Stellar model in a fourth order theory of gravity

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Within a fourth order theory of gravity, we obtain the approximation equations in first order of the coupling constant α of the quadratic term in the curvature and apply these equations to discuss a spherically symmetric perfect fluid stellar model in a weak field limit up to second order in the mass density ρ. We find, unlike general relativity (GR), that the continuity of the metric does not allow for a discontinuous mass density; i.e., for any bounded distribution of matter the pressure *and* the mass density have to be zero at the boundary. We show that the active mass of the fourth order theory is different than the active mass in GR. Furthermore, for a hard core star model, we find the explicit solution for the pressure and investigate the upper bound on the active mass of the star by assuming that matter couples minimally in the Jordan conformal frame and by applying the dominant energy condition to the perfect fluid at the center of the star. We show that there exist values of α and of the radius R for which this mass of the system does not have an upper bound. [S0556-2821(98)02602-2]

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I. INTRODUCTION

Higher order theories of gravity are the generally covariant extensions of general relativity (GR) when we consider in the Lagrangian density nonlinear terms in the curvature. The field equations derived by second order variations of this Lagrangian contain derivatives of the metric of an order higher than the second. [However, the field equations are of second order when we use the first order formalism (independent variations of the metric and the connection) [1,2].]

The most general action containing the Einstein plus Gauss-Bonnet terms is (for a vacuum)

$$S=\int \sqrt{-g}(R+\alpha R^2+\beta R_{cd}R^{cd})d^4x, \tag{1}$$

where we have not considered surface terms since they will not contribute to the analysis of the field equations we will perform. The factors α and β are new universal constants (a Riemann-squared term can be eliminated using the Gauss-Bonet identity; the term linear in R is necessary for a proper Newtonian limit [3]).

The field equations derived by extremizing the action are given by

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}+\alpha K_{\mu\nu}+\beta L_{\mu\nu}=8\pi GT_{\mu\nu}, \tag{2}$$

where

$$K_{\mu\nu}=-2R_{;\mu\nu}+2g_{\mu\nu}\Box R-\tfrac{1}{2}g_{\mu\nu}R^2+2RR_{\mu\nu}, \tag{3}$$

$$L_{\mu\nu}=-2R^{\alpha}_{\mu;\nu\alpha}+\Box R_{\mu\nu}+\tfrac{1}{2}g_{\mu\nu}\Box R+2R^{\alpha}_{\mu}R_{\alpha\nu} -\tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}. \tag{4}$$

On the classical level, according to Noakes [4], the fourth order theories satisfy one major criterion for the physical acceptability of a gravitational theory; that is, the initial value formulation is well posed.

On the other hand, attempts to quantize GR or to regularize the stress-energy-momentum tensor of quantum fields propagating in curved spacetimes have led investigators to consider gravitational actions involving curvature squared terms [5]. Higher derivative theories appear to enjoy better renormalizability properties than GR [6]. Although the correct quantum theory is not yet known, the string theory, for example, suggests in addition to the Einstein action a quadratic and even higher order terms in the curvature.

In modern cosmology inflation has become standard since there is no doubt that the existence of an inflationary period [7–9] (exponential expansion of the cosmic scale factor) solves a lot of problems connected with the standard big bang model of the Universe. So it is no wonder that the Starobinsky model (in which curvature-squared terms lead automatically to the desired inflationary period) enjoyed so much interest in recent years.

We propose here to analyze the properties of the effective gravity theory characterized by the action (1). We study the case $\beta=0$ and consider only small values of the parameter α ($|\alpha|\ll 1$) [8]. Thus, in Sec. II we obtain approximate solutions to the field equations in first order of α. In Sec. III we particularize these equations for a spherically symmetric stellar model (as a perfect fluid source) and in Sec. IV we obtain solutions for this model in different approximations of ρ (the mass density) and p (the pressure).

