources}} + \delta_\varepsilon \Delta_k S \rangle = 0, \tag{3.1}$$

where

$$\langle \mathcal{O} \rangle \equiv e^{-W_k} \int \mathcal{D}h \mathcal{D}C \mathcal{D}\bar{C} \mathcal{O} e^{-S_{\mathrm{tot}}}. \tag{3.2}$$

Our goal is to convert Eq. (3.1) to a statement about the average action Γ_k. Because the BRS transformation (2.17) is off-shell nilpotent when acting on $h_{\mu\nu}$ and on C^μ (but not on $\bar{C}_\mu$) one has

$$\delta_\varepsilon S_{\mathrm{sources}} = -\varepsilon \kappa^{-2} \int d^d x \sqrt{\bar{g}} \{ t^{\mu\nu} \mathcal{L}_C(\bar{g}_{\mu\nu} + h_{\mu\nu}) - \bar{\sigma}_\mu C^\nu \partial_\nu C^\mu - \kappa \alpha^{-1} \sigma^\mu F_\mu(\bar{g}, h) \}. \tag{3.3}$$

If we take the expectation value of Eq. (3.3) and express W_k in terms of Γ_k we find

$$\langle \delta_\varepsilon S_{\mathrm{source}} \rangle = \frac{\epsilon}{\kappa^2} \int d^d x \, \frac{1}{\sqrt{\bar{g}(x)}} \left\{ \frac{\delta \Gamma_k'}{\delta \bar{h}_{\mu\nu}} \frac{\delta \Gamma_k'}{\delta \beta^{\mu\nu}} + \frac{\delta \Gamma_k'}{\delta \xi^\mu} \frac{\delta \Gamma_k'}{\delta \tau_\mu} \right\} + \frac{\varepsilon}{\kappa^2} \tilde{Y}_k , \tag{3.4}$$

with

$$\tilde{Y}_k \equiv \int d^d x \left\{ \frac{1}{\sqrt{\bar{g}}} \left(\frac{\delta \Delta_k S}{\delta \bar{h}_{\mu\nu}} \frac{\delta \Gamma_k'}{\delta \beta^{\mu\nu}} + \frac{\delta \Delta_k S}{\delta \xi^\mu} \frac{\delta \Gamma_k'}{\delta \tau_\mu} \right) - \sqrt{2} \, \frac{\kappa}{\alpha} \sqrt{\bar{g}} F_\mu(\bar{g}, \bar{h}) R_k^{\mathrm{gh}} \xi^\mu \right\}. \tag{3.5}$$

Here we defined

$$\Gamma_k' \equiv \Gamma_k - S_{\mathrm{gf}}[\bar{h}; \bar{g}] \tag{3.6}$$

and we exploited the equation of motion $\langle \delta S_{\mathrm{tot}} / \delta \bar{C}_\mu \rangle = 0$ which can be cast in the form

$$\left[\frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta}{\delta \bar{\xi}_\mu(x)} - \sqrt{2} \bar{g}^{\mu\nu} \mathcal{F}^{\rho\sigma}_\nu \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta}{\delta \beta^{\rho\sigma}(x)} \right] \Gamma_k[\bar{h}, \xi, \bar{\xi}; \beta, \tau; \bar{g}] = 0. \tag{3.7}$$

The variation of the cutoff terms gives rise to

$$\langle \delta_\varepsilon \Delta_k S \rangle = -\frac{\varepsilon}{\kappa^2} (Y_k + \tilde{Y}_k), \tag{3.8}$$

with

$$Y_k \equiv \kappa^2 \, \mathrm{Tr} \left[(R_k^{\mathrm{grav}})^{\mu\nu\rho\sigma} (\Gamma_k^{(2)} + \hat{R}_k)^{-1}_{\bar{h}_{\rho\sigma}\varphi} \frac{\delta^2 \Gamma_k}{\sqrt{\bar{g}} \, \delta\varphi \sqrt{\bar{g}} \, \delta \beta^{\mu\nu}} \right] - \sqrt{2} \, \mathrm{Tr} \left[R_k^{\mathrm{gh}} (\Gamma_k^{(2)} + \hat{R}_k)^{-1}_{\xi^\mu \varphi} \frac{\delta^2 \Gamma_k}{\sqrt{\bar{g}} \, \delta\varphi \sqrt{\bar{g}} \, \delta \tau_\mu} \right]$$
$$+ 2\alpha^{-1} \kappa^2 \, \mathrm{Tr}[R_k^{\mathrm{gh}} \mathcal{F}^{\rho\sigma}_\mu (\Gamma_k^{(2)} + \hat{R}_k)^{-1}_{\bar{h}_{\rho\sigma} \bar{\xi}_\mu}], \tag{3.9}$$

where $\varphi \equiv \{\overline{h}, \xi, \bar{\xi}\}$ is summed over. From Eq. (3.4) and Eq. (3.8) we obtain the Ward identities in their final form:

$$\int d^d x \frac{1}{\sqrt{\bar{g}}} \left\{ \frac{\delta \Gamma_k'}{\delta \overline{h}_{\mu\nu}} \frac{\delta \Gamma_k'}{\delta \beta^{\mu\nu}} + \frac{\delta \Gamma_k'}{\delta \xi^\mu} \frac{\delta \Gamma_k'}{\delta \tau_\mu} \right\} = Y_k. \tag{3.10}$$

Equation (3.10) has to be compared to the ordinary gravitational Ward identities [26] which are similar to Eq. (3.10) but with a vanishing RHS. In fact, the contribution Y_k is due to the cutoff and therefore it vanishes in the limit $k \rightarrow 0$ because $R_k \sim k^2 \rightarrow 0$ in this limit. Hence the standard effective action $\lim_{k\rightarrow 0} \Gamma_k$ is guaranteed to obey its usual Ward identities, and BRS invariance is restored for $k \rightarrow 0$.

Because the Ward identity (3.10) is derived from the same functional integral as the evolution equation, it is automatically satisfied for the exact solution of the evolution equation. For approximate solutions of the evolution equation their consistency with the Ward identity is not guaranteed, and one may even use Eq. (3.10) to judge the quality of the approximation [12,22].

The most important strategy for finding approximate (but still nonperturbative) solutions to the evolution equation is to truncate the space of action functionals. Typically one works on a finite-dimensional subspace parametrized by only a few generalized couplings. As a first step towards such a truncation one can try to neglect the evolution of the ghost action. This amounts to making an ansatz of the following form:

$$\Gamma_k[g, \bar{g}, \xi, \bar{\xi}; \beta, \tau] = \overline{\Gamma}_k[g] + \hat{\Gamma}_k[g, \bar{g}] + S_{\rm gf}[g - \bar{g}; \bar{g}] + S_{\rm gh}[g - \bar{g}, \xi, \bar{\xi}; \bar{g}] - \int d^d x \sqrt{\bar{g}} \{\beta^{\mu\nu} \mathcal{L}_\xi g_{\mu\nu} + \tau_\mu \xi^\nu \partial_\nu \xi^\mu\}. \tag{3.11}$$

