

# Lagrangian and Hamiltonian formalism for discontinuous fluid and gravitational field

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The barotropic ideal fluid with step and  $\delta$ -function discontinuities coupled to Einstein's gravity is studied. The discontinuities represent star surfaces and thin shells; only nonintersecting discontinuity hypersurfaces are considered. No symmetry (such as, e.g., the spherical symmetry) is assumed. The symplectic structure as well as the Lagrangian and the Hamiltonian variational principles for the system are written down. The dynamics is described completely by the fluid variables and the metric on the fixed background manifold. The Lagrangian and the Hamiltonian are given in two forms: the volume form, which is identical to that corresponding to the smooth system, but employs distributions, and the surface form, which is a sum of volume and surface integrals and employs only smooth variables. The surface form is completely four or three covariant (unlike the volume form). The spacelike surfaces of time foliations can have a cusp at the surface of discontinuity. Geometrical meaning of the surface terms in the Hamiltonian is given. Some of the constraint functions that result from the shell Hamiltonian cannot be smeared so as to become differentiable functions on the (unconstrained) phase space. Generalization of the formulas to more general fluid is straightforward. [S0556-2821(98)03202-0]

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## I. INTRODUCTION

Spherically symmetric thin shells or dust stars (such as the Oppenheimer-Snyder one) are popular models used extensively in the study of a number of phenomena: properties of classical gravitational collapse [1], properties of classical black holes [2], quantum gravitational collapse [3], the dynamics of domain walls in the early Universe [4], the back reaction in the Hawking effect [5], entropy on black holes [6], or quantum theory of black holes [7,8], to mention just a few examples.

The classical dynamics of objects with discontinuities in matter density is well understood; it is determined by Einstein's equations, the matter dynamical equations, and some jump conditions at the discontinuity. The jump conditions for the steplike discontinuity require that there are coordinates in which the metric is  $C^1$  at the discontinuity surface [9]—for the thin shells, they were first formulated by Dautcourt [10]; Dautcourt's equations have been rewritten in a covariant form by Israel [11].

In many investigations, however, a variation principle or a Hamiltonian is needed from which this classical dynamics follows. Often, such principles (suitable, e.g., for spherically symmetric models) are just guessed from the dynamical equations; some attempts to obtain them from more general variational principles are given in [12] and [5]. Indeed, this is an interesting problem by itself: how is the large number of different one-degree-of-freedom Hamiltonians scattered in the literature related to the Einstein-Hilbert action? Our overall picture of the world has to be self-consistent, even if we indulge in using a number of different models, each just applicable for a situation under study.

In the present paper, we reformulate the dynamics of

gravitation and the ideal discontinuous fluid in the Hamiltonian form. That is, we identify the canonical variables ( $p$ 's and  $q$ 's) and Lagrange multipliers and write down a Hamiltonian functional of these variables; we show that the constraints and the canonical equations resulting from this Hamiltonian are equivalent to the system of Einstein equations and the ideal fluid dynamical equations (plus the Israel equations in the case of a thin shell).

To identify the suitable symplectic structure and find the variational formulas, we employ the methods described in detail in [13] and their application to general relativity as given in [14]. We will, however, keep the paper self-contained by motivating and explicitly performing all relevant derivations.

The model of matter used extensively in this paper is that of the simplest kind: the barotropic ideal fluid. This can be formulated as a Lagrangian field theory without any constraints [15]. Generalization to ideal fluid with internal degrees of freedom (such as [16]) or to any conservative continuum should be straightforward; in any case, the gravitational parts of our Lagrangians and Hamiltonians (which represent the solution to the main problem) have general validity.

In each particular case, the classical dynamics can be obtained from a variational principle that has the same form as the corresponding variational principle for a smooth system, if some particular generalized functions are allowed to describe the matter distribution: the step function for star boundaries and the  $\delta$  function for thin shells [12]. This simplicity is, however, traded for the freedom in the choice of coordinates: the generalized function approach works only if the metric is  $C^1$  for the step, and  $C^0$  for the  $\delta$  discontinuity. We transform, therefore, the Lagrangians and the Hamilto-

nians to the so-called *surface form* containing only smooth variables; such Lagrangians and Hamiltonians as well as symplectic forms decompose into sums of volume and surface integrals. The transformation can best be done in the so-called *adapted coordinates*; these are coordinates in which the embedding functions of the surfaces of discontinuity acquire the simplest possible form. The result, however, is covariant in the sense that arbitrary smooth coordinates can be chosen inside of each separated volume (left or right of the discontinuity surface) as well as along the discontinuity surface itself.

An important trick is used throughout the paper: we work in coordinate systems which are always adapted to the position of the discontinuity surface. This way the discontinuity surface may be considered as a fixed submanifold of the spacetime. Thus, the dynamics of the star surface or thin shell is not described by the spacetime coordinates of these objects but by the evolution of the physical fields such as the metric of matter fields along the surfaces. Then, for example, the variations and time derivatives of the embedding functions of the two-surfaces of discontinuity in the three-surfaces of constant time both vanish identically. Our formulas are written only for one hypersurface of discontinuity; an extension to an arbitrary number of hypersurfaces is easy if they do not intersect each other.

Two interesting problems arise. First, we do not show that the dynamics makes sense even on-shell. By that, we mean that there is to be a well-posed initial value problem. One ought to be able to define some nice space of initial data, consisting of those values of the canonical variables that satisfy some well-defined set of constraints, jump, and falloff conditions so that a unique solution to the dynamical equations will exist in a neighborhood of the initial surface. In this paper, we shall just assume that the dynamics is all right. At least in some special cases (such as spherical symmetry), the space of classical solutions is well known and it is as large as one expects.

The second problem is to show that the Hamiltonian formalism defines a (regular) constrained system. This means that one can find a phase space (possibly an extension of ours), a complete set of constraints, and a Hamiltonian satisfying the following conditions: (1) the constraints and Hamiltonian must be differentiable functions on the phase space so that their Poisson brackets are well defined and (2) the Hamiltonian must be first class and the constraint set must be split nicely into first and second class constraints (Bergmann-Dirac analysis [17]). Of course, such an “off-shell” formulation is necessary as a starting point for Dirac quantization. The difficulty is that some constraints at the shell are not differentiable functions on the phase space even if they are smeared along the shell, because the smearing is then only two dimensional, whereas the differentiability would require a three-dimensional smearing. Without an off-shell formulation, the way to quantum theory need not be barred, however. One can try to solve the singular constraints and to substitute the solution back into the action so that a variational principle results which leads to equivalent dynamics without the singular constraints [18]. A problem with such a procedure seems to be that equations quickly grow very messy.

The plan of the paper is as follows. Section II is devoted

to the step and Sec. III to the  $\delta$ -function discontinuities. Section II A introduces the ideal fluid model and its dynamics in a fixed spacetime (metric) background. Basic formulas of the Lagrange and Euler pictures concerning Lagrangians, Hamiltonians, stress-energy tensors, and equations of motion are derived; these equations apply to both the step and  $\delta$ -function discontinuity. The method of *variation formulas* is presented, which enables us to find the symplectic structure as well as to generate the equations of motion. The surface of discontinuity can be moved without problems as long as the metric is fixed. In Sec. II B, the fluid is coupled to the dynamical gravity. Relevant formulas concerning the variation of the Einstein-Hilbert action are collected. The surfaces of discontinuity are now fixed. This helps us to avoid some formal problems. The variation formulas for the system are written in Lagrangian and Hamiltonian form.

In Sec. III A, an action for the thin shell and dynamical gravity is written down in the Lagrangian formalism; the shells are fixed and generalized functions are employed. In Sec. III B, the adapted coordinates are used to transform the (Lagrange formalism) action into a sum of volume and surface integrals disposing of the generalized functions and gaining more coordinate (gauge) freedom (four-covariance): arbitrary coordinates can be chosen left to the shell, right to the shell, and along the shell. In Sec. III C, the variation of the action in the surface form is calculated and the obtained dynamical equations are listed; they contain the Israel equation. A variation formula is derived which is only three-covariant: the foliation by spacelike surfaces  $t = \text{const}$  must be such that the  $t$  surfaces are continuous but can have a cusp at the discontinuity surface; the embedding functions of the discontinuity two-surface in the  $t$  surfaces must be time independent. Section III D contains a Legendre transformation to a Hamiltonian formalism; the general form of the Hamiltonian for the system of thin shells and gravity is presented. In Sec. III E, the explicit functional dependence of the Hamiltonian on the dynamical variables is written down and the geometrical meaning of the surface terms in the Hamiltonian is disclosed. In Sec. III F, the variation of the Hamiltonian is explicitly calculated so that all canonical equations and constraints following from the Hamiltonian can be listed. This not only enables us to check that the Hamiltonian generates the desired dynamics (including the Israel equation) but also to classify the resulting equations into “canonical equations” and “constraints.” For example, the six relations that are equivalent to the Israel equation consist of one super-Hamiltonian constraint, two supermomentum constraints, two singular constraints (these cannot be made differentiable by smearing), and one canonical equation. A, necessarily preliminary, discussion of the result is given.