The fourth order equations corresponding to the above Lagrangian density share with GR its vacuum solutions. This may suggest that the classical test of GR is automatically satisfied through the Schwarzschild solution [10,11]. However, the empty space solutions are to be matched to interior solutions and it may well occur that the matching conditions are not satisfied [12,13]. Higher order theories have a richer set of vacuum solutions than GR [14,15]; in other words, the vacuum solutions of GR are also solutions of higher order theories but the converse is in general not true. If we call Σ_{VHO} the set of vacuum solutions of higher order theory and Σ_{VGR} the same set but to GR, its difference $\Sigma_{\Delta V}=\Sigma_{VHO}-\Sigma_{VGR}$ is in general a nonempty set. Unlike lower derivative corrections, however, it is false to assume that adding a higher derivative correction term, with a small coefficient, will only perturb the original theory. The presence of an unconstrained higher derivative term, no matter how small it may naively appear, makes the new theory dramatically different from the original. We give an example of this remark later on in Sec. IV.

Finally, in Sec. V, for a hard core star model we find the explicit solution for the pressure and investigate the upper bound on the mass of the star, by assuming the validity of the dominant energy condition at its center. We show that there exist values of α and of the radius R for which the active mass of the system does not have an upper bound.

II. APPROXIMATE FIELD EQUATIONS

In our case the field equation is

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+2\alpha R(R_{\mu\nu}-\tfrac{1}{4}g_{\mu\nu}R)-2\alpha(R_{,\mu;\nu}-g_{\mu\nu}\Box R)=8\pi G T_{\mu\nu}. \tag{5}$$

The contraction of this equation is

$$R=6\alpha\Box R-8\pi G T. \tag{6}$$

Using Eq. (6) on the left-hand side of Eqs. (5) we get, to order α (from now on we shall only keep terms up to order α),

$$R_{\mu\nu}(1-16\pi G\alpha T)-3\alpha g_{\mu\nu}\Box R=8\pi G(T_{\mu\nu}-\tfrac{1}{2}Tg_{\mu\nu})+16\pi G\alpha(2\pi G g_{\mu\nu}T^2+g_{\mu\nu}\Box T-T_{,\mu;\nu}). \tag{7}$$

Dividing this equation by $(1-16\pi G\alpha T)$, we obtain

$$R_{\mu\nu}-3\alpha g_{\mu\nu}\Box R=8\pi G(T_{\mu\nu}-\tfrac{1}{2}T)+16\pi G\alpha T(8\pi G T_{\mu\nu}-4\pi G g_{\mu\nu}T)+16\pi\alpha G(2\pi G T^2 g_{\mu\nu}+g_{\mu\nu}\Box T-T_{,\mu;\nu}). \tag{8}$$

Using the d'Alembertian of Eq. (6), we can put the last equation, in terms of $R_{\mu\nu}$, T, and $T_{\mu\nu}$, as

$$R_{\mu\nu}=8\pi G(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T)+16\pi\alpha G T(8\pi G T_{\mu\nu}-2\pi G g_{\mu\nu}T)-16\pi\alpha G(\tfrac{1}{2}g_{\mu\nu}\Box T+T_{,\mu;\nu}). \tag{9}$$

The last equation is the first order approximation of our method; if we put $\alpha=0$, we obtain GR.

III. SPHERICAL SYMMETRIC EQUATIONS

Let us consider now a static, spherically symmetric spacetime manifold with a metric in the "standard" form:

$$ds^2=A(r)dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)-B(r)dt^2, \tag{10}$$

with functions $A(r)$ and $B(r)$, which are to be determined by solving the field equations (9). Since we are interested in the gravitational field of a bounded source, we shall further assume that the manifold is asymptotically flat at spatial infinity. We assume a perfect fluid energy-momentum tensor

$$T_{\mu\nu}=pg_{\mu\nu}+(p+\rho)U_\mu U_\nu, \tag{11}$$

with p the pressure, ρ the total energy density, and U^μ the velocity four-vector, defined so that $U_\mu U^\mu=-1$. Since the fluid is at rest, we take

$$U_r=U_\theta=U_\phi=0, \quad U_t=-\sqrt{B(r)}. \tag{12}$$

Our assumptions of time independence and spherical symmetry imply that p and ρ are functions only of the radial coordinate r. A first step in solving Eq. (9) is to derive an equation for $A(r)$ alone, by forming the quantity

$$\frac{R_{rr}}{2A}+\frac{R_{tt}}{2B}+\frac{R_{\theta\theta}}{r^2}=\frac{A_{,r}}{rA^2}+\frac{1}{r^2}-\frac{1}{Ar^2}. \tag{13}$$