In Eq. (3.11) we pulled out the classical $S_{\rm gf}$ and $S_{\rm gh}$ from Γ_k, and the coupling to the BRS variations also has the same form as in the bare action. The remaining functional depends on both $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. It is further decomposed as $\overline{\Gamma}_k + \hat{\Gamma}_k$ where $\overline{\Gamma}_k$ is defined as in Eq. (2.26) and $\hat{\Gamma}_k$ contains the deviations for $\bar{g} \neq g$. Hence by definition

$$\hat{\Gamma}_k[g, g] = 0. \tag{3.12}$$

$\hat{\Gamma}_k$ can be viewed as a quantum correction the gauge fixing term which also vanishes for $\bar{g} = g$. The ansatz (3.11) satisfies the initial condition (2.38) if

$$\overline{\Gamma}_\Lambda = S, \quad \hat{\Gamma}_\Lambda = 0 \tag{3.13}$$

and it satisfies the quantum equation of motion (3.7) exactly. Equation (3.13) suggests setting $\hat{\Gamma}_k = 0$ for all k in a first approximation. In this case it can be checked that if the ansatz (3.11) is inserted into the Ward identity (3.10) its LHS vanishes identically. Including $\hat{\Gamma}_k$ the Ward identity assumes the form

$$\int d^d x \mathcal{L}_\xi g_{\mu\nu} \frac{\delta \hat{\Gamma}_k[g, \bar{g}]}{\delta g_{\mu\nu}(x)} = -Y_k. \tag{3.14}$$

We see that $\hat{\Gamma}_k = 0$ is a good approximation provided we may neglect Y_k. The traces which define Y_k amount to loop integrals, and if we think in terms of a loop expansion Y_k is certainly a higher loop effect and may be neglected in a first approximation. At the nonperturbative level one can still try to set $\hat{\Gamma}_k = 0$ and investigate the consequences in concrete examples. In Yang-Mills theory the analogous truncation has led to rather encouraging results already [5,6,9]. In the next section we shall perform an explicit calculation in this approximation.

If one inserts the ansatz (3.11) into the evolution equation (2.32) one finds the following equation for the evolution of Γ_k in the subspace spanned by the ansatz:

$$\begin{aligned} \partial_t \Gamma_k[g, \bar{g}] = \frac{1}{2} \operatorname{Tr}[(\kappa^{-2} \Gamma_k^{(2)}[g, \bar{g}] + R_k^{\rm grav}[\bar{g}])^{-1} \partial_t R_k^{\rm grav}[\bar{g}]] \\ - \operatorname{Tr}[(-\mathcal{M}[g, \bar{g}] + R_k^{\rm gh}[\bar{g}])^{-1} \partial_t R_k^{\rm gh}[\bar{g}]]. \end{aligned} \tag{3.15}$$

This equation is written down in terms of

$$\begin{aligned} \Gamma_k[g, \bar{g}] &= \Gamma_k[g, \bar{g}, 0, 0; 0, 0] \\ &= \overline{\Gamma}_k[g] + S_{\rm gf}[g - \bar{g}; \bar{g}] + \hat{\Gamma}_k[g, \bar{g}]. \end{aligned} \tag{3.16}$$

$\Gamma_k^{(2)}$ is the Hessian of $\Gamma_k[g, \bar{g}]$ with respect to $g_{\mu\nu}$ at fixed $\bar{g}_{\mu\nu}$. For the harmonic coordinate condition, the classical kinetic term of the ghosts, $\mathcal{M}$, is given by Eq. (2.11).

IV. THE EINSTEIN-HILBERT TRUNCATION

In this section we illustrate the use of Eq. (3.15) by means of a simple example. At the UV scale Λ we start from the classical Einstein-Hilbert action in d dimensions,

$$S = \frac{1}{16\pi \bar{G}} \int d^d x \sqrt{g} \{-R(g) + 2\bar{\lambda}\}, \tag{4.1}$$

and we evolve it down to smaller scales $k < \Lambda$. For the time being we shall not try to send Λ to infinity, so the nonrenormalizability of the theory is not an issue here. We are going to use a truncation which replaces the bare Newton constant $\bar{G}$ and the bare cosmological constant $\bar{\lambda}$ in Eq. (4.1) by k-dependent functions

$$G_k \equiv Z_{Nk}{}^{-1}\bar{G} \tag{4.2}$$

and $\bar{\lambda}_k$, respectively:

$$\Gamma_k[g,\bar{g}] = 2\kappa^2 Z_{Nk}\int d^dx\,\sqrt{g}\{-R(g)+2\bar{\lambda}_k\} + \kappa^2 Z_{Nk}\int d^dx\,\sqrt{\bar{g}}\;\bar{g}^{\mu\nu}(\mathcal{F}_\mu^{\alpha\beta}g_{\alpha\beta})(\mathcal{F}_\nu^{\rho\sigma}g_{\rho\sigma}). \tag{4.3}$$

This ansatz is of the form (3.16) with $\hat{\Gamma}_k$ neglected and the classical gauge-fixing term given by Eq. (2.4) with Eqs. (2.8), (2.9) and $\alpha = 1/Z_{Nk}$. (Note that $\mathcal{F}_\mu^{\alpha\beta}g_{\alpha\beta} = \mathcal{F}_\mu^{\alpha\beta}\bar{h}_{\alpha\beta}$ because $\bar{D}_\mu \bar{g}_{\alpha\beta} = 0$.) In order to determine the functions Z_{Nk} and $\bar{\lambda}_k$ we have to project the evolution equation on the space spanned by the operators $\sqrt{g}$ and $\sqrt{g}R$. After having inserted the ansatz into the evolution equation we may set $\bar{g}_{\mu\nu} = g_{\mu\nu}$ so that the gauge-fixing term in Eq. (4.3) vanishes. The LHS of the evolution equation then reads

$$\partial_t \Gamma_k[g,g] = 2\kappa^2\int d^dx\,\sqrt{g}[-R(g)\partial_t Z_{Nk} + 2\partial_t(Z_{Nk}\bar{\lambda}_k)]. \tag{4.4}$$

On the RHS of Eq. (3.15) we have to perform a derivative expansion and retain only the terms proportional to $\int\sqrt{g}$ and $\int\sqrt{g}R$. Equating the result to Eq. (4.4) we can read off the system of ordinary differential equations for Z_{Nk} and $\bar{\lambda}_k$. They have to be solved subject to the initial conditions $Z_{N\Lambda} = 1$ and $\bar{\lambda}_\Lambda = \bar{\lambda}$. In this manner the renormalization group flow in the space of all action functionals is projected onto the 2-dimensional subspace parametrized by $\bar{G}$ and $\bar{\lambda}$.

In the evolution equation we need the second functional derivative of $\Gamma_k[g,\bar{g}]$ at fixed $\bar{g}_{\mu\nu}$. We expand

$$\Gamma_k[\bar{g}+\bar{h},\bar{g}] = \Gamma_k[\bar{g},\bar{g}] + O(\bar{h}) + \Gamma_k^{\text{quad}}[\bar{h};\bar{g}] + O(\bar{h}^3) \tag{4.5}$$

and we find for the piece which is quadratic in $\bar{h}_{\mu\nu}$:

$$\Gamma_k^{\text{quad}}[\bar{h};\bar{g}] = Z_{Nk}\kappa^2\int d^dx\,\sqrt{\bar{g}}\,\bar{h}_{\mu\nu}[-K^{\mu\nu}{}_{\rho\sigma}\bar{D}^2 + U^{\mu\nu}{}_{\rho\sigma}]\bar{h}^{\rho\sigma}. \tag{4.6}$$

Here indices are raised and lowered with $\bar{g}_{\mu\nu}$, and the tensors K and U are given by

$$K^{\mu\nu}{}_{\rho\sigma} = \frac{1}{4}[\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}] \tag{4.7}$$

and

$$U^{\mu\nu}{}_{\rho\sigma} = \frac{1}{4}[\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}](\bar{R}-2\bar{\lambda}_k) + \frac{1}{2}[\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma} + \bar{g}_{\rho\sigma}\bar{R}^{\mu\nu}] - \frac{1}{4}[\delta^\mu_\rho\bar{R}^\nu{}_\sigma + \delta^\mu_\sigma\bar{R}^\nu{}_\rho + \delta^\nu_\rho\bar{R}^\mu{}_\sigma + \delta^\nu_\sigma\bar{R}^\mu{}_\rho] - \frac{1}{2}[\bar{R}^\nu{}_\rho{}^\mu{}_\sigma + \bar{R}^\nu{}_\sigma{}^\mu{}_\rho]. \tag{4.8}$$