## II. FLUID WITH A STEP DISCONTINUITY

Our point of departure in this section is the description of relativistic barotropic perfect fluid as given in Ref. [15] (observe that this description is easily extended to any conservative continuum). We will extend and modify the method so that it allows for discontinuous matter distributions admitting situations such as a jump of density at the boundary of a star (a step-function type of discontinuity along a timelike hypersurface).

## A. Fluid in gravitational field

### 1. The description of the fluid

The fluid that have just ‘‘mechanical’’ degrees of freedom consists of identifiable elementary volumes—mass points of the fluid. It can, therefore, be completely described by specifying the mass and the spacetime coordinates of each of these mass points. All mass points form the so-called matter space  $Z$ , which is a three-dimensional manifold; let  $z^a$ ,  $a=1,2,3$ , be some coordinates in  $Z$ . Let us denote the spacetime by  $M$  and let  $x^\mu$ ,  $\mu=0,1,2,3$  be some coordinates in  $M$ . The state of the fluid can then be described by a map  $\zeta: M \rightarrow Z$ , in coordinates  $z^a(x^\mu)$ ; the particle trajectories are then determined by  $z^a(x) = \text{const}$ . The matter space  $Z$  is equipped with a scalar density  $h(z)$ , which determines the mole or particle density of the fluid, so that the number  $N$  of particles or moles in the volume  $V_z \subset Z$  is given by  $N(V_z) = \int_{V_z} d^3z h$ . We assume further that  $h$  has a step discontinuity at a two surface  $\Sigma_z$  in  $Z$ , defined by the equation  $F(z) = 0$ , where  $F$  is a smooth function with nonzero gradient  $F_a$ . Let  $\Sigma := \zeta^{-1}(\Sigma_z)$  be a timelike three-surface separating  $M$  in two open subsets  $V^+$  and  $V^-$  so that  $h[z(x)] > 0$  for  $x \in V^-$  and  $h[z(x)] = 0$  for  $x \in V^+$ . One can make many more general assumptions (e.g., allowing for several matter filled regions), but this will only complicate the description without requiring any new method of approach.

The map  $\zeta$  and the density  $h$  define mole (particle) current  $j^\mu$  in  $M$  by

$$j^\mu = h \epsilon^{\mu\nu\kappa\lambda} z_\nu^1 z_\kappa^2 z_\lambda^3, \quad (1)$$

where

$$z_\mu^a := \frac{\partial z^a}{\partial x^\mu}.$$

$j^\mu(x)$  is discontinuous at  $\Sigma$ ,  $j^\mu \neq 0$  in  $V^-$ ,  $j^\mu = 0$  in  $V^+$ , and  $j^\mu$  is tangential to  $\Sigma$ . (We denote the limits to  $\Sigma$  from inside by the superscript minus.)  $j^\mu$  is a vector density; it is easy to show that  $j^\mu$  is identically conserved everywhere in  $M$ ,  $j^\mu_{;\mu} = 0$ .

The current  $j^\mu(x)$  defines the spacetime four-velocity  $u^\mu(x)$  and the rest mole (particle) scalar density  $n$  of the fluid in  $V^-$  and at  $\Sigma$  by

$$j^\mu = \sqrt{|g|} n u^\mu, \quad (2)$$

where  $g := \det(g_{\mu\nu})$  and  $g_{\mu\nu} u^\mu u^\nu = -1$ . Hence,

$$n = \frac{1}{\sqrt{|g|}} \sqrt{-g_{\mu\nu} j^\mu j^\nu}, \quad (3)$$

$n$  has a discontinuity of step type at  $\Sigma$ .

In [15], it is shown that the fluid equations of motion can be obtained from the Lagrange density  $L_m$  which is given by

$$L_m = -\sqrt{|g|} n e(n), \quad (4)$$

where  $e(n)$  is the energy per mole in the rest frame of the fluid and  $L_m$  is considered as a function of  $z^a$ ,  $z_\mu^a$ , and  $g_{\mu\nu}$ .

As the specific volume  $V$  (i.e., the volume of one mole in the rest frame) is  $1/n$ , we obtain for the pressure  $p$  of the fluid

$$p = -\frac{\partial e}{\partial V} = n^2 e', \quad (5)$$

in  $V^-$ .

### 2. Stress energy density

By definition, the stress-energy tensor density of ideal fluid (see, e.g., [9]) has the form

$$T^{\mu\nu} = \sqrt{|g|} ((\rho + p) u^\mu u^\nu + p g^{\mu\nu}), \quad (6)$$

where

$$\rho = n e(n) \quad (7)$$

is the rest mass density;  $T^{\mu\nu}$  has a step discontinuity at  $\Sigma$ . In this section, we collect some important formulas valid for this tensor density.

Let us vary the action of the fluid

$$I_m = - \int_{V^-} d^4x \sqrt{|g|} n e(n) \quad (8)$$

with respect to  $g_{\mu\nu}$ . Using Eq. (3), we have

$$\delta(\sqrt{|g|} n) = \delta \sqrt{-g_{\mu\nu} j^\mu j^\nu} = -\frac{1}{2\sqrt{|g|}} n^{-1} j^\mu j^\nu \delta g_{\mu\nu},$$

which, together with Eq. (2) and the well-known variation formula for determinants yields

$$\delta n = -\frac{1}{2} n (g^{\mu\nu} + u^\mu u^\nu) \delta g_{\mu\nu}. \quad (9)$$

Then,

$$\begin{aligned} \delta I_m = \int_{V^-} d^4x \sqrt{|g|} & \left( -\frac{1}{2} g^{\mu\nu} n e + \frac{1}{2} n (g^{\mu\nu} + u^\mu u^\nu) \right. \\ & \left. \times (e + n e') \right) \delta g_{\mu\nu}, \end{aligned}$$

and a straightforward calculation using Eqs. (7), (5), and (6) leads to

$$T^{\mu\nu}(x) = 2 \frac{\delta I_m}{\delta g_{\mu\nu}(x)}. \quad (10)$$

The next important relation is the Belinfante-Rosenfeld theorem [19–21] applied to our case: the Lagrange density  $L_m$  must satisfy the following identity:

$$\frac{\partial L_m}{\partial z_\mu^a} z_\nu^a + 2 \frac{\partial L_m}{\partial g_{\mu\rho}} g_{\nu\rho} = L_m \delta_\nu^\mu. \quad (11)$$

This equation is equivalent to the requirement that  $L_m$  is a scalar density, and its derivation is straightforward. From the identity (11) and the formula (10), we obtain immediately that

$$T_\nu^\mu = L_m \delta_\nu^\mu - \frac{\partial L_m}{\partial z_\mu^a} z_\nu^a. \quad (12)$$

Thus, the so-called canonical stress-energy tensor density on the right-hand side is equal to the source of gravitational field.

The formulas (10) and (12) imply the Noether identity

$$\nabla_\mu T_\nu^\mu = \left( -\partial_\mu \frac{\partial L_m}{\partial z_\mu^a} + \frac{\partial L_m}{\partial z^a} \right) z_\nu^a. \quad (13)$$

There are counterparts to Eqs. (10)–(13) within any description of any type of ideal fluid. Derivation of Eq. (13) starts from the equation

$$\nabla_\mu T_\nu^\mu = \partial_\mu T_\nu^\mu - \Gamma_{\mu\nu}^\rho T_\rho^\mu.$$

If one substitutes for  $T_\nu^\mu$  from Eq. (12) into the first term on the right-hand side and from Eq. (10) into the second one, the identity follows. One consequence of the Noether identity is that the four components of the covector  $\nabla_\mu T_\nu^\mu$  are not independent:

$$j^\nu \nabla_\mu T_\nu^\mu = 0,$$

because the definition of  $j^\mu$  implies the identity  $j^\mu z_\mu^a \equiv 0$ . Hence, the equation system  $\nabla_\mu T_\nu^\mu = 0$  contains only three independent equations (Euler equation); the energy conservation equation

$$\nabla_\mu (neu^\mu) = -p \nabla_\mu u^\mu,$$

for the fluid is satisfied identically *within our description*.