Using Eq. (9) and substituting the covariant derivative of T, we get

$$\frac{1}{A}-1-\frac{rA,r}{A^2}-16\pi\alpha G r^2\frac{T_{,rr}}{A}-\frac{8\pi\alpha G}{r^2}\left(\frac{r^4}{A}\right)_{,r}T_{,r}=8\pi G[T^t_t r^2-4\pi\alpha G r^2 T(T-4T^t_t)]. \tag{14}$$

It is not difficult to rewrite this equation as a linear differential equation:

$$S_{,r}+S\left(\frac{1-16\pi\alpha G(r^2T_{rr}+2rT_{,r})}{r-8\pi\alpha G r^2 T_{,r}}\right)=8\pi G\left(\frac{r^2T^t_t-4\pi\alpha G r^2 T(T-4T^t_t)}{r(1-8\pi\alpha G r T_{,r})}\right)+\frac{1}{r(1-8\pi\alpha G T_{,r})}, \tag{15}$$

where $S=1/A$.

Now we can expand Eq. (15) up to first order in the parameter α to obtain

$$S_{,r}+SQ(r)=q_0+q_1, \tag{16}$$

where

$$Q(r)=\frac{1}{r}-16\pi\alpha G(rT_{rr}+2T_{,r})+8\pi\alpha G T_{,r},$$

$$q_0=8\pi G r T^t_t+\frac{1}{r}, \tag{17}$$

$$q_1=8\pi\alpha G[8\pi G T_{,r}r^2T^t_t-4\pi r T G(T-4T^t_t)].$$

The solution of Eq. (16) is

$$S(r)=e^{-\int Q(r)dr}\int_0^r e^{\int Q(\acute{r})d\acute{r}}[q_0(\acute{r})+q_1(\acute{r})]d\acute{r}. \tag{18}$$

The metric have to be smooth at $r=0$, thus we integrate between zero and r. In the first order approximation Eq. (18) becomes

$$S=\frac{1}{r}\int_0^r q_0(\acute{r})\acute{r}d\acute{r}+\frac{16\pi\alpha G(rT_{,r}+1/2T)}{r}\int_0^r q_1(\acute{r})\acute{r}d\acute{r}$$
$$-\frac{16\pi\alpha G}{r}\int_0^r q_0(\acute{r})\acute{r}(\acute{r}T_{,\acute{r}}+1/2T)d\acute{r}+\frac{1}{r}\int_0^r q_1(\acute{r})\acute{r}d\acute{r}. \tag{19}$$

Using Eq. (11) and substituting Eq. (17) in the last expression, we get

$$S=1-\frac{2Gm(r)}{r}+16\pi\alpha rT_{,r}-32\pi\alpha G^2\frac{m(r)}{r}(rT_{,r}+1/2T)$$
$$-\frac{(8\pi G)^2\alpha}{r}\int_0^r T_{,\acute{r}}\acute{r}^3T_t^t d\acute{r}+\frac{(8\pi\alpha G)^2}{r}\int_0^r \acute{r}^2TT_t^t d\acute{r}$$
$$-\frac{(8\pi\alpha G)^2}{2r}\int_0^r 4\pi\acute{r}^2T^2d\acute{r}, \tag{20}$$

where

$$m(r)=\int_0^r 4\pi\acute{r}^2\rho(\acute{r})d\acute{r}. \tag{21}$$

It is important to notice that there is a factor α in front of the terms $O(\rho^2)$.

On the other hand, the $R_{\theta\theta}$ corresponding equation is

$$-1+S\left(\frac{B_{,r}r}{2B}+1\right)+\frac{rS_{,r}}{2}=-8\pi G\left(T_{\theta\theta}-\frac{1}{2}g_{\theta\theta}T\right)$$
$$-16\pi\alpha GT(8\pi GT_{\theta\theta}$$
$$-2\pi Gg_{\theta\theta}T)$$
$$+16\pi G\alpha(\tfrac{1}{2}g_{\theta\theta}\Box T+SrT_{,r}). \tag{22}$$