In Eq. (4.8) all geometrical quantities are constructed from the background metric.[4] In order to partially diagonalize the quadratic form (4.6) we write $\bar{h}_{\mu\nu}$ as the sum of a traceless tensor $\hat{h}_{\mu\nu}$ and a trace part involving $\phi \equiv \bar{g}^{\mu\nu}\bar{h}_{\mu\nu}$:

$$\bar{h}_{\mu\nu} = \hat{h}_{\mu\nu} + d^{-1}\bar{g}_{\mu\nu}\phi, \quad \bar{g}^{\mu\nu}\hat{h}_{\mu\nu} = 0. \tag{4.9}$$

As a consequence, Eq. (4.6) becomes

$$\Gamma_k^{\text{quad}}[\bar{h};\bar{g}] = Z_{Nk}\kappa^2\int d^dx\,\sqrt{\bar{g}}\left\{\frac{1}{2}\hat{h}_{\mu\nu}[-\bar{D}^2 - 2\bar{\lambda}_k + \bar{R}]\hat{h}^{\mu\nu} - \left(\frac{d-2}{4d}\right)\phi\left[-\bar{D}^2 - 2\bar{\lambda}_k + \frac{d-4}{d}\bar{R}\right]\phi - \bar{R}_{\mu\nu}\hat{h}^{\nu\rho}\hat{h}^\mu{}_\rho + \bar{R}_{\alpha\beta\nu\mu}\hat{h}^{\beta\nu}\hat{h}^{\alpha\mu} + \frac{d-4}{d}\phi\bar{R}_{\mu\nu}\hat{h}^{\mu\nu}\right\}. \tag{4.10}$$

The equations for Z_{Nk} and $\bar{\lambda}_k$ obtain by comparing the coefficients of $\int\sqrt{g}$ and $\int\sqrt{g}R$ on both sides of the evolution equation at $\bar{g}_{\mu\nu} = g_{\mu\nu}$. For this purpose we may insert an arbitrary family of metrics $g_{\mu\nu}$ which is general enough to identify the terms

[4]We use the conventions $R^\sigma{}_{\rho\mu\nu} = -\partial_\nu\Gamma^\sigma_{\mu\rho} + \ldots$, $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$.

$\int\sqrt{g}$ and $\int\sqrt{g}R$ and to distinguish them from higher order terms in the derivative expansion, such as $\int\sqrt{g}R^2$ or $\int\sqrt{g}R^{\mu\nu}D_\mu D_\nu R$, for instance. We exploit this freedom by assuming that $\overline{g}_{\mu\nu}$ corresponds to a maximally symmetric space, i.e., that

$$\overline{R}_{\mu\nu\rho\sigma}=\frac{1}{d(d-1)}[\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma}-\overline{g}_{\mu\sigma}\overline{g}_{\nu\rho}]\overline{R}$$
$$\overline{R}_{\mu\nu}=\frac{1}{d}\,\overline{g}_{\mu\nu}\overline{R}. \tag{4.11}$$

From now on the curvature scalar $\overline{R}$ parametrizes the family of metrics inserted, and it should be regarded as an externally prescribed number rather than a functional of the metric. For a maximally symmetric background the quadratic action boils down to

$$\Gamma_k^{\rm quad}[\overline{h};\overline{g}]=\frac{1}{2}\,Z_{Nk}\kappa^2\int d^dx\sqrt{\overline{g}}\left\{\hat{h}_{\mu\nu}[-\overline{D}^2-2\overline{\lambda}_k+C_T\overline{R}]\hat{h}^{\mu\nu}-\left(\frac{d-2}{2d}\right)\phi[-\overline{D}^2-2\overline{\lambda}_k+C_S\overline{R}]\phi\right\}, \tag{4.12}$$

with

$$C_T\equiv\frac{d(d-3)+4}{d(d-1)},\quad C_S\equiv\frac{d-4}{d}. \tag{4.13}$$

Before continuing we have to specify the precise form of the cutoff operators $R_k^{\rm grav}$ and $R_k^{\rm gh}$ to be used in the evolution equation (3.15). Both of them have the structure (2.13) whereby $\mathcal{Z}_k$ should be adjusted in such a way that for every low-momentum mode the cutoff combines with the kinetic term of this mode to $-\overline{D}^2+k^2$ times a constant. Looking at Eq. (4.12) we see that the respective kinetic terms for $\hat{h}_{\mu\nu}$ and ϕ differ by a factor of $-(d-2)/2d$. This suggests the following choice:

$$(\mathcal{Z}_k^{\rm grav})^{\mu\nu\rho\sigma}=\left[(I-P_\phi)^{\mu\nu\rho\sigma}-\frac{d-2}{2d}P_\phi^{\mu\nu\rho\sigma}\right]Z_{Nk}. \tag{4.14}$$

Here

$$(P_\phi)_{\mu\nu}{}^{\rho\sigma}=d^{-1}\overline{g}_{\mu\nu}\overline{g}^{\rho\sigma} \tag{4.15}$$

is the projector on the trace part of the metric. For the traceless tensor Eq. (4.14) coincides with Eq. (2.15) for $Z_k^{\rm grav}=Z_{Nk}$, and for ϕ the different relative normalization is taken into account. Thus we obtain in the $\hat{h}$ and the ϕ sector, respectively:

$$(\kappa^{-2}\Gamma_k^{(2)}[g,g]+R_k^{\rm grav})_{\hat{h}\hat{h}}=Z_{Nk}[-D^2+k^2R^{(0)}(-D^2/k^2)-2\overline{\lambda}_k+C_TR],$$
$$(\kappa^{-2}\Gamma_k^{(2)}[g,g]+R_k^{\rm grav})_{\phi\phi}=-\frac{d-2}{2d}\,Z_{Nk}[-D^2+k^2R^{(0)}(-D^2/k^2)-2\overline{\lambda}_k+C_SR]. \tag{4.16}$$

From now on we may set $\overline{g}=g$ and we omit the bars from the metric and the curvature.

The last missing ingredient for the evolution equation is the Faddeev-Popov operator. From Eq. (2.11) one obtains, at $\overline{g}=g$,

$$\mathcal{M}[g,g]^\mu{}_\nu=\delta_\nu^\mu D^2+R^\mu{}_\nu=-\delta_\nu^\mu[-D^2+C_VR], \tag{4.17}$$

with

$$C_V\equiv-\frac{1}{d}. \tag{4.18}$$

In the second part of Eq. (4.17) we used Eq. (4.11) for a maximally symmetric background. Since we did not take into account any renormalization effects in the ghost action we set $Z_k^{\rm gh}\equiv1$ in $R_k^{\rm gh}$ and obtain

$$-\mathcal{M}+R_k^{\rm gh}=-D^2+k^2R^{(0)}(-D^2/k^2)+C_VR. \tag{4.19}$$

Let us write $\mathcal{S}_k(R)$ for the RHS of the renormalization group equation (3.15) with $\overline{g}=g$. Inserting Eq. (4.16) and Eq. (4.19) there, we arrive at

$$\mathcal{S}_k(R)=\mathrm{Tr}_T[\mathcal{N}(\mathcal{A}+C_TR)^{-1}]+\mathrm{Tr}_S[\mathcal{N}(\mathcal{A}+C_SR)^{-1}]$$
$$-2\,\mathrm{Tr}_V[\mathcal{N}_0(\mathcal{A}_0+C_VR)^{-1}], \tag{4.20}$$