### 3. The variational formula

Let us consider the four-dimensional volume  $V$  enclosed between two Cauchy surfaces  $S_1$  and  $S_2$ ; the boundary  $\Sigma$  of the fluid divides  $V$  into two parts  $V^-$  and  $V^+$  and similarly  $S_i$  into  $S_i^-$  and  $S_i^+$ ,  $i=1,2$ . We assume that  $S_1$  and  $S_2$  are  $C^1$  surfaces, that is, the induced metric on, as well as the unit normal vector to  $S_1$  and  $S_2$  are both  $C^1$ .

Let us vary the matter action  $I_m$  with respect to  $z^a(x)$  and  $g_{\mu\nu}$ ; we obtain

$$\begin{aligned} \delta I_m = & \int_{\delta V^-} d^4x L_m + \int_{V^-} d^4x \partial_\mu \left( \frac{\partial L_m}{\partial z_\mu^a} \delta z^a \right) + \int_{V^-} d^4x \\ & \times \left( \frac{\partial L_m}{\partial z^a} - \partial_\mu \frac{\partial L_m}{\partial z_\mu^a} \right) \delta z^a + \frac{1}{2} \int_{V^-} d^4x T^{\mu\nu} \delta g_{\mu\nu}. \end{aligned} \quad (14)$$

The first two integrals can be transformed to surface integrals along  $\Sigma$ ,  $S_1$ , and  $S_2$ . With this aim in mind, we use the coordinates  $x^\mu$  in  $M$  that are adapted to the surfaces. This means that  $x^0 = t_i$  along  $S_i$ ,  $i=1,2$  and

$$F(z^a(x)) = x^3 \quad (15)$$

along  $\Sigma$ , so that  $x^k$ ,  $k=1,2,3$ , are coordinates on  $S_i$  and  $x^\alpha$ ,  $\alpha=0,1,2$ , are coordinates on  $\Sigma$ . Then, the change of  $F$  if we vary  $z^a$  is given by

$$F(z^a(x) + \delta z^a(x)) = x^3 + F_a \delta z^a,$$

and the coordinate  $x^3$  of  $\Sigma$  changes by the amount

$$\delta x^3 = -F_a \delta z^a \Big|_{x^3=0}.$$

Thus, we obtain for the first term

$$\int_{\delta V^-} d^4x L_m = - \int_\Sigma dx^0 dx^1 dx^2 L_m F_a \delta z^a.$$

For the second term, we have

$$\begin{aligned} \int_{V^-} d^4x \partial_\mu \left( \frac{\partial L_m}{\partial z_\mu^a} \delta z^a \right) = & \int_\Sigma d\Sigma \frac{\partial L_m}{\partial z_3^a} \delta z^a + \int_{S_2} dS \frac{\partial L_m}{\partial z_0^a} \delta z^a \\ & - \int_{S_1} dS \frac{\partial L_m}{\partial z_0^a} \delta z^a, \end{aligned}$$

where the abbreviations  $d\Sigma = dx^0 dx^1 dx^2$  and  $dS = dx^1 dx^2 dx^3$  are used. Equation (15) implies that  $F_a z_\mu^a = \delta_\mu^3$ , so we can write

$$\frac{\partial L_m}{\partial z_3^a} = \frac{\partial L_m}{\partial z_\mu^a} F_b z_\mu^b.$$

Collecting these results, we obtain the equation

$$\begin{aligned} \delta I_m = & \int_{V^-} d^4x \left( \frac{\partial L_m}{\partial z^a} - \partial_\mu \frac{\partial L_m}{\partial z_\mu^a} \right) \delta z^a + \int_\Sigma d\Sigma \left( \frac{\partial L_m}{\partial z_\mu^a} z_\mu^b \right. \\ & \left. - L_m \delta_a^b \right) F_b \delta z^a + \int_{S_2} dS \frac{\partial L_m}{\partial z_0^a} \delta z^a - \int_{S_1} dS \frac{\partial L_m}{\partial z_0^a} \delta z^a \\ & + \frac{1}{2} \int_{V^-} d^4x T^{\mu\nu} \delta g_{\mu\nu}. \end{aligned} \quad (16)$$

Thus, the field equations consist of *volume* equations that hold in  $V^-$ ,

$$\frac{\partial L_m}{\partial z^a} - \partial_\mu \frac{\partial L_m}{\partial z_\mu^a} = 0, \quad (17)$$

and *surface* equations that hold at  $\Sigma$ ,

$$\left( \frac{\partial L_m}{\partial z_\mu^a} z_\mu^b - L_m \delta_a^b \right) F_b = 0. \quad (18)$$

The surface of the star is an observer independent dynamical element of the system.

Let us discuss the meaning of the field equations. For the volume equation (17), we just invoke Noether's identity, Eq. (13); we can then see that they are equivalent to the conservation equations  $\nabla_\mu T_\nu^\mu = 0$ . The surface equations can be rewritten as follows. First, using Eq. (12), we have

$$T_\nu^\mu z_\mu^b = \left( -\frac{\partial L_m}{\partial z_\mu^a} z_\mu^b + L_m \delta_a^b \right) z_\nu^a.$$

Hence,

$$T_\nu^\mu F_b z_\mu^b = \left( -\frac{\partial L_m}{\partial z_\mu^a} z_\mu^b + L_m \delta_a^b \right) F_b z_\nu^a. \quad (19)$$

However,  $F_b z_\mu^b$  is covector normal to  $\Sigma$ , so the three surface equations (18) can be written in a covariant form as

$$T_\mu^\perp|_\Sigma = 0, \quad (20)$$

where  $T_\mu^\perp = T_\mu^\nu \tilde{m}_\nu$  and  $\tilde{m}_\nu$  is any normal covector to  $\Sigma$ . Of these four equations, only three are independent, because  $T_{\nu j}^\perp$  is identically zero, as one easily verifies; in fact, the three equations  $T_k^\perp|_\Sigma = 0$  imply Eq. (20). It follows further from Eq. (6) and from  $F_b z_\mu^b u^\mu = 0$  that Eq. (20) is equivalent to the condition that the pressure vanishes,  $p = 0$ , at the surface.

Formula (16) is only valid in the adapted coordinates; in particular, the Lagrangian density  $L_m$  must be expressed in these coordinates. Let us pass to more general coordinates. In fact, the volume integrals are already in a covariant form, so we have just to transform the boundary integrals. However, we will need explicitly only the integrals over the Cauchy surfaces.

Let us define

$$p_a := \frac{\partial L_m}{\partial z_0^a}. \quad (21)$$

The reader can easily verify that Eq. (21), which is written in the adapted coordinates, defines a three-density  $p_a$  along  $S_t$  (independent of the adapted coordinates), because the quantity  $L_m$  is a four-density.

The field equations (both volume and surface) are, therefore, equivalent to the formula

$$\begin{aligned} \delta \int_{V^-} d^4 x L_m &= \int_{S_2} d^3 y p_a \delta z^a - \int_{S_1} d^3 y p_a \delta z^a \\ &+ \frac{1}{2} \int_{V^-} d^4 x T^{\mu\nu} \delta g_{\mu\nu}, \end{aligned}$$

and this formula is valid in any coordinates  $x^\mu$  in  $V^-$  and  $y^k$ ,  $k = 1, 2, 3$ , along  $S_t$ . Let us denote the matter occupied part of the Cauchy surface  $x^0 = t$  by  $S_t$  and the corresponding part of the matter space by  $Z^-$ . The following coordinates will simplify all calculations: the intersection  $S_t \cap \Sigma$  is given by  $x^3 = x^3(t)$  and  $S_t$  by  $x^3 < x^3(t)$ .  $x^k$ ,  $k = 1, 2, 3$ , are coordinates on  $S_t$ ; we will call them ‘‘time-dependent adapted coordinates.’’ Then, we can write

$$\begin{aligned} \delta \int_{V^-} d^4 x L_m &= \int_{t_1}^{t_2} dt \left( \frac{d}{dt} \int_{x^3 < x^3(t)} d^3 x p_a \delta z^a \right. \\ &\left. + \frac{1}{2} \int_{x^3 < x^3(t)} d^3 x T^{\mu\nu} \delta g_{\mu\nu} \right). \end{aligned}$$

If we define the Lagrangian  $\mathcal{L}_m$  by

$$\mathcal{L}_m = \int_{S_t} d^3 x L_m$$

and go to the limit  $S_2 \rightarrow S_1$ , we obtain the *variation formula* for the Lagrangian

$$\begin{aligned} \delta \mathcal{L}_m &= \int_{S_t} d^3 x (p_a \delta z^a) + \int_{\partial S_t} d^2 x p_a \delta z^a \dot{x}^3 \\ &+ \frac{1}{2} \int_{S_t} d^3 x T^{\mu\nu} \delta g_{\mu\nu}. \end{aligned} \quad (22)$$

The variation formula has been derived by careful inclusion of all ‘‘boundary terms’’; this will be the main strategy for our derivation of the Lagrangian and Hamiltonian formalism that admit discontinuities. The role of the variational formula is to generate all dynamical equations (including the definition of momenta): the variation of the left-hand side (LHS) is to be calculated and compared with the RHS.