In this theory, as in GR, the Bianchi identity gives

$$\frac{B_{,r}}{B}=-\frac{2p_{,r}}{p+\rho}. \tag{23}$$

The d'Alembertian of T, $\Box T$, becomes

$$\Box T=ST_{,rr}+\left(\frac{1}{2}S_{,r}+\frac{2}{r}S+\frac{SB_{,r}}{2B}\right)T_{,r}. \tag{24}$$

Using Eqs. (23) and (24) in Eq. (22), and writing $p=p_0+\alpha p_1$, we obtain, in first order of the parameter α,

$$-1+\frac{rS_{,r}}{2}+S\left(1-\frac{p_{,r}r}{p+\rho}\right)=-8\pi G\left[r^2p-\frac{r^2}{2}(3p-\rho)\right]-16\pi\alpha G(3p_0-\rho)[8\pi Gr^2p_0-2\pi Gr^2(3p_0-\rho)]$$
$$+16\pi G\alpha\left\{\frac{1}{2}r^2\left[\left(1-\frac{2Gm}{r}\right)(3p_{0,rr}-\rho_{,rr})+\left(\frac{1}{2}S_{,r}+\frac{2S}{r}+\frac{SB_{,r}}{2B}\right)T_{,r}\right]-SrT_{,r}\right\}. \tag{25}$$

This is the generalization of the Oppenheimer-Volkoff equation. For $\alpha=0$ we obtain the corresponding GR equation, as it is expected:

$$p_{0,r}=-(p_0+\rho)G\frac{m(r)+4\pi r^3p_0}{r(r-2Gm)}. \tag{26}$$

Thus, for fluid matter with a given equation of state, $p=p(\rho)$, the equilibrium configurations can be determined in a similar way as in GR: We arbitrarily prescribe a central density ρ_c, and hence a central pressure $p_c=p(\rho_c)$. Then we integrate Eqs. (25), (20), and (23) outward until we reach the surface of the star, $p=0,\rho=0$, where we join the interior solution to a vacuum solution.

The solutions of Eq. (25) are, in general, very difficult to obtain; we solve it, in the next section, in a weak field approximation beyond the linear case.

IV. WEAK FIELD APPROXIMATION

We want to obtain an approximation better than the linear one which has just been done by Stelle [6] and other authors [1]. Let us consider the case $0<\rho\ll 1$. We will assume that the energy-momentum tensor $T_{\mu\nu}$ obeys the dominant energy condition [16,18], so that $|p|\leqslant\rho$. Then, we have $p^2\leqslant|p|\rho\leqslant\rho^2\leqslant\rho$.

If one includes in the total energy-momentum tensor the effective scalar degree of freedom of the fourth order theory [19], then the dominant energy condition is never satisfied. Our aim, however, is to explore the consequences of the ''Jordan conformal frame'' being the physical one, and thus matter will be assumed to couple minimally as above, with energy conditions imposed only on the energy-momentum tensor of the perfect fluid (see also the last comment in the Conclusions).

We shall develop the approximation up to terms of order $O(p\rho)$. Using Eq. (26) we can write

$$p_{0,r}\propto O(\rho^2),\quad p_{0,rr}\propto O(\rho^2), \tag{27}$$

where we have assumed that $\rho_{,r}\propto O(\rho)$. Then, Eq. (20) up to this order is

$$S=1-\frac{2Gm(r)}{r}+16\alpha Gr\pi(3p_0-\rho)_{,r}+\frac{32\pi\alpha G^2m(r)}{r}(r\rho_{,r}+\rho/2)-\frac{1}{2}\alpha(8\pi Gr\rho)^2+\frac{2(8\pi G)^2\alpha}{r}\int_0^r \acute{r}^2\rho^2 d\acute{r}+O(p\rho). \quad (28)$$

Using this result, we can calculate the d'Alembertian of T:

$$\Box T=(3p_{0,rr}-\rho_{,rr})+\frac{2Gm\rho_{,rr}}{r}+\frac{6p_{0,r}}{r}-\rho_{,r}\left(\frac{2}{r}-\frac{3Gm}{r^2}-4\pi Gr\rho-\frac{p_{0,r}}{\rho+p_0}\right)+O(p\rho). \quad (29)$$