with

$$\mathcal{A}\equiv-D^2+k^2R^{(0)}(-D^2/k^2)-2\overline{\lambda}_k$$

$$\mathcal{N} \equiv (2Z_{Nk})^{-1}\partial_t[Z_{Nk}k^2R^{(0)}(-D^2/k^2)]$$
$$=\left[1-\frac{1}{2}\,\eta_N(k)\right]k^2R^{(0)}(-D^2/k^2)+D^2R^{(0)\prime}\times(-D^2/k^2) \quad (4.21)$$

where a prime denotes the derivative with respect to the argument and

$$\eta_N(k)\equiv -\partial_t \ln Z_{Nk} \quad (4.22)$$

is the anomalous dimension of the operator $\sqrt{g}R$. The operators $\mathcal{N}_0$ and $\mathcal{A}_0$ are defined similarly to Eq. (4.21) but with $\lambda=0$ and $Z_{Nk}=1$, i.e., $\eta_N(k)=0$. Equation (4.20) involves traces of functions of the covariant Laplacian $D^2 \equiv g^{\mu\nu}D_\mu D_\nu$ acting on traceless symmetric tensors (''T''), scalars (''S'') and vectors (''V''). Because we need only the zeroth and the first order in the curvature scalar we can expand

$$\mathcal{S}_k(R)=\mathrm{Tr}_T[\mathcal{N}\mathcal{A}^{-1}]+\mathrm{Tr}_S[\mathcal{N}\mathcal{A}^{-1}]-2\,\mathrm{Tr}_V[\mathcal{N}_0\mathcal{A}_0^{-1}]$$
$$-R(C_T\,\mathrm{Tr}_T[\mathcal{N}\mathcal{A}^{-2}]+C_S\,\mathrm{Tr}_S[\mathcal{N}\mathcal{A}^{-2}]$$
$$-2C_V\,\mathrm{Tr}_V[\mathcal{N}_0\mathcal{A}_0^{-2}])+O(R^2). \quad (4.23)$$

The traces in Eq. (4.23) can be evaluated by taking advantage of the heat kernel expansion

$$\mathrm{Tr}[e^{-isD^2}]=\left(\frac{i}{4\pi s}\right)^{d/2}\mathrm{tr}(I)\times\int d^dx\sqrt{g}\left\{1-\frac{1}{6}isR+O(R^2)\right\}. \quad (4.24)$$

Here I denotes the unit matrix of the space of fields on which D^2 acts. Hence $\mathrm{tr}(I)$ is the number of independent field components and in particular

$$\mathrm{tr}_S(I)=1$$
$$\mathrm{tr}_V(I)=d \quad (4.25)$$
$$\mathrm{tr}_T(I)=\frac{1}{2}(d-1)(d+2).$$

Considering an arbitrary function W with a Fourier transform $\widetilde{W}$, the expansion of the trace

$$\mathrm{Tr}[W(-D^2)]=\int_{-\infty}^{\infty}ds\widetilde{W}(s)\mathrm{Tr}[e^{-isD^2}] \quad (4.26)$$

is given by

$$\mathrm{Tr}[W(-D^2)]=(4\pi)^{-d/2}\,\mathrm{tr}(I)\left\{Q_{d/2}[W]\int d^dx\,\sqrt{g}+\frac{1}{6}Q_{d/2-1}[W]\int d^dx\sqrt{g}R+O(R^2)\right\}, \quad (4.27)$$

with

$$Q_n[W]\equiv\int_{-\infty}^{\infty}ds(-is)^{-n}\widetilde{W}(s). \quad (4.28)$$

Reexpressing Eq. (4.28) in terms of W leads to the Mellin transform ($n>0$)

$$Q_0[W]=W(0)$$
$$Q_n[W]=\frac{1}{\Gamma(n)}\int_0^{\infty}dzz^{n-1}W(z). \quad (4.29)$$

The next step is to use Eq. (4.27) in order to evaluate Eq. (4.23) and to combine $\mathcal{S}(R)$ with the LHS of the evolution equation, Eq. (4.4). From the coefficients of $\int\sqrt{g}$ we can read off the following equation:

$$\partial_t(Z_{Nk}\overline{\lambda}_k)=(4\kappa^2)^{-1}(4\pi)^{-d/2}\{\mathrm{tr}_T(I)Q_{d/2}[\mathcal{N}/\mathcal{A}]+\mathrm{tr}_S(I)Q_{d/2}[\mathcal{N}/\mathcal{A}]-2\,\mathrm{tr}_V(I)Q_{d/2}[\mathcal{N}_0/\mathcal{A}_0]\}. \quad (4.30)$$

Likewise $\int\sqrt{g}R$ gives rise to

$$\partial_tZ_{Nk}=-(12\kappa^2)^{-1}(4\pi)^{-d/2}[\mathrm{tr}_T(I)\{Q_{d/2-1}[\mathcal{N}/\mathcal{A}]-6C_TQ_{d/2}[\mathcal{N}/\mathcal{A}^2]\}+\mathrm{tr}_S(I)\{Q_{d/2-1}[\mathcal{N}/\mathcal{A}]-6C_SQ_{d/2}[\mathcal{N}/\mathcal{A}^2]\}$$
$$-2\,\mathrm{tr}_V(I)\{Q_{d/2-1}[\mathcal{N}_0/\mathcal{A}_0]-6C_VQ_{d/2}[\mathcal{N}_0/\mathcal{A}_0^2]\}]. \quad (4.31)$$

In Eqs. (4.30) and (4.31), $\mathcal{N}$ and $\mathcal{A}$ are considered c-number functions of z which replaces $-D^2$ in Eq. (4.21). For every cutoff $R^{(0)}$ we define the functions ($p=1,2,\dots$)

$$\Phi_n^p(w)=\frac{1}{\Gamma(n)}\int_0^{\infty}dz\,z^{n-1}\frac{R^{(0)}(z)-zR^{(0)\prime}(z)}{[z+R^{(0)}(z)+w]^p}$$
$$\widetilde{\Phi}_n^p(w)=\frac{1}{\Gamma(n)}\int_0^{\infty}dz\,z^{n-1}\frac{R^{(0)}(z)}{[z+R^{(0)}(z)+w]^p} \quad (4.32)$$

for $n>0$, and[5]

$$\Phi_0^p(w)=\widetilde{\Phi}_0^p(w)=(1+w)^{-p}. \quad (4.33)$$

In terms of the Φ's, Eq. (4.30) assumes the form

[5]Actually Eq. (4.33) follows from Eq. (4.32) in the limit $n\searrow 0$.

$$\partial_t(Z_{Nk}\overline{\lambda_k})=(16\kappa^2)^{-1}(4\pi)^{-d/2}k^d[2d(d+1)\Phi^1_{d/2}(-2\overline{\lambda_k}/k^2)-8d\Phi^1_{d/2}(0)-d(d+1)\eta_N\tilde{\Phi}^1_{d/2}(-2\overline{\lambda_k}/k^2)] \quad (4.34)$$

and Eq. (4.31) becomes

$$\partial_t Z_{Nk}=-(24\kappa^2)^{-1}(4\pi)^{-d/2}k^{d-2}\Bigg[d(d+1)\left\{\Phi^1_{d/2-1}(-2\overline{\lambda_k}/k^2)-\frac{1}{2}\eta_N\tilde{\Phi}^1_{d/2-1}(-2\overline{\lambda_k}/k^2)\right\}$$
$$-6d(d-1)\left\{\Phi^2_{d/2}(-2\overline{\lambda_k}/k^2)-\frac{1}{2}\eta_N\tilde{\Phi}^2_{d/2}(-2\overline{\lambda_k}/k^2)\right\}-4d\Phi^1_{d/2-1}(0)-24\Phi^2_{d/2}(0)\Bigg]. \quad (4.35)$$