Equation (22) is also the point of departure for the transformation to the Hamiltonian formalism.

#### 4. The Hamiltonian formalism

Let us disregard the surface term in the formula (22) and define the Hamiltonian density  $H_m$  by a Legendre transformation of the form

$$H_m := p_a \dot{z}^a - L_m.$$

Then we obtain for the variation of the Hamiltonian  $\mathcal{H}_m$ , which is defined by

$$\mathcal{H}_m := \int_{S_t} d^3 x H_m,$$

the relation

$$\delta \mathcal{H}_m = \delta \int_{S_t} d^3 x p_a \dot{z}^a - \delta \int_{S_t} d^3 x L_m.$$

Performing carefully the variation in the first term and substituting from Eq. (22) for the second, we have

$$\begin{aligned} \delta \mathcal{H}_m &= \int_{S_t} d^3 x (\delta p_a \dot{z}^a - \dot{p}_a \delta z^a) + \int_{x^3 = x^3(t)} d^2 x p_a (\dot{z}^a \delta x^3 \\ &- \dot{x}^3 \delta z^a) - \frac{1}{2} \int_{S_t} d^3 x T^{\mu\nu} \delta g_{\mu\nu}. \end{aligned} \quad (23)$$

From this equation, not only the canonical equations are obtained, which will consist of volume and of surface equations, but also the symplectic structure of the system can be read off, which is given by the first two integrals on the RHS; these integrals can be interpreted as  $\omega(\delta z^a, \delta p_a; \dot{z}^a, \dot{p}_a)$ , where  $\omega$  is the symplectic form and  $(\delta z^a(x), \delta p_a(x))$ ,  $(\dot{z}^a(x), \dot{p}_a(x))$  are two vectors; notice that  $\omega$  has a surface part. This is a general and very important observation, which will, for instance, help to decide what are the canonical variables for the shell in the second part of the paper. The last integral in Eq. (23) represents variation with respect to non-dynamical ‘‘parameters’’  $g_{\mu\nu}(x)$ .

We will need the following relations between the Hamiltonian density and the stress-energy density:

$$T_0^0 = -H_m, \quad (24)$$

$$T_k^0 = -p_a z_k^a, \quad (25)$$

$$T_l^k = \left( \frac{\partial H_m}{\partial p_a} p_a - H_m \right) \delta_l^k + \frac{\partial H_m}{\partial z_k^a} z_l^a, \quad (26)$$

which are valid in the time-dependent adapted coordinates. The first two equations are obtained immediately from Eq. (12) and the definition of  $p_a$ . To derive the last equation, we first notice that Eq. (23) has the following consequence:

$$T^{\mu\nu} = -2 \frac{\partial H_m}{\partial g_{\mu\nu}}. \quad (27)$$

Second, we derive an equation analogous to Eq. (11) for  $H_m$ ; we use the fact that  $H_m$  behaves as a three-density if we change the coordinates  $x^k$  keeping  $t$  fixed and that  $H_m$  is a function of  $z^a$ ,  $z_m^a$ ,  $p_a$  and  $g_{\mu\nu}$ ,  $H_m = H_m(z^a, z_m^a, p_a, g_{\mu\nu})$ :

$$\frac{\partial H_m}{\partial z_l^a} z_k^a + \frac{\partial H_m}{\partial p_a} p_a \delta_k^l + 2 \frac{\partial H_m}{\partial g_{\rho l}} g_{\rho k} = H_m \delta_k^l.$$

Then, Eq. (26) follows immediately. Again, analogs of Eqs. (24)–(27) are valid for all types of ideal fluids.

To finish the Legendre transformation, we have also to express the velocity  $\dot{z}^a$  in terms of  $p_a$ ,  $z^a$ , and  $z_k^a$  in the Hamiltonian. This is not completely straightforward. To begin with, we substitute for  $L_m$  from Eq. (4) into the definition (21) of  $p_a$ :

$$p_a = -\sqrt{|g|} \rho' \frac{\partial n}{\partial z^a},$$

where

$$\rho' = \frac{d\rho}{dn}.$$

Equations (2) and (3) imply

$$\frac{\partial n}{\partial z^a} = -\frac{1}{\sqrt{|g|}} u^\mu \frac{\partial j^\mu}{\partial z^a}.$$

The definition of  $j^\mu$  implies the following identity [15]:

$$\frac{\partial j^\mu}{\partial z_\nu^a} z_\kappa^a = j^\mu \delta_\kappa^\nu - j^\nu \delta_\kappa^\mu. \quad (28)$$

Combining the three equations, we easily find

$$u_k = -\frac{p_a z_k^a}{\rho' j^0}, \quad (29)$$

where  $j^0$  depends only on  $z^a$  and  $z_k^a$  [cf. Eq. (1)]:

$$j^0 = h(z) \det(z_k^a).$$

The (3+1) decomposition of the metric (see, e.g., [9])

$$g^{00} = -N^{-2}, \quad g_{0k} = N_k,$$

$$g_{kl} = q_{kl}, \quad g = -qN^2,$$

gives, with the help of Eq. (2),

$$q^{kl} u_k u_l = -1 - \frac{(u^0)^2}{g^{00}} = -1 + \frac{1}{N^2} \left( \frac{j^0}{\sqrt{q}} \right)^2.$$

Substituting for  $u_k$  from Eq. (29), we obtain the identity

$$\frac{1}{n^2} \left( \frac{j^0}{\sqrt{q}} \right)^2 = 1 + \frac{q^{kl} z_k^a z_l^b}{\rho'^2 (j^0)^2} p_a p_b. \quad (30)$$

This equation determines  $n$  as a function of  $p_a$ ,  $z^a$ , and  $z_k^a$ . The solution depends on the unknown function  $\rho(n)$  and is determined only implicitly, in general.

The identity (28) implies that

$$\frac{\partial j^k}{\partial z^a} = -j^0 x_a^k,$$

where  $x_a^k$  is the matrix inverse to  $z_k^a$ . As  $j^k$  depends linearly on  $\dot{z}^a$ , we have

$$j^k = -j^0 x_a^k \dot{z}^a$$

or

$$\dot{z}^a = -\frac{z_a^k j^k}{j^0}.$$

Substituting for  $j^k$  from

$$j^k = g^{kl} (j_l - N_{lj} j^0),$$

for  $j_l$  from Eq. (1) and for  $u_k$  from Eq. (29), we obtain that

$$\dot{z}^a = N \frac{\sqrt{q} n}{\rho' (j^0)^2} q^{kl} z_k^a z_l^b p_b + N^k z_k^a.$$

Then, Eqs. (12) and (21) yield

$$T_0^0 = -N \sqrt{q} \left( \rho + \frac{n}{\rho' (j^0)^2} q^{kl} z_k^a z_l^b p_a p_b \right) - N^k z_k^a p_a. \quad (31)$$

According to the formula (24), this determines the form of the Hamiltonian  $\mathcal{H}_m$ .

### 5. The Euler picture

At this stage, we have derived all of the important formulae of the Hamiltonian formalism in which the fluid is described by the functions  $z^a(x)$ ; these are Lagrange coordinates [16] and we can call the formalism the “*Lagrange picture*.” Sometimes the *Euler picture* is more practical, however. This can be obtained by the following canonical transformation. The new fields  $x^k(z, t)$  (Euler coordinates) are defined by

$$x^k(z(x, t), t) = x^k, \quad \forall x^k, t \quad (32)$$

and the conjugate momenta  $P_k$  by

$$P_k(z, t) := -X(z, t)z_k^a(x(z, t), t)p_a(x(z, t), t), \quad (33)$$

where

$$X := \det\left(\frac{\partial x^k}{\partial z^a}\right).$$

One easily checks that Eqs. (32) and (33) define a canonical transformation.