Finally, we write Eq. (25) in the form

$$-\left(1-\frac{2Gm}{r}\right)\frac{rp_{,r}}{p+\rho}=\frac{Gm}{r}+4\pi Gr^2p-8\pi G\alpha r\rho_{,r}-\frac{32\pi G^2\alpha m(r)\rho}{r}-8\pi\alpha G^2 13m(r)\rho_{,r}-(8\pi G)^2\alpha r^2\rho^2-\frac{(8\pi G)^2}{r}\alpha\int_0^r \acute{r}^2\rho^2 d\acute{r}+O(p\rho). \quad (30)$$

Now we are ready to solve Eq. (23). The solution is

$$\ln B=-\int_r^\infty\left(\frac{2Gm}{\acute{r}^2}+8\pi G\acute{r}p+\frac{4G^2m^2}{\acute{r}^3}-80\pi G^2\alpha\frac{m\rho}{\acute{r}^2}-2\alpha(8\pi G)^2\acute{r}\rho^2-16\pi\alpha G\rho_{,\acute{r}}-224\pi G^2\alpha\frac{m\rho_{,\acute{r}}}{\acute{r}}-\frac{2(8\pi G)^2}{\acute{r}^2}\alpha\int_0^{\acute{r}}\tilde{r}^2\rho^2 d\tilde{r}\right)d\acute{r}+O(p\rho), \quad (31)$$

where we have chosen the limits of the integral to satisfy the boundary condition $B\to 1$ when $r\to\infty$. On the other hand,

$$\int_r^\infty \frac{m}{\acute{r}}\rho_{,\acute{r}}d\acute{r}=-\frac{\rho m}{r}+\int_r^\infty\left(\frac{\rho m}{\acute{r}^2}-4\pi\acute{r}\rho^2\right)d\acute{r}. \quad (32)$$

From Eqs. (31) and (32) to order $O(p\rho)$ we obtain

$$B=1-\int_r^\infty\left(\frac{2Gm}{\acute{r}^2}+8\pi G\acute{r}p\right)d\acute{r}-16\pi G\alpha\rho\left(1+\frac{14mG}{r}\right)+2G\left(\left[\int_r^\infty\frac{m}{\acute{r}^2}d\acute{r}\right]^2-2\int_r^\infty\frac{m^2}{\acute{r}^3}d\acute{r}\right)-\int_r^\infty\left(896G^2\pi^2\alpha\acute{r}\rho^2-304\pi G^2\alpha\frac{m\rho}{\acute{r}^2}-2(8\pi G)^2\alpha\acute{r}\rho^2-\frac{2(8\pi G)^2\alpha}{\acute{r}^2}\int_0^{\acute{r}}\tilde{r}^2\rho^2 d\tilde{r}\right)d\acute{r}. \quad (33)$$

From Eq. (33) we can see that B would be continuous if ρ is continuous. This is an important property of fourth order theories. Unlike GR, the continuity of the metric does not allow for a discontinuous mass density; i.e., for any bounded distribution of matter the pressure *and* the mass density have to be zero at the boundary. Let us assume that the star has a finite radius R and define $M=m(R)$. Then the metric *outside* the star $(r>R)$, where $p=0$ and $\rho=0$, will be

$$B(r)=1-\frac{2G}{r}\mathcal{M}+O(p\rho),\quad A^{-1}=S(r)=B(r),\quad \mathcal{M}=M-(8\pi)^2G\alpha\int_0^R\acute{r}^2\rho^2d\acute{r}. \quad (34)$$

$\mathcal{M}$ is the mass that governs the Keplerian motion of a test body in the distant Newtonian gravitational field; i.e., it is the star active mass and it is different to the active mass of GR identified with M. In a general nonlinear theory the active mass $\mathcal{M}$ is not necessarily equal to the total mass and energy (or inertial mass), obtained as the conserved quantity associated with the time symmetry of the model [17].