Let us introduce the dimensionless, renormalized Newton constant

$$g_k\equiv k^{d-2}G_k\equiv k^{d-2}Z_{Nk}{}^{-1}\overline{G} \quad (4.36)$$

and the dimensionless cosmological constant

$$\lambda_k\equiv k^{-2}\overline{\lambda_k}. \quad (4.37)$$

Here $G_k\equiv Z_{Nk}{}^{-1}\overline{G}$ is the dimensionful renormalized Newton constant at scale k. The evolution of g_k is governed by the equation

$$\partial_t g_k=[d-2+\eta_N(k)]g_k. \quad (4.38)$$

From Eq. (4.35) we obtain for the anomalous dimension $\eta_N(k)$:

$$\eta_N(k)=g_kB_1(\lambda_k)+\eta_N(k)g_kB_2(\lambda_k), \quad (4.39)$$

with

$$B_1(\lambda_k)\equiv\frac{1}{3}(4\pi)^{1-d/2}[d(d+1)\Phi^1_{d/2-1}(-2\lambda_k)-6d(d-1)\Phi^2_{d/2}(-2\lambda_k)-4d\Phi^1_{d/2-1}(0)-24\Phi^2_{d/2}(0)]$$

$$B_2(\lambda_k)\equiv-\frac{1}{6}(4\pi)^{1-d/2}[d(d+1)\tilde{\Phi}^1_{d/2-1}(-2\lambda_k)-6d(d-1)\tilde{\Phi}^2_{d/2}(-2\lambda_k)]. \quad (4.40)$$

We can solve Eq. (4.39) for the anomalous dimension in terms of g_k and λ_k:

$$\eta_N=\frac{g_kB_1(\lambda_k)}{1-g_kB_2(\lambda_k)}. \quad (4.41)$$

The scale derivative of λ_k is related to Eq. (4.34) according to

$$\partial_t\lambda_k=-(2-\eta_N)\lambda_k+32\pi g_k\kappa^2k^{-d}\partial_t(Z_{Nk}\overline{\lambda_k}), \quad (4.42)$$

so that

$$\partial_t\lambda_k=-(2-\eta_N)\lambda_k+\frac{1}{2}g_k(4\pi)^{1-d/2}[2d(d+1)$$
$$\times\Phi^1_{d/2}(-2\lambda_k)-8d\Phi^1_{d/2}(0)-d(d+1)\eta_N$$
$$\times\tilde{\Phi}^1_{d/2}(-2\lambda_k)]. \quad (4.43)$$

Equations (4.38) and (4.43) with Eq. (4.41) is the set of differential equations we wanted to derive. Once the initial values g_Λ and λ_Λ are given, it determines the value of the running Newton's constant and cosmological constant at any scale $k\leqslant\Lambda$. Although they were derived from a relatively simple truncation, the above evolution equations encapsulate nonperturbative effects which go beyond a simple one-loop calculation. This is particularly obvious if one expands Eq. (4.41), for instance, for small values of g_k:

$$\eta_N=g_kB_1(\lambda_k)[1+g_kB_2(\lambda_k)+g_k^2B_2^2(\lambda_k)+\cdots]. \quad (4.44)$$

We observe that η_N receives contributions from arbitrarily high orders of perturbation theory.

V. RUNNING NEWTON'S CONSTANT AND COSMOLOGICAL CONSTANT

A. Near two dimensions

In $d=2$ dimensions $\int\sqrt{g}R$ is a topological invariant proportional to the Euler number and the quantum theory under consideration has at most finitely many (topological) degrees of freedom. In $d=2+\varepsilon$ dimensions, on the other hand, one finds a dynamically nontrivial theory with a nonzero β-function for g_k [27–29]:

$$\partial_t g_k=[\varepsilon+\eta_N]g_k. \quad (5.1)$$

Gravity in $2+\varepsilon$ dimensions provides an interesting laboratory for a first test of the evolution equation because here the conformal factor of the metric can have both a conventional ($\varepsilon<0$) and a ''wrong-sign'' ($\varepsilon>0$) kinetic term; see Eq. (4.12).

The anomalous dimension has a power series expansion

$$\eta_N=\eta_N^{(0)}+\eta_N^{(1)}\varepsilon+\eta_N^{(2)}\varepsilon^2+\cdots \tag{5.2}$$

and therefore

$$\partial_t g_k=[(1+\eta_N^{(1)})\varepsilon+\eta_N^{(0)}]g_k+O(\varepsilon^2). \tag{5.3}$$

Expanding the functions (4.40) as $B_{1,2}=B_{1,2}^{(0)}+B_{1,2}^{(1)}\varepsilon+\cdots$ one has

$$\eta_N^{(0)}=\frac{g_k B_1^{(0)}}{1-g_k B_2^{(0)}}$$

$$\eta_N^{(1)}=\frac{g_k B_1^{(1)}}{1-g_k B_2^{(0)}}+\frac{g_k^2 B_1^{(0)} B_2^{(1)}}{(1-g_k B_2^{(0)})^2}. \tag{5.4}$$

The lowest order terms are

$$B_1^{(0)}(\lambda_k)=2(1-2\lambda_k)^{-1}-4\Phi_1^2(-2\lambda_k)-\frac{32}{3}$$

$$B_2^{(0)}(\lambda_k)=2\tilde{\Phi}_1^2(-2\lambda_k)-(1-2\lambda_k)^{-1}. \tag{5.5}$$

We remark that for vanishing cosmological constant, $B_1^{(0)}$ is a universal quantity, i.e., it does not depend on the precise form of $R^{(0)}$:

$$B_1^{(0)}(0)=-\frac{38}{3}. \tag{5.6}$$

The reason is that the integrand in the integral representation of $\Phi_1^2(0)$ equals the derivative of $z(z+R^{(0)}(z))^{-1}$; hence it is sufficient to know that $R^{(0)}$ is bounded everywhere in order to establish that

$$\Phi_1^2(0)=1. \tag{5.7}$$

Unlike $\Phi_1^2(0)$, $\tilde{\Phi}_1^2(\lambda_k)$ is sensitive to the shape of $R^{(0)}$ even for $\lambda_k=0$. In order to be more explicit we evaluate Eq. (5.5) at $\lambda\neq 0$ for the constant cutoff function $R^{(0)}(z)=1$. Though it does not vanish for $z\to\infty$, it yields at least qualitatively correct results [6,9] as long as it does not introduce UV divergences into the integral under consideration. For Φ_1^2 and $\tilde{\Phi}_1^2$ this is not the case and one finds

$$\Phi_1^2(w)=\tilde{\Phi}_1^2(w)=(1+w)^{-1}, \tag{5.8}$$

so that

$$B_1^{(0)}(\lambda_k)=-2(1-2\lambda_k)^{-1}-\frac{32}{3}$$

$$B_2^{(0)}(\lambda_k)=(1-2\lambda_k)^{-1}. \tag{5.9}$$

As a consequence, we obtain the following answer for the anomalous dimension:

$$\eta_N^{(0)}=-\frac{38}{3}g_k\,\frac{1-\frac{32}{19}\lambda_k}{1-g_k-2\lambda_k}. \tag{5.10}$$

Equation (5.10) improves on earlier results in Refs. [27, 28, 29]. It takes into account partially resummed higher loop effects (higher powers of g_k) and it includes the effect of the running cosmological constant.