Let us first derive some useful relations. By differentiating Eq. (32) with respect to  $t$  at constant  $x^k$ , we obtain

$$\dot{x}^k = -x_a^k \dot{z}^a, \quad (34)$$

where

$$x_a^k := \frac{\partial x^k}{\partial z^a}.$$

The derivative of the same equation with respect to  $x^k$  at constant  $t$  gives

$$x_a^k z_l^a = \delta_l^k. \quad (35)$$

If the field  $z^a(x, t)$  is changed to  $\tilde{z}^a(x, t)$ , then  $x^k(z, t)$  is changed to  $\tilde{x}^k(z, t)$  satisfying

$$\tilde{x}^k(\tilde{z}(x, t), t) = x^k, \quad \forall x^k, t.$$

Thus, if  $\tilde{z}^a(x, t) = z^a(x, t) + \delta z^a(x, t)$ , the above equation implies that

$$\delta_* x^k = -x_a^k \delta z^a. \quad (36)$$

The symbol  $\delta_*$  is to stress and remind us that this variation is of a different kind than  $\delta$ , if applied to fields: the former is obtained by comparing the values of the field at the same point of the matter space, that is, at different points of the spacetime; the latter compares the values of the field at the same point of the spacetime.

With the help of the above relations, we can transform all formulas of the Hamiltonian formalism. Let us start with Eq. (23). First, the inverse transformation for the momenta follows from Eq. (33):

$$p_a(x, t) = -X^{-1}(x, t)x_a^k(z(x, t), t)P_k(z(x, t), t). \quad (37)$$

The time derivative of this equation at constant  $x^k$  can be calculated with the result

$$\begin{aligned} \dot{p}_a &= X^{-1}(x_a^k x_b^m z_{lm}^n P_k \dot{x}^l + z_l^b x_{ab}^k P_k \dot{x}^l + x_a^k z_l^b P_k \partial_b \dot{x}^l \\ &\quad + x_a^k z_l^b \partial_b P_k \dot{x}^l - P_l \partial_a \dot{x}^l - x_a^k \dot{P}_k), \end{aligned} \quad (38)$$

where we introduced the abbreviation

$$z_{kl}^a := \frac{\partial^2 z^a}{\partial x^k \partial x^l}$$

and similarly  $x_{ab}^k$ . An analogous formula holds for the variation  $\delta p_a$ , one just has to replace dots by  $\delta$ 's. Employing these equations, we obtain after a lengthy but straightforward calculation

$$\begin{aligned} \delta p_a \dot{z}^a - \dot{p}_a \delta z^a &= X^{-1}(\delta_* P_k \dot{x}^k - \dot{P}_k \delta_* x^k) \\ &\quad + X^{-1} \partial_a (P_k z_l^a \delta_* x^k \dot{x}^l) \\ &\quad - X^{-1} \partial_a (P_l z_k^a \delta_* x^k \dot{x}^l), \end{aligned}$$

or

$$\begin{aligned} \int_{S_t} d^3 x (\delta p_a \dot{z}^a - \dot{p}_a \delta z^a) &= \int_{Z^-} d^3 z (\delta_* P_k \dot{x}^k - \dot{P}_k \delta_* x^k) \\ &\quad + \int_{Z^-} d^3 z \partial_a ((P_k z_l^a \\ &\quad - P_l z_k^a) \delta_* x^k \dot{x}^l). \end{aligned} \quad (39)$$

Then, we transform the second integral on the RHS of Eq. (23):

$$\begin{aligned} &\int_{x^3=x^3(t)} d^2 x p_a (\dot{z}^a \delta x^3 - \dot{x}^3 \delta z^a) \\ &= \int_{S_t} d^3 x \partial_l (p_a (\dot{z}^a \delta_* x^l - \dot{x}^l \delta z^a)) \\ &= \int_{Z^-} d^3 z X z_l^b \partial_b (X^{-1} P_k (\dot{x}^k \delta_* x^l - \dot{x}^l \delta_* x^k)); \end{aligned}$$

we have used Eqs. (34), (36), (35), and (37). Because of the identity  $\partial_b (X z_l^b) = 0$ , we obtain finally

$$\begin{aligned} \int_{x^3=x^3(t)} d^2 x p_a (\dot{z}^a \delta x^3 - \dot{x}^3 \delta z^a) &= - \int_{Z^-} d^3 z \partial_b ((P_k z_l^b \\ &\quad - P_l z_k^b) \delta_* x^k \dot{x}^l). \end{aligned} \quad (40)$$

Equations (23), (39), and (40) imply

$$\begin{aligned} \delta \mathcal{H}_m &= \int_{Z^-} d^3 z (\delta_* P_k \dot{x}^k - \dot{P}_k \delta_* x^k) \\ &\quad + \frac{1}{2} \int_{Z^-} d^3 z X T^{\mu\nu} \delta g_{\mu\nu}(x(x, t), t). \end{aligned} \quad (41)$$

Thus, the symplectic form has no surface term in the Euler picture. The variation of the metric in the last term on the RHS is independent of the other variations, and it is defined by comparing values of the metric at the same spacetime points.

Let us suppose that  $\delta g_{\mu\nu}(x) = 0$ , and let us introduce the transformed Hamiltonian density  $H'_m$  by  $H'_m = X H_m$ , so that

$$\mathcal{H}_m = \int_{Z^-} d^3 z H'_m.$$

Then,

$$\delta \mathcal{H}_m = \int_{Z^-} d^3 z \delta_* H'_m,$$

and we have

$$\int_{Z^-} d^3z \delta_* H'_m = \int_{Z^-} d^3z (\delta_* P_k \dot{x}^k - \dot{P}_k \delta_* x^k). \quad (42)$$

In this form, the variational formula is suitable for derivation of the canonical equations. To this aim, let us calculate  $\delta_* H'_m$ ;  $H'_m$  is of the form  $H'_m(x^k, x^k_a, P_k)$ , hence

$$\begin{aligned} \delta_* H'_m &= \left( \frac{\partial H'_m}{\partial x^k} - \partial_a \frac{\partial H'_m}{\partial x^k_a} \right) \delta_* x^k + \frac{\partial H'_m}{\partial P_k} \delta_* P_k \\ &+ \partial_a \left( \frac{\partial H'_m}{\partial x^k_a} \delta_* x^k \right). \end{aligned}$$

Thus, the field equations consist of the volume equations

$$\dot{x}^k = \frac{\partial H'_m}{\partial P_k}, \quad (43)$$

$$-\dot{P}_k = \frac{\partial H'_m}{\partial x^k} - \partial_a \frac{\partial H'_m}{\partial x^k_a}, \quad (44)$$

and the surface equations

$$F_a \left. \frac{\partial H'_m}{\partial x^k_a} \right|_{\Sigma_z} = 0. \quad (45)$$

Let us check that Eq. (45) is equivalent to Eq. (20). We have

$$H'_m = X H_m \left( z^a(x^l), z^a_k(x^l_b), p_a(z^a(x^l), z^a_k(x^l_b), P_k), g_{\mu\nu}(x^l) \right),$$

so that

$$\left. \frac{\partial H'_m}{\partial x^k_a} \right|_{x,P} = X z^a_k H'_m + X \left. \frac{\partial H'_m}{\partial z^b_l} \right|_{z,p} \frac{\partial z^b_l}{\partial x^k_a} + X \left. \frac{\partial H'_m}{\partial p_b} \right|_{z,z_k} \frac{\partial p_b}{\partial x^k_a}.$$

Equations (35) and (37) imply

$$\begin{aligned} \frac{\partial z^b_l}{\partial x^k_a} &= -z^b_l z^a_k, \\ \frac{\partial p_b}{\partial x^k_a} &= -p_b z^a_k + \delta^a_b p_c z^c_k, \end{aligned}$$

and we obtain easily

$$\left. \frac{\partial H'_m}{\partial x^k_a} \right|_{x,P} = X z^a_l \left( \left( H_m - \frac{\partial H_m}{\partial p_b} p_b \right) \delta^l_k - \frac{\partial H_m}{\partial z^b_l} z^b_k \right) + X \frac{\partial H_m}{\partial p_a} p_b z^b_k.$$

Application of Eqs. (25) and (26) as well as Eq. (23) simplify the expression to

$$\left. \frac{\partial H'_m}{\partial x^k_a} \right|_{x,P} = -X z^a_\mu T^{\mu k}_k. \quad (46)$$

Hence, Eq. (45) becomes

$$F_a z^a_\mu T^{\mu k}_k \Big|_{\Sigma_z} = 0,$$

which is equivalent to Eq. (20).