Let us study now the *interior* solution $(r<R)$. From Eq. (33), we have

$$B(r)=1-\int_r^R\left(\frac{2Gm}{\acute{r}^2}+8\pi G\acute{r}p\right)d\acute{r}-\frac{2GM}{R}-16\pi G\alpha\rho\left(1+\frac{14m}{r}\right)+2G\left[\left(\int_r^R\frac{m}{\acute{r}^2}d\acute{r}\right)^2+\frac{M}{r}\int_r^R\frac{m}{\acute{r}^2}d\acute{r}-2\int_r^R\frac{m^2}{\acute{r}^3}d\acute{r}\right]-\int_r^R 768\alpha\pi^2G^2\acute{r}\rho^2d\acute{r}+\int_r^R 304\pi\alpha G^2\frac{m\rho}{\acute{r}^2}d\acute{r}+2\alpha(8\pi G)^2\frac{\int_0^R\acute{r}^2\rho^2d\acute{r}}{R}+2\alpha(8\pi G)^2\int_r^R\left(\frac{1}{\acute{r}^2}\int_0^{\acute{r}}\tilde{r}^2\rho^2d\tilde{r}\right)d\acute{r} \quad (35)$$

At the border of the star, $r=R$, the function $B(r)$ becomes

$$B(R)=1-\frac{2GM}{R}-16\pi G\alpha\rho\left(1+\frac{14m}{r}\right)+2\alpha(8\pi G)^2\frac{\int_0^R r^2\rho^2 dr}{R}. \quad (36)$$

If ρ is continuous through the star border, then $\rho(R)=0$ and the last equation becomes Eqs. (34), which represents the exterior solution at the same point. To obtain a particular solution it is necessary to give a state equation and solve the generalization of the Oppenheimer-Volkoff equation. We will present a particular case in the next section.

If we take the Newtonian limit $p\ll\rho\ll 1$, Eqs. (33) and (28) become

$$B=1-2\int_r^\infty \frac{Gm(\acute{r})}{\acute{r}^2}d\acute{r}-16\pi\alpha G\rho, \quad (37)$$

$$S(r)=\left(1-\frac{2Gm(r)}{r}\right)-16\pi\alpha G r\rho_{,r}. \quad (38)$$

From $T^{\mu}_{;\mu}=0$ applied to a test particle in the Newtonian limit, we obtain

$$\frac{d^2x^i}{dt}=\frac{1}{2}\frac{\partial\Phi}{\partial x^i}, \quad (39)$$

where

$$\Phi=-\int_r^\infty \frac{Gm(\acute{r})}{\acute{r}^2}d\acute{r}-8\pi\alpha G\rho\equiv V_N-8\pi\alpha G\rho. \quad (40)$$

We have a departure of the acceleration of a test body from the Newtonian theory value which is proportional to the gradient of the mass density. This departure from the Newtonian value has to be measured when the body is moving ‘‘through’’ a matter-filled region. Of course this statement can only be considered in the statistical sense. This result is also in agreement with the Newtonian limit of the generalization of the Oppenheimer-Volkoff equation:

$$p_{,r}=-\rho\frac{Gm}{r^2}+8\pi\alpha G\rho\rho_r, \quad (41)$$

where we have used that in the Newtonian approximation $m(r)\ll r$.

On the other hand, it is interesting to notice [2] that in the Newtonian limit of the fourth order theory, the gravitational potential is [6]

$$V=-G\int\left(\frac{1+\frac{1}{3}e^{-r/\sqrt{6\alpha}}}{r}\right)\rho d^3x; \quad (42)$$

thus, in the linear order in α we obtain

$$V\rightarrow V_N-\frac{G}{3}6\alpha\lim_{\alpha\rightarrow 0}\int\frac{e^{-r/\sqrt{6\alpha}}}{6\alpha r}\rho d^3x=V_N-8\pi\alpha G\rho. \quad (43)$$

V. INCOMPRESSIBLE FLUID SOLUTIONS

Just as in GR we find an interesting application of the present theory to the class of stable stars consisting of an incompressible fluid, with equation of state $\rho=\rho_0\theta(R-r)$. Actually these stars do not represent realistic models, but they are simple enough to allow us to find the explicit solutions to the above equations, and to set an upper limit on the value of MG/R, and through it on $\mathcal{M}$. In this case, Eq. (30) becomes (we drop the index zero in ρ_0 for simplicity)

$$\frac{-p_{,r}}{p+\rho}=4\pi Gr\frac{\rho/3+p-32\pi\alpha G\rho^2}{1-\frac{8}{3}\pi G\rho r^2}+O(p\rho), \quad (44)$$

where $r<R$. This equation is separable for p; the solution is

$$\left(\frac{\rho/3-32\pi\alpha G\rho^2+p}{p+\rho}\right)=C\left(1-\frac{8}{3}\pi Gr^2\rho\right)^{(1/2+24\pi\alpha G\rho)}. \quad (45)$$