One of the interesting features of Einstein-Hilbert gravity in $2+\varepsilon$ dimensions is that the evolution of Newton's constant is governed by a fixed point g_* at which the β-function (5.3) vanishes. To lowest order in ε it is given by

$$g_*=-\varepsilon B_1^{(0)}(\lambda_k)^{-1}. \tag{5.11}$$

The λ dependence of g_* is non–universal. For $R^{(0)}=1$ we obtain

$$g_*=\frac{3}{38}\,\varepsilon\,\frac{1-2\lambda_k}{1-\frac{32}{19}\lambda_k}. \tag{5.12}$$

Equation (5.12) is reliable for $\lambda_k\ll 1$. In this regime the fixed point g_* is UV stable if $\varepsilon>0$ and it is IR stable for $\varepsilon<0$. For $\varepsilon>0$ and $\lambda_k\equiv 0$ this fixed point was discussed by Weinberg [28] in the context of the asymptotic safety scenario for quantum gravity. Our result for the dependence of g_k on the cosmological constant can only be obtained in a framework with a proper infrared regularization because we are investigating the influence of the relevant dimension-two operator on a marginal coupling. (In a sense, the role played by the running cosmological constant is similar to the quadratic mass renormalization in four-dimensional scalar theories.) For $\varepsilon>0$ the theory is asymptotically free. Near the fixed point the dimensionful Newton constant $G_k=g_*/k^\varepsilon$ vanishes for $k\to\infty$.

The evolution of λ_k itself is governed by Eq. (4.43). For $g_k\approx g_*$, where g_k and η_N are of order ε, one finds that also the β-function of λ has a zero of order ε:

$$\lambda_*=-\frac{3}{38}\Phi_1^1(0)\varepsilon. \tag{5.13}$$

This fixed point of the λ evolution is UV stable for either sign of ε. We conclude that to first order in ε and for $\varepsilon>0$ the combined (λ,g) system has an UV stable fixed point given by Eq. (5.13) together with $g_*=(3/38)\varepsilon$.

B. Four dimensions

In $d=4$ dimensions, the running of Newton's constant is governed by the following functions of the cosmological constant:

$$B_1(\lambda)=-\frac{1}{3\pi}[18\Phi_2^2(-2\lambda)-5\Phi_1^1(-2\lambda)$$

$$+6\Phi_2^2(0)+4\Phi_1^1(0)] \tag{5.14}$$

$$B_2(\lambda)=\frac{1}{6\pi}[18\tilde{\Phi}_2^2(-2\lambda)-5\tilde{\Phi}_1^1(-2\lambda)]. \tag{5.15}$$

The dimensionful quantity G_k evolves according to

$$\partial_t G_k = \eta_N G_k, \tag{5.16}$$

with the anomalous dimension given by Eq. (4.41). In order to get a feeling for the behavior of G_k, let us restrict our attention to the lowest order in g_k which amounts to keeping only the first nontrivial correction of the expansion in $\overline{G}k^2$. Then $\eta_N = B_1(\lambda_k) g_k + \cdots$, or with $g_k = k^2 G_k = k^2 \overline{G} + O(\overline{G}^2)$,

$$\eta_N = B_1(\lambda_k)\overline{G}k^2 + O(\overline{G}^2). \tag{5.17}$$

First we consider the case where the cosmological constant is much smaller than k^2. Then we may approximate $\lambda_k \approx 0$ in Eq. (5.17), and Eq. (5.16) has the solution

$$G_k = G_0[1 - \omega\ \overline{G}k^2 + O(\overline{G}^2 k^4)]. \tag{5.18}$$

Here

$$\omega \equiv -\frac{1}{2} B_1(0) = \frac{1}{6\pi}[24\Phi_2^2(0) - \Phi_1^1(0)] \tag{5.19}$$

is a pure number, which depends on the function $R^{(0)}$, however. For the exponential cutoff (2.14) we have $\Phi_1^1(0) = \pi^2/6$ and $\Phi_2^2(0) = 1$, so that

$$\omega = \frac{4}{\pi}\left(1 - \frac{\pi^2}{144}\right) > 0. \tag{5.20}$$

For different cutoff functions the numerical value of ω will be slightly different but it will still be positive. Therefore Eq. (5.18) tells us that Newton's constant decreases as k^2 increases; it is small in the UV and grows larger as we evolve it towards the infrared. The sign of this effect is the same as for the non-Abelian gauge coupling in Yang-Mills theory and it is opposite to the one in QED. The main difference is that G_k depends quadratically on k while, to lowest order, the gauge coupling in Yang-Mills theory runs only logarithmically. We see that gravity is ''antiscreening'' in the sense that at large distances Newton's constant is larger than at small distances. This confirms the intuitive picture that the gravitational charge (mass) is not screened by quantum fluctuations but rather receives an additional positive contribution from the virtual particles surrounding it.

Let us consider a gravitational (thought) experiment which involves a typical length scale r, the distance of two heavy test particles, for instance. If $r \equiv k^{-1}$ acts as the effective IR cutoff scale, Eq. (5.18) suggests the following form of a distance-dependent Newton's constant (with factors of $\hbar$ and c restored):

$$G(r) = G(\infty)\left[1 - \omega\,\frac{\overline{G}\hbar}{r^2 c^3} + O\left(\frac{1}{r^4}\right)\right]. \tag{5.21}$$

We expect[6] that, to leading order in $1/r$, the quantum corrected static Newtonian potential of two test masses should be closely related to $V(r) = -G(r) m_1 m_2 / r$. It is interesting to compare Eq. (5.21) to what is actually obtained by a diagrammatic calculation of the lowest order correction to the potential. Recently Donoghue [31] has pointed out that quantized Einstein gravity makes a well-defined prediction for this quantity which is unaffected by the nonrenormalizability of the theory. One finds a result of the form

$$V(r) = -G\frac{m_1 m_2}{r}\left[1 - \frac{G(m_1+m_2)}{2c^2 r} - \tilde{\omega}\,\frac{G\hbar}{r^2 c^3}\right]. \tag{5.22}$$

The term proportional to $(m_1+m_2)/r$ is a kinematic effect of classical general relativity; it is independent of $\hbar$ and is not related to the β-function of G_k therefore. However, the last term in Eq. (5.22), proportional to $G\hbar/r^2$, has precisely the same structure as Eq. (5.21). The most recent calculation of $\tilde{\omega}$ was performed in Ref. [32] with the result

$$\tilde{\omega} = \frac{118}{15\pi} > 0. \tag{5.23}$$

This number has the same sign and is of the same order of magnitude as the value found originally in Ref. [31], but there is no precise agreement yet. In Ref. [33], $\tilde{\omega}$ was calculated using different methods [34,35] and a negative value was found; this would correspond to ''screening'' rather than ''antiscreening.'' Possible reasons for this discrepancy were discussed in Ref. [32]. While the issue is not fully settled yet, it is believed that by correctly identifying and evaluating the set of relevant Feynman diagrams, quantum Einstein gravity gives rise to an unambiguous value for $\tilde{\omega}$. From our investigation of the renormalization group flow we expect this value to be positive.

One can use the full nonperturbative information contained in Eq. (4.41) in order to extend the domain of validity of our result towards larger values of g_k or smaller distances r. This would involve a numerical solution of Eq. (4.38) on which we shall not embark at this point.