Finally, the transformed Eq. (30) reads

$$n^2 + \frac{n^2}{h^2 \rho^2} q^{kl} P_k P_l = \frac{h^2}{q X^2}. \quad (47)$$

For example, in the case of dust,  $\rho = \mu n$ , where  $\mu$  is a constant (rest mass per mole or particle) and Eq. (47) can be solved explicitly:

$$n = \frac{1}{\sqrt{qX}} \frac{\mu h^2}{\sqrt{\mu^2 h^2 + q^{kl} P_k P_l}}.$$

For dust, Eq. (31), which determines the form of the matter Hamiltonian, specializes to

$$T_0^0 = -N \sqrt{\mu^2 (J^0)^2 + q^{kl} z^a_k z^b_l p_a p_b} - N^k z^a_k p_a. \quad (48)$$

## B. Gravity becomes dynamical

In the previous sections, gravity was just an external field. Here, it will become dynamical: the metric  $g_{\mu\nu}(x)$  will satisfy Einstein's equations with the fluid stress energy tensor as a source.

### 1. Description of the system

The main problem which we shall meet is the following. If the Einstein equations are satisfied, the discontinuity in the distribution of the fluid leads to a discontinuity in derivatives of the metric. Thus, we must allow for such discontinuity from the very beginning. Moreover, a general variation of the metric, which includes a shift of the coordinates of the discontinuity, will have a jump of higher order than the metric itself: if the second derivatives of the metric have a jump, then the first derivative of its variation will have a jump, etc. If we write naively the usual expression for the variation of the action in the case of  $\delta$  function fluid distribution, then many terms in it look meaningless within the theory of distributions ( $\delta$  functions multiplied by discontinuous functions, etc.). Some ingenious calculation of all variations might still lead to meaningful expressions. Instead, we resort to a simple trick by which the problem is avoided: we fix the spacetime coordinates of the discontinuity surface  $\Sigma$ . In this way, the surface of the discontinuity is formally made to an "externally given" boundary. The fields  $z^a(x)$  and  $g_{\mu\nu}(x)$  will satisfy simple boundary conditions at  $\Sigma$ , and these conditions will be "inherited" by their variations. Such a strategy is possible within the general relativity, because it can be considered as a partial fixing of gauge. Indeed, any change of the coordinates of the discontinuity surface  $\Sigma$  can be considered as a superposition of a transformation of coordinates in a neighborhood of  $\Sigma$  keeping the physical fields fixed, and a change of the physical fields keeping the coordinates fixed; the first step is just a change of gauge. The dynamics of the surface is determined by the form of the metric near and at the surface.

To be more specific about the boundary conditions, let us choose the coordinates  $z^a$  in  $Z$  such that  $\Sigma_z$  is given by  $z^3 = 0$ , and the coordinates  $x^\mu$  in  $M$  such that  $\Sigma$  is defined by  $x^3 = 0$ . Thus, for the matter fields, we require

$$z^3|_{\Sigma} = 0, \quad \delta z^3|_{\Sigma} = 0. \quad (49)$$

It follows that



$$x^3|_{\Sigma_z} = 0, \quad \delta x^3|_{\Sigma_z} = 0 \quad (50)$$

in the Euler picture. We further assume the following.

*Condition 1.* The spacetime  $(M, g)$  is asymptotically flat and globally hyperbolic.

*Condition 2.* The metric  $g_{\mu\nu}(x)$  is piecewise  $C^\infty$  in  $M$ , the only discontinuity being that its second derivatives jump at  $\Sigma$ .

Then the variation  $\delta g_{\mu\nu}(x)$  satisfies analogous condition 2.

The total action for our fluid-gravity system is  $I = \bar{I}_m + I_g$ . Here

$$I_g = \frac{1}{16\pi G} \int_V d^4x \sqrt{|g|} R,$$

where  $G$  is the Newton constant and  $R$  is the curvature scalar of  $g_{\mu\nu}$ . The function  $R(x)$  can have a step discontinuity at  $\Sigma$ .  $\bar{I}_m$  is obtained from  $I_m$  of Eq. (8) after the following substitution:

$$z^3(x^\alpha, x^3=0) = 0, \quad \dot{z}^3(x^\alpha, x^3=0) = 0, \quad z_A^3(x^\alpha, x^3=0) = 0, \quad (51)$$

$\alpha = 0, 1, 2$  and  $A = 1, 2$ . Thus,  $\bar{I}_m$  contains less variables than  $I_m$ .

The integration volume  $V$  is chosen to be bounded by two Cauchy surfaces  $S_1$  and  $S_2$  and by a timelike surface  $\Sigma^+$  (which will be eventually pushed to the infinity). Let the coordinates  $x^\mu$  be adapted also to  $\Sigma^+$  so that  $\Sigma^+$  is defined by  $x^3 = r^+$ . The matter boundary  $\Sigma$  divides  $V$  into  $V^-$  and  $V^+$ , and  $S_i$  into  $S_i^-$  and  $S_i^+$ .

## 2. The variational formula

The variation of the gravity action  $I_g$  can be obtained from the following fundamental lemma that has been shown in [14].

*Lemma 1.* Let the integration volume  $V$  of the action  $I_g$  be bounded by two spacelike surfaces  $S_1$  and  $S_2$ , and by a smooth timelike surface  $\Sigma$ ; let  $x^\mu$  be some coordinates in  $V$ ,  $y^k$  in  $S_i$ ,  $\xi^\alpha$  in  $\Sigma$ , and  $\eta^A$  in  $\partial S_i = \Sigma \cap S_i$ . Then

$$\begin{aligned} \delta I_g = & -\frac{1}{16\pi G} \int_V d^4x G^{\mu\nu} \delta g_{\mu\nu} - \frac{1}{16\pi G} \int_{S_2} d^3y q_{kl} \delta \pi^{kl} \\ & + \frac{1}{16\pi G} \int_{S_1} d^3y q_{kl} \delta \pi^{kl} + \frac{1}{8\pi G} \int_{\partial S_2} d^2\eta \sqrt{\lambda} \delta \alpha \\ & - \frac{1}{8\pi G} \int_{\partial S_1} d^2\eta \sqrt{\lambda} \delta \alpha - \frac{1}{16\pi G} \int_\Sigma d^3\xi \gamma_{\alpha\beta} \delta Q^{\alpha\beta}, \end{aligned} \quad (52)$$

where

$$G^{\mu\nu} := \sqrt{|g|} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right), \quad (53)$$

$R^{\mu\nu}$  is the Ricci tensor of the metric  $g_{\mu\nu}$ ,  $q_{kl}$  is the induced metric on  $S_i$  written with respect to the coordinates  $y^k$ ,  $q$  its determinant,

$$\pi^{kl} := \sqrt{q} (K q^{kl} - K^{kl}),$$

$$K_{kl} := -n_{\mu;\nu} \frac{\partial x^\mu}{\partial y^k} \frac{\partial x^\nu}{\partial y^l},$$

$$K = q^{kl} K_{kl},$$

$n_\mu$  is the future directed unit normal to  $S_i$  so that  $K_{kl}$  is the second fundamental form of the surface  $S_i$ ,  $\gamma_{\alpha\beta}$  is the metric induced on  $\Sigma$  written with respect to the coordinates  $\xi^\alpha$ ,  $\gamma$  its determinant

$$Q^{\alpha\beta} := \sqrt{|\gamma|} (L \gamma^{\alpha\beta} - L^{\alpha\beta}),$$

$$L_{\alpha\beta} := \tilde{m}_{\mu;\nu} \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta},$$

$$L = \gamma^{\alpha\beta} L_{\alpha\beta}, \quad (54)$$

$\tilde{m}_\mu$  is the external (with respect to the volume  $V$ ) unit normal to  $\Sigma$  so that  $L_{\alpha\beta}$  is the second fundamental form of  $\Sigma$ ,  $\lambda$  is the determinant of the two-metric  $\lambda_{AB}$  induced on  $\partial S_i$  written with respect to the coordinates  $\eta^A$  and  $\alpha$  is defined by

$$\alpha := -\operatorname{arcsinh}(g_{\mu\nu} n^\mu \tilde{m}^\nu).$$

Lemma 1 is completely general, independent of the form and description of the matter; it determines the ‘‘gravitational part’’ of the variation formula that we are going to derive. For the ‘‘matter part,’’ we can use the formula (16) in which the surface integral along  $\Sigma$  is left out. Indeed,  $F(z) = z^3$  for our special coordinates and the boundary condition (49) gives  $F_a \delta z^a = 0$ . Hence,

$$\begin{aligned} \delta I_m = & \int_{V^-} d^4x \left( \frac{\partial \bar{L}_m}{\partial z^a} - \partial_\mu \frac{\partial \bar{L}_m}{\partial z^a_\mu} \right) \delta z^a + \int_{S_2^-} dS \frac{\partial \bar{L}_m}{\partial z^a_0} \delta z^a \\ & - \int_{S_1^-} dS \frac{\partial \bar{L}_m}{\partial z^a_0} \delta z^a + \frac{1}{2} \int_{V^-} d^4x T^{\mu\nu} \delta g_{\mu\nu}. \end{aligned} \quad (55)$$