The constant C can be obtained from the condition $p(R)=0$, and then

$$C=(1/3-32\pi\alpha G\rho)(1-8/3\pi GR^2\rho)^{-(1/2+24\pi\alpha G\rho)}. \quad (46)$$

Using this value of C and Eq. (45), we obtain

$$p(r)=\rho\frac{N}{D}, \quad (47)$$

where

$$N=(1-8/3\pi Gr^2\rho)^{(1/2+24\pi\alpha G\rho)}-(1-8/3\pi GR^2\rho)^{(1/2+24\pi\alpha G\rho)},$$

$$D=(1/3-32\pi\alpha G\rho)^{-1}(1-8/3\pi GR^2\rho)^{(1/2+24\pi\alpha G\rho)}-(1-8/3\pi Gr^2\rho)^{(1/2+24\pi\alpha G\rho)}.$$

The above solution, according to Eq. (44), is valid up to order $O(\rho^4)$; then we have to expand Eq. (47) up to this order:

$$p(r)=\frac{2}{3}\pi G\rho^2(R^2-r^2)+\frac{4}{3}\pi^2G^2\rho^3(R^2-r^2)\left(R^2-\frac{r^2}{3}-48\alpha\right)+\frac{4}{9}G^2\pi^2(R^4-r^4)\rho^3, \quad r<R. \quad (48)$$

It is interesting to notice that the first terms in Eq. (48) are the corresponding Newtonian solution as would be expected. We assume now, in agreement with the GR limit, that the energy-momentum tensor satisfies the strong energy condition [18] and that $|\alpha|G\rho^2\ll\rho$; thus, from Eqs. (44) and (48) we may infer that the pressure is a positive definite function, decreasing with r, and it has a maximum at the origin:

$$0<p(0)=\tfrac{2}{3}\pi G\rho^2R^2+\tfrac{16}{9}\pi^2G^2\rho^3R^4-64\pi^2\alpha G^2R^2\rho^3. \tag{49}$$

Let us apply now the dominant energy condition [18] at the origin:

$$\rho>p(0). \tag{50}$$

Using Eq. (49) in condition (50) we have

$$1-\frac{1}{2}\left(\frac{GM}{R}\right)+\left(\frac{36\alpha}{R^2}-1\right)\left(\frac{GM}{R}\right)^2>0. \tag{51}$$

As we said at the beginning the constant $|\alpha|$ has to be very small, according to experimental evidence [8]. However, we may study condition (51) for any value of α/R^2. To this end it is convenient to introduce the auxiliary variables $x=GM/R>0$ and $b=(36\alpha/R^2)-1$. A straightforward analysis shows the following.

(i) $b>0$. For $b>1/16$ the condition (50) is always satisfied, all values of x are possible, and the system is free from singularities [20]. For $0<b\leqslant 1/16$ there exist values (x_1,x_2) such that for $x_1\leqslant x\leqslant x_2$ the dominant energy condition is violated; x_1 and x_2 are the roots of Eq. (51), equal zero, and they satisfy $2\leqslant x_1\leqslant 4$, $4\leqslant x_2<\infty$. We can conjecture that given a value of R, we would have stable configurations for $x\notin[x_1,x_2]$; i.e., if $x\in[x_1,x_2]$, the (mini)star may become stable by accretion of mass or through an explosion to get rid of any excess of mass to reach an $x<x_1$.

(ii) $b\to 0$. In this case $x_1\to 2$ and $x_2\to\infty$ and the only stable solutions are for $x<2$; i.e., $GM/R<2$.

(iii) $b<0$. There exists $x_0(|b|)>0$ such that for $x\geqslant x_0$ the energy condition is again violated. The function $x_0(|b|)=[(1+16|b|)^{1/2}-1]/4|b|$. We have $0<x_0\leqslant 2$ for $\infty>|b|\geqslant 0$.