In our approach we can study the influence of the cosmological constant on the running of G_k. It is interesting to ask, for instance, whether a large λ_k can destroy the antiscreening character of the gravitational interaction ($\eta_N < 0$). Let us look at Eq. (5.17) with $B_1(\lambda_k)$ given in Eq. (5.14). If a regime exists with $\eta_N > 0$ (screening) then $B_1(\lambda)$ must be positive there. This can only happen if the term $5\Phi_1^1(-2\lambda_k)$ in the brackets on the RHS of Eq. (5.14) is larger than the sum of the other terms because the Φ's are always positive. However, $\Phi_n^p(w)$ decreases for increasing w and finally vanishes for $w \to \infty$. Therefore a negative cosmological constant will not change the sign of η_N since $B_1(\lambda_k) < 0$ for $\lambda_k \leqslant 0$.

For $\lambda_k > 0$, the Φ's in Eq. (5.14) are evaluated at negative arguments $w \equiv -2\lambda_k$. From Eq. (4.32) it is clear that $\Phi_n^p(w)$ blows up for $w \to -1$. [The function $z + R^{(0)}(z)$ assumes its minimum value 1 at $z=0$ and increases monotonically for $z > 0$.] This signals that our approximation breaks down for $\lambda_k \approx 1/2$ or $\overline{\lambda}_k \approx k^2/2$. For moderately large values of λ_k, $B_1(\lambda_k)$ is still negative. As λ_k approaches 1/2 from below, only the first two terms on the RHS of Eq. (5.14) are impor-

[6]Recall that in QED the analogous substitution $e^2/r \to e^2(r^{-1})/r$ correctly reproduces the leading term of the Uehling potential if the one-loop formula for the running coupling $e^2(\mu)$ is used [30].

tant. It might be that B_1 turns negative then, but this would be in a regime where our truncation is no longer reliable, and the sign would even depend on $R^{(0)}$ in general.

At this point a general remark concerning the domain of validity of our truncation might be in order. In Sec. III we showed that truncations of the form (3.11) with $\hat{\Gamma}_k=0$ are consistent with the modified Ward identities provided Y_k is small. For the Einstein-Hilbert truncation we can evaluate the traces in Eq. (3.9) and we can express Y_k in terms of the functions $\Phi_n^p(w)$. It is clear, therefore, that Y_k becomes large for $w\to-1$, and that our truncation cannot account for this regime.

The running of the (dimensionful) cosmological constant itself is governed by the equation

$$\partial_t\overline{\lambda}_k=\eta_N\overline{\lambda}_k+\frac{1}{2\pi}k^4G_k[10\Phi_2^1(-2\overline{\lambda}_k/k^2)-8\Phi_2^1(0)$$
$$-5\eta_N\Phi_2^1(-2\overline{\lambda}_k/k^2)] \tag{5.24}$$

If we switch off the renormalization group improvement for a moment and set $\eta_N=0$, $\overline{\lambda}_k=0$ on the RHS of Eq. (5.24), it has the solution

$$\overline{\lambda}_k=\frac{1}{4\pi}\Phi_2^1(0)\overline{G}(k^4-\Lambda^4)+\overline{\lambda}_\Lambda. \tag{5.25}$$

We observe the canonical scale dependence $\overline{\lambda}_k\sim k^4$ which one expects in any naive one-loop calculation: if $\overline{\lambda}_k$ starts off positive at $k=\Lambda$, its absolute value decreases when k is lowered until it reaches zero and then $\overline{\lambda}_k$ becomes negative (for Λ large enough). It is obvious that any attempt to fine tune $\overline{\lambda}_\Lambda$ in such a way that $\lim_{k\to 0}\ \overline{\lambda}_k=0$ cannot have a universal meaning because $\Phi_2^1(0)$ depends on the form of the cutoff. The evolution equation (5.24) improves on the one-loop result in two respects: it includes the effect of the running G_k, and via the "threshold function" Φ_2^1 it describes the back reaction of the changing $\overline{\lambda}_k$ on its β-function. In particular, for $\overline{\lambda}_k<0$ and $k^2\ll|\overline{\lambda}_k|$ the relevant IR cutoff in the graviton propagator is $|\overline{\lambda}_k|$ rather than k^2. Then the graviton modes do not contribute to the running of $\overline{\lambda}_k$ any longer, and their decoupling is described by the function $\Phi_2^1(w)$. If, on the other hand, the evolution starts with $\overline{\lambda}_k>0$, the threshold functions make the coefficient of the k^4 term in Eq. (5.24) even larger, and the running towards zero is faster than in Eq. (5.25). This effect is counteracted by the term $\eta_N\overline{\lambda}_k$ which is negative for $\eta_N<0$. It cannot prevent $\overline{\lambda}_k$ from overshooting zero, however.

VI. CONCLUSION

In this paper we proposed a general framework for the treatment of quantum gravity along the lines of the Wilsonian renormalization group. We introduced a scale-dependent effective action and we derived an exact renormalization group equation which describes its dependence on the built-in infrared cutoff. The effective action is invariant under general coordinate transformations; no symmetry-violating terms are generated during the evolution. It satisfies a set of modified gravitational Ward identities which ensure that, in the limit of a vanishing cutoff, the conventional Ward identities are recovered. By virtue of the diffeomorphism invariance of the effective action, fairly simple invariant truncations of the space of actions are sufficient to describe the essential physics in a nonperturbative way. The modified Ward identities provide a check for the quality of the truncations. The evolution equation can be used both for the quantization of fundamental theories ($\Lambda\to\infty$) and for the evolution of effective theories (Λ finite). It is defined in terms of manifestly finite, ultraviolet convergent functional traces. The evolution equation by itself is meaningful even if the action is not positive definite. In this case the original Euclidean functional integral formulation might be problematic, and the precise relation between the two approaches is not entirely clear yet.

As a first application, we have tested our method within a simple truncation which retains only the invariants $\int\sqrt{g}R$ and $\int\sqrt{g}$. Nevertheless, the resulting evolution equations for Newton's constant and the cosmological constant contain nonperturbative information. In $2+\varepsilon$ dimensions we found corrections to the β-function for G_k and we determined its dependence on the cosmological constant. In 4 dimensions we saw that the β-function for G_k depends on k quadratically, and that Newton's constant increases at large distances. Within its restricted domain of validity, this result confirms earlier speculations by Polyakov [36] on a possible gravitational antiscreening.

It would be interesting to allow for a more general truncation and to include more complicated invariants in the ansatz for Γ_k. Not only higher powers of the curvature should be kept but also, and perhaps more importantly, nonlocal terms must be included (similar to the 2D induced gravity action $\int RD^{-2}R$, for instance). This would lead to a better understanding of quantum gravity in the extreme infrared, and might help to clarify certain issues in quantum cosmology. For instance, it has been proposed that quantum gravitational effects at large distances should be important both in the context of the dark matter problem [37] and the cosmological constant problem [14,36]. In fact, it is quite clear that the nature of the IR divergences, and hence of the renormalization group flow for $k\to 0$, is quite different depending on whether λ is zero or not [13]. In a perturbative expansion, one of the traces on the RHS of the evolution equation consists of graviton loops attached to external graviton lines. The most singular (for $k\to 0$) diagrams are those which involve the vertices obtained by expanding $\lambda\int\sqrt{g}$, because they do not contain any momentum factors. Hence for $\lambda\neq 0$ the renormalization effects should be much stronger than for $\lambda=0$, and this could eventually drive the cosmological constant to zero. We hope to come back to this point elsewhere.