Our next task is to rewrite the surface integrals in Eq. (55) in a covariant way. We define, in analogy with Eq. (21),

$$\bar{p}_a := \frac{\partial \bar{L}_m}{\partial z^a}.$$

By a similar argument as in Sec. II A 3,  $\bar{p}_a$  are surface densities, and the covariant form of the integrals is

$$\begin{aligned} & \int_{S_2^-} dS \frac{\partial \bar{L}_m}{\partial z^a_0} \delta z^a - \int_{S_1^-} dS \frac{\partial \bar{L}_m}{\partial z^a_0} \delta z^a \\ & = \int_{S_2^-} d^3y \bar{p}_a \delta z^a - \int_{S_1^-} d^3y \bar{p}_a \delta z^a. \end{aligned} \quad (56)$$

The relations between the old and new matter momenta will play some role. They can be summarized as follows. In  $V^-$ , we simply have

$$\bar{p}_a = p_a, \quad (57)$$

whereas at  $\Sigma$ ,

$$\bar{p}_A = p_A, \quad p_3 = p(\bar{p}_A, z^a, z_k^a), \quad (58)$$

where  $p$  is some function of the variables indicated. Equation (57) and the first expression of Eq. (58) follow directly from the definitions if the substitution (51) is made in the expressions on the RHSs. As  $\bar{L}_m$  does not depend on  $\dot{z}^3$  at the boundary, there is *no*  $\bar{p}_3|_\Sigma$ ;  $p_3|_\Sigma$  as given by Eq. (21) with the substitutions (51); it is, however, nonzero and it can be expressed as in the second expression of Eq. (58). Let us give a proof. The solution of Eq. (21) with respect to  $\dot{z}^a$  reads

$$\dot{z}^a = \dot{z}^a(p_a, z^a, z_k^a).$$

At  $\Sigma$ , we must have  $\dot{z}^3 = 0$ , so we obtain one constraint for the functions  $p_a|_\Sigma$ :

$$\dot{z}^3(p_a, z^a, z_k^a) = 0. \quad (59)$$

This can be solved for  $p_3|_\Sigma$ ; the second expression of Eq. (58) is the solution.

As an example, we work out the explicit form of Eq. (59) for dust. We easily obtain from Eq. (1)

$$j^k = -hX^{-1}x_a^k \dot{z}^a,$$

if we observe that the intermediately resulting terms can be expressed by means of the  $z_k^a$  derivatives of the determinant of the matrix  $z_k^a$ . Equation (3) yields

$$n = \frac{h}{X\sqrt{|g|}} \sqrt{-g_{00} + 2\bar{N}_a \dot{z}^a - \bar{q}_{ab} \dot{z}^a \dot{z}^b},$$

where

$$\begin{aligned} \bar{N}_a &:= N_k x_a^k, \\ \bar{q}_{ab} &:= g_{kl} x_a^k x_b^l. \end{aligned}$$

Then, for  $\rho = \mu n$ , we have from Eqs. (21) and (4)

$$p_a = -\mu \frac{h^2}{X^2 \sqrt{|g|}} \frac{1}{n} (\bar{N}_a - \bar{q}_{ab} \dot{z}^b),$$

so that

$$\dot{z}^a = \bar{q}^{ab} \left( \bar{N}_b + \frac{X^2 \sqrt{|g|}}{\mu h^2} n p_b \right),$$

and the desired constraint (59) reads

$$\bar{N}^3 + n \frac{X^2 N \sqrt{q}}{\mu h^2} \bar{q}^{3a} p_a = 0, \quad (60)$$

where

$$\bar{q}^{ab} := q^{kl} z_k^a z_l^b$$

and

$$\bar{N}^a := N^k z_k^a.$$

For  $n$ , we have to insert from Eq. (30)

$$\frac{1}{n} = \frac{X^2 \sqrt{q}}{\mu h^2} \sqrt{\mu^2 h^2 X^{-2} + \bar{q}^{ab} p_a p_b}.$$

Thus, the constraint (60) can be written as a quadratic equation for  $p_3$ , whose general solution is

$$p_{3\pm} = -\frac{\bar{q}^{3A}}{\bar{q}^{33}} p_A \pm \bar{N}^3 \sqrt{\frac{\bar{\lambda}^{AB} p_A p_B + \mu^2 h^2 X^{-2}}{\bar{q}^{33} (N^2 \bar{q}^{33} - (\bar{N}^3)^2)}},$$

where

$$\bar{\lambda}^{AB} = \lambda^{CD} z_C^A z_D^B$$

and

$$\lambda_{AB} = q_{AB}$$

is the metric induced on  $\partial S$ . Only the lower sign is admissible, as we can easily see from Eq. (60). Hence, finally,

$$p_3|_\Sigma = -\frac{\bar{q}^{3A}}{\bar{q}^{33}} p_A - \bar{N}^3 \sqrt{\frac{\bar{\lambda}^{AB} p_A p_B + \mu^2 h^2 X^{-2}}{\bar{q}^{33} (N^2 \bar{q}^{33} - (\bar{N}^3)^2)}}.$$

Equations (52), (55), and (56) imply the final *variational formula* for our gravity-fluid system

$$\begin{aligned} \delta I = & \int_{V^-} d^4 x \left( \frac{\partial \bar{L}_m}{\partial z^a} - \partial_\mu \frac{\partial \bar{L}_m}{\partial z_\mu^a} \right) \delta z^a - \frac{1}{16\pi G} \int_V d^4 x G^{\mu\nu} \delta g_{\mu\nu} \\ & + \int_{S_2^-} d^3 y \bar{p}_a \delta z^a - \int_{S_1^-} d^3 y \bar{p}_a \delta z^a \\ & - \frac{1}{16\pi G} \int_{S_2} d^3 y q_{kl} \delta \pi^{kl} + \frac{1}{16\pi G} \int_{S_1} d^3 y q_{kl} \delta \pi^{kl} \\ & + \frac{1}{8\pi G} \int_{\partial S_2} d^2 \eta \sqrt{\lambda} \delta \alpha - \frac{1}{8\pi G} \int_{\partial S_1} d^2 \eta \sqrt{\lambda} \delta \alpha \\ & - \frac{1}{16\pi G} \int_\Sigma d^3 \xi \gamma_{\alpha\beta} \delta Q^{\alpha\beta} + \frac{1}{2} \int_{V^-} d^4 x T^{\mu\nu} \delta g_{\mu\nu}. \quad (61) \end{aligned}$$

From the formula (61) we can read off the field equations; within  $V^-$ , we have

$$\frac{\partial L_m}{\partial z^a} - \partial_\mu \frac{\partial L_m}{\partial z_\mu^a} = 0, \quad (62)$$

$$G^{\mu\nu} = 8\pi G T^{\mu\nu} \quad (63)$$

and within  $V^+$ , we have

$$G^{\mu\nu} = 0. \quad (64)$$

Apparently, the surface equation (18) has been lost. However, using the boundary condition 2, we easily find that  $G_\mu^\perp$  is continuous at  $\Sigma$ . Hence,

$$\lim_{x^3=0^-} G_\mu^\perp = 0$$

and the surface field equation follows from Eq. (63). We also observe that the dynamics can be completely shifted to the gravity if the ideal fluid is described by comoving coordinates everywhere in  $V^-$ .

Putting everything together, we obtain in a way analogous to that in Sec. II A 3 that the field equations are equivalent to the following relation (which is an analogon of Eq. (5.16) of [14]):

$$\begin{aligned} \delta\mathcal{L} = & \int_{S^-} d^3y (\bar{p}_a \delta z^a) \cdot - \frac{1}{16\pi G} \int_S d^3y (q_{kl} \delta \pi^{kl}) \cdot \\ & + \frac{1}{8\pi G} \int_{\partial S} d^2\eta (\sqrt{\lambda} \delta \alpha) \cdot - \frac{1}{16\pi G} \int_{\partial S} d^2\eta \gamma_{\alpha\beta} \delta Q^{\alpha\beta}, \end{aligned} \tag{65}$$

where  $\mathcal{L}$  is the Lagrangian of the system,

$$\mathcal{L} := \int_S dS \bar{L}.$$

Equation (65) is the *variation formula* for our system.