VI. CONCLUSIONS

Within a fourth order theory of gravity we have obtained an approximation in first order of the coupling constant of the quadratic term in the curvature and in second order in the mass density for a perfect fluid source representing a bounded object (the star). We have found, unlike GR, that the continuity of the metric does not allow for a discontinuous mass density; i.e., for any bounded distribution of matter the pressure *and* the mass density have to be zero at the boundary. We have shown that the active mass of the fourth order theory is different to the active mass of the same source in GR. The main equation derived in our work is a generalization of the Oppenheimer-Volkoff equation which we solve for the pressure in the hard core model (constant density). In the $\rho=$const model, using the dominant energy condition at the center of the star we find that there exist values of α and of the radius R for which the system is free from singularities for any value of the total active mass of the star.

Finally, we comment briefly on the possibility of describing our system in a different frame. Namely, it has been known for some time that nonlinear theories of gravity with Lagrangian $L=f(R)\sqrt{-g}$ share the property that acting on the metric by a suitable conformal transformation, the field equations can be recast into the Einstein equations for the rescaled metric, interacting with a scalar field. The original set of variables is commonly called the Jordan conformal frame, while the transformed set, whose dynamics is described by Einstein equations, is called the Einstein conformal frame. A problem thus arises of which is the metric structure of space-time that is the physical one.

We have chosen to study our model in the Jordan frame in which gravity is entirely described by the (original) metric tensor g. The Lagrangian for gravity and matterfields [19] can be written as

$$L=[R+\alpha R^2+2L_{mat}]\sqrt{-g}. \tag{52}$$

The gravitational field equations are given by Eq. (5). Moreover, we only consider the approximate field equations, in order α, which are obtained from Eq. (5) as given by Eq. (9).

On the other hand, using the general procedure described by Magnano and Sokolowski [19], upon rescaling the metric, we get the Einstein frame Lagrangian dynamically equivalent to Eq. (52) which contains a minimally coupled scalar field and a matter Lagrangian which contains explicitly the scalar field.

The field equations can be written in the form [19]

$$\tilde{G}_{ab}=t_{ab}+e^{-\sqrt{2/3}\phi}T_{ab}(\text{mat var},\ \phi,\tilde{g}),$$

where

$$t_{ab}=\phi_{,a}\phi_{,b}-\frac{1}{2}\tilde{g}_{ab}\tilde{g}^{cd}\phi_{,c}\phi_{,d}-V(\phi)\tilde{g}_{ab},$$

$$\tilde{\Box}\phi=\frac{dV}{d\phi}+\frac{1}{\sqrt{6}}e^{-\sqrt{2/3}\phi}\tilde{g}^{ab}T_{ab}(\text{mat var},\ \phi,\tilde{g}),$$

$$V(\phi)=\frac{(e^{\sqrt{2/3}\phi}-1)^2}{8\alpha e^{2/3\phi^2}}.$$

In this picture the scalar field influences the motion of any gravitating matter, in particular the perfect fluid of our model.

However, another viewpoint is possible: If we assume that the rescaled metric $\tilde{g}$ is the physical one and the vacuum Lagrangian is transformed first into the corresponding Einstein-frame Lagrangian, and then a minimal coupling Lagrangian for matter is added, we obtain a theory in which the stress-energy tensor for matter is conserved. The field equations are:

$$\tilde{G}_{ab}=t_{ab}+T_{ab}(\text{mat var},\tilde{g}),$$

$$\tilde{\Box}\phi=\frac{dV}{d\phi}.$$

In the Einstein conformal frame the departures from general relativity and the standard Newtonian limit which we have found can therefore be viewed as a direct result of the nonminimal coupling between the effective scalar degree of freedom ϕ and ''ordinary'' matter, which leads to a deviation from the equivalence principle. The authors of [19] might argue that, on grounds of positivity of energy, the Einstein conformal frame is the preferred one and matter couplings should be made minimal in that frame, rather than in the Jordan conformal frame as is the case here. That would lead to an entirely different model which we are not considering in this paper.

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[20] It is interesting to compare this result, for a (approximated) nonempty solution, with the existence of a family of exact vacuum solutions which are regular at the origin, as discussed by Stelle [6].