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[1] R. P. Feynman, Acta Phys. Pol. **24**, 697 (1963); B. DeWitt, Phys. Rev. **160**, 1113 (1967); **162**, 1195 (1967); **162**, 1239 (1967); G. 't Hooft and M. Veltman, Ann. Inst. Henri Poincaré, Sect. A **20**, 69 (1974); S. Deser and P. van Nieuwenhuizen, Phys. Rev. D **10**, 401 (1974); **10**, 410 (1974).
[2] For an introduction see J. F. Donoghue, gr-qc/9512024.
[3] K. G. Wilson, Phys. Rev. B **4**, 3174 (1971); **4**, 3184 (1971); K. G. Wilson and J. Kogut, Phys. Rep., Phys. Lett. **12C**, 75 (1974); F. Wegner and A. Houghton, Phys. Rev. A **8**, 401 (1973); S. Weinberg, Erice Subnucl. Phys. **1** (1976); J. Polchinski, Nucl. Phys. **B231**, 269 (1984).
[4] C. Wetterich, Phys. Lett. B **301**, 90 (1993).
[5] M. Reuter and C. Wetterich, Nucl. Phys. **B417**, 181 (1994).
[6] M. Reuter and C. Wetterich, Nucl. Phys. **B391**, 147 (1993); **B408**, 91 (1993); **B427**, 291 (1994).
[7] N. Tetradis and C. Wetterich, Nucl. Phys. **B398**, 659 (1993); Int. J. Mod. Phys. A **9**, 4029 (1994); Nucl. Phys. **B422**, 541 (1994); M. Gräter and C. Wetterich, Phys. Rev. Lett. **75**, 378 (1995); N. Tetradis and D. Litim, Nucl. Phys. **B464**, 492 (1996); B. Bergerhoff, F. Freire, D. Litim, S. Lola, and C. Wetterich, Phys. Rev. B **53**, 5734 (1996); N. Tetradis, Nucl. Phys. **B488**, 92 (1997).
[8] M. Reuter, Phys. Rev. D **53**, 4430 (1996).
[9] M. Reuter, DESY Report No. 96-065; hep-th/9604124.
[10] A. Hasenfratz and P. Hasenfratz, Nucl. Phys. **B270**, 685 (1986); B. Warr, Ann. Phys. (N.Y.) **183**, 1 (1988), **183**, 59 (1988); T. Hurd, Commun. Math. Phys. **124**, 153 (1989); G. Keller and C. Kopper, Phys. Lett. B **273**, 323 (1991); U. Ellwanger, Z. Phys. C **58**, 619 (1993); U. Ellwanger and L. Vergara, Nucl. Phys. **B398**, 52 (1993); M. Bonini, M. D'Attanasio, and G. Marchesini, *ibid.* **B409**, 441 (1993); **B418**, 81 (1994); **B437**, 163 (1995); M. Alford, Phys. Lett. B **336**, 237 (1994); M. Alford and J. March-Russell, Nucl. Phys. **B417**, 527 (1994); S.-B. Liao, J. Polonyi, and D. Xu, Phys. Rev. D **51**, 748 (1995); P. E. Haagensen, Y. Kubyshin, J. I. Latorre, and E. Moreno, Phys. Lett. B **323**, 330 (1994); U. Kerres, G. Mack, and G. Palma, Nucl. Phys. **B467**, 510 (1996); J. Comellas, Y. Kubyshin, and E. Moreno, hep-th/9601112; M. Griessl, G. Mack, Y. Xylander, and G. Palma, Nucl. Phys. **B477**, 878 (1996).
[11] T. R. Morris, Phys. Lett. B **345**, 139 (1995); **329**, 241 (1994); **334**, 355 (1994).
[12] U. Ellwanger, Phys. Lett. B **335**, 364 (1994); U. Ellwanger, M. Hirsch, and A. Weber, Z. Phys. C **69**, 687 (1996).
[13] N. C. Tsamis and R. P. Woodard, Phys. Lett. B **301**, 351 (1993); Ann. Phys. (N.Y.) **238**, 1 (1995); A. D. Dolgov, M. B. Einhorn, and V. I. Zakharov, Phys. Rev. D **52**, 717 (1995).
[14] I. Antoniadis and E. Mottola, Phys. Rev. D **45**, 2013 (1992); E. Mottola, J. Math. Phys. **36**, 2470 (1995).
[15] A. Ringwald and C. Wetterich, Nucl. Phys. **B334**, 506 (1990); C. Wetterich, *ibid.* **B352**, 529 (1991).
[16] C. Wetterich, Z. Phys. C **60**, 461 (1993).
[17] A. Bonanno, Phys. Rev. D **52**, 969 (1995).
[18] R. Floreanini and R. Percacci, Nucl. Phys. **B436**, 141 (1995).
[19] E. Elizalde, S. Odintsov, and I. Shapiro, Class. Quantum Grav. **11**, 1607 (1994); I. Antoniadis and S. Odintsov, Phys. Lett. B **343**, 76 (1995); E. Elizalde *et al.*, Phys. Rev. D **52**, 2202 (1995).
[20] I. Antoniadis, P. O. Mazur, and E. Mottola, Phys. Lett. B **323**, 284 (1994).
[21] D. Dalvit and F. Mazzitelli, Phys. Rev. D **50**, 1001 (1994); **52**, 2577 (1995).
[22] M. Reuter and C. Wetterich, hep-th/9605039 [Nucl. Phys. B (to be published)].
[23] M. Reuter, in Proceedings of the 5th Hellenic School and Workshop on Elementary Particle Physics, Corfu, Greece, 1995 (unpublished), hep-th/9602012.
[24] S. L. Adler, Rev. Mod. Phys. **54**, 729 (1982).
[25] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Effective Action in Quantum Gravity* (IOP, Bristol, 1992).
[26] N. Nakanishi and I. Ojima, *Covariant Operator Formalism of Gauge Theories and Quantum Gravity* (World Scientific, Singore, 1990).
[27] R. Gastmans, R. Kallosh, and C. Truffin, Nucl. Phys. **B133**, 417 (1978); S. M. Christensen and M. J. Duff, Phys. Lett. **79B**, 213 (1978).
[28] S. Weinberg, in *General Relativity, an Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[29] H. Kawai and M. Ninomiya, Nucl. Phys. **B336**, 115 (1990); I. Jack and D. R. T. Jones, *ibid.* **B358**, 695 (1991); H. Kawai, Y. Kitazawa, and M. Ninomiya, *ibid.* **B404**, 684 (1993); T. Aida, Y. Kitazawa, J. Nishimura, and A. Tsuchiya, *ibid.* **B444**, 353 (1995).
[30] W. Dittrich and M. Reuter, *Effective Lagrangians in Quantum Electrodynamics* (Springer, Berlin, 1985).
[31] J. F. Donoghue, Phys. Rev. Lett. **72**, 2996 (1994); Phys. Rev. D **50**, 3874 (1994).
[32] H. W. Hamber and S. Liu, Phys. Lett. B **357**, 51 (1995).
[33] I. J. Muzinich and S. Vokos, Phys. Rev. D **52**, 3472 (1995).
[34] G. Modanese, Phys. Lett. B **325**, 354 (1994); Nucl. Phys. **B434**, 697 (1995); hep-th/9306078.
[35] H. W. Hamber and R. M. Williams, Nucl. Phys. **B435**, 361 (1995).
[36] A. M. Polyakov, in *Gravitation and Quantizations*, Proceedings of the Les Houches Summer School, Les Houches, France, 1992, edited by J. Zinn-Justin and B. Julià, Les Houches Summer School Proceedings Vol. 57 (North Holland, Amsterdam, 1995); hep-th/9304146.
[37] T. Goldman, J. Perez-Mercader, F. Cooper, and M. M. Nieto, Phys. Lett. B **281**, 219 (1992); A. Bottino, C. W. Kim, and J. Song, *ibid.* **351**, 116 (1995); O. Bertolami and J. Garcia-Bellido, Int. J. Mod. Phys. D **5**, 363 (1996).