### 3. The Hamiltonian formalism

Equation (65) is a good starting point for the Legendre transformation to a Hamiltonian formalism. We define one of the conceivable total Hamiltonians for our system by

$$\begin{aligned} \check{\mathcal{H}} := & \int_{S^-} d^3y \bar{p}_a \dot{z}^a - \frac{1}{16\pi G} \int_S d^3y q_{kl} \dot{\pi}^{kl} \\ & + \frac{1}{8\pi G} \int_{\partial S} d^2\eta \sqrt{\lambda} \dot{\alpha} - \mathcal{L}. \end{aligned} \tag{66}$$

Then the field equations can be obtained from the *variation formula*

$$\begin{aligned} \delta\check{\mathcal{H}} = & \int_{S^-} d^3y (\dot{z}^a \delta \bar{p}_a - \dot{\bar{p}}_a \delta z^a) + \frac{1}{16\pi G} \int_S d^3y (\dot{q}_{kl} \delta \pi^{kl} \\ & - \dot{\pi}^{kl} \delta q_{kl}) + \frac{1}{8\pi G} \int_{\partial S} d^2\eta \frac{1}{2\sqrt{\lambda}} (\dot{\alpha} \delta \lambda - \lambda \delta \alpha) \\ & + \frac{1}{16\pi G} \int_{\partial S} d^2\eta \gamma_{\alpha\beta} \delta Q^{\alpha\beta}. \end{aligned} \tag{67}$$

To find the explicit form of the Hamiltonian we use the following lemma.

*Lemma 2.* In the adapted coordinates  $x^\mu$  defined in Sec. II B 1, the following identity holds at any  $S = S_t$  in the volume  $V$ :

$$\begin{aligned} & \int_S dS q_{kl} \dot{\pi}^{kl} - 2 \int_{\partial S} d\partial S \sqrt{\lambda} \dot{\alpha} \\ & = -2 \int_S dS \sqrt{|g|} R_0^0 - 2 \int_{\partial S} d\partial S \sqrt{|\gamma|} L_0^0, \end{aligned} \tag{68}$$

where  $L^{\alpha\beta}$  is the second fundamental form of the boundary  $\partial V$  corresponding to the normal oriented outwards from  $V$ ,  $dS = dx^1 dx^2 dx^3$  and  $d\partial S = dx^1 dx^2$ .

The derivation of this identity is given in [14] [Eq. (6.3)]; the form (68) is easily obtained if one uses the equation  $Q^{AB} g_{AB} - Q^{00} g_{00} = \sqrt{|\gamma|} L_0^0$ , which follows from Eq. (54). We also observe that

$$\int_{S^-} dS \bar{p}_a \dot{z}^a - \bar{\mathcal{L}}_m = \bar{\mathcal{H}}_m,$$

where  $\bar{\mathcal{H}}_m$  is obtained from  $\mathcal{H}_m$  as given in Sec. II A 4 by the substitutions (51), (57), and (58). Thus, the substitution for  $p_3$  is discontinuous near  $\Sigma$ . It might, therefore, seem that the corresponding Hamiltonian density  $\bar{H}_m$  would not be continuous at  $\Sigma$ , but this is not true. The reason is that  $H_m|_\Sigma$  does not depend on  $p_3$  if the conditions (49) are satisfied:

$$\left. \frac{\partial H_m}{\partial p_a} \right|_\Sigma = \dot{z}^3|_\Sigma = 0.$$

Hence, we have from Eq. (24)

$$\bar{\mathcal{H}}_m = - \int_S dS T_0^0.$$

Collecting all results, we obtain finally

$$\check{\mathcal{H}} = \int_S dS \left( -T_0^0 + \frac{1}{8\pi G} G_0^0 \right) + \frac{1}{8\pi G} \int_{\partial S} d\partial S \sqrt{|\gamma|} L_0^0. \tag{69}$$

This is the full ‘‘off-shell’’ Hamiltonian of our system. If the Einstein equations hold, its value is just the surface integral.

Let us rewrite the volume integral in a covariant form. Any tensor density  $W_\nu^\mu$  satisfies the identity

$$W_0^0 = \frac{1}{g^{00}} (W^{00} - g^{0k} W_k^0).$$

The unit future-oriented normal covector  $n_\mu$  to  $S$  has the components  $n_\mu = -N \delta_\mu^0$  with respect to the adapted coordinates; it follows that

$$W_0^0 = -\sqrt{q} (N w^{\perp\perp} + N^k w_k^\perp), \tag{70}$$

where

$$w^{\perp\perp} := \frac{1}{\sqrt{|g|}} W^{\mu\nu} n_\mu n_\nu,$$

$$w_k^\perp := \frac{1}{\sqrt{|g|}} W_k^\mu n_\mu.$$

Thus, the volume integral can be written as

$$\int_S dS \left( -T_0^0 + \frac{1}{8\pi G} G_0^0 \right) = \frac{1}{8\pi G} \int_S d^3y \sqrt{q} (NC + N^k C_k), \tag{71}$$

where

$$C := -\frac{1}{\sqrt{|g|}}(G^{\mu\nu} - 8\pi GT^{\mu\nu})n_\mu n_\nu \quad (72)$$

and

$$C_k := -\frac{1}{\sqrt{|g|}}(G_k^\mu - 8\pi GT_k^\mu)n_\mu \quad (73)$$

are the *super-Hamiltonian* and the *supermomentum* of our system (or scalar and vector constraint functions).

Let us return to the formula (67), which not only implies the field equations, if we perform the variation on the LHS and compare the result with the RHS, but it also determines the so-called *control mode* (see, e.g., [13]) and the type of boundary value problem for the field equations. We observe that this mode is a kind of ‘‘curvature-control-mode’’; it amounts to keeping fixed (controlling) the external curvature  $Q^{\alpha\beta}$  at the boundary  $\Sigma^+$  (see [14]). Such a boundary problem for the Einstein equations has not been studied. To pass to a more natural, and in fact more conventional, approach, we have to perform an additional Legendre transformation at the boundary [14]:

$$\mathcal{H} = \check{\mathcal{H}} - \frac{1}{16\pi G} \int_{\partial S} d^2 \eta \gamma_{AB} Q^{AB}$$

so that we have finally, also using Eq. (71),

$$\mathcal{H} = \frac{1}{8\pi G} \int_S dS \sqrt{q} (NC + N^k C_k) - \frac{1}{16\pi G} \int_{\partial S} d\partial S Q^{00} \gamma_{00} \quad (74)$$

and

$$\begin{aligned} \delta\mathcal{H} = & \int_{S^-} dS (\dot{z}^a \delta \bar{p}_a - \dot{\bar{p}}_a \delta z^a) + \frac{1}{16\pi G} \int_S dS (\dot{q}_{kl} \delta \pi^{kl} \\ & - \dot{\pi}^{kl} \delta q_{kl}) + \frac{1}{8\pi G} \int_{\partial S} d\partial S (\dot{\alpha} \delta \sqrt{\lambda} - \sqrt{\lambda} \delta \alpha) \\ & + \frac{1}{16\pi G} \int_{\partial S} d\partial S (\gamma_{00} \delta Q^{00} + 2\gamma_{0A} \delta Q^{0A} - Q^{AB} \delta \gamma_{AB}). \end{aligned} \quad (75)$$

The last surface integral in Eq. (74) will result in the Arnowitt-Deser-Misner (ADM) energy, if the limit  $\Sigma^+ \rightarrow \infty$  is carefully performed (this has been shown in [14]). The last one in Eq. (75) defines the way of control:  $Q^{00}$ ,  $Q^{0A}$ , and  $\gamma_{AB}$  are kept fixed at the boundary.

The transformation to the Euler picture in the matter part of the Hamiltonian is straightforward; let us denote the resulting Hamiltonian density by  $\bar{H}'_m$ . Most formulas of Sec. II A 5 will result in the analogous formulas for  $\bar{H}'_m$ , if the substitutions (51), (57), and (58) are performed in them. For example, we have to use the modified formula (25), which will read at  $\Sigma$

$$T_A^0 = -\bar{p}_B z_A^B, \quad T_3^0 = -\bar{p}_B z_3^B - p(\bar{p}, z) z_3^3,$$

etc. Only those formulas that contain derivatives of  $H'_m$  with respect to the variables which are not contained in  $\bar{H}'_m$  (such as  $p_3|_\Sigma$  and  $x_A^3|_\Sigma$ ) need some care to be properly transformed.

We observe finally that the field equations derived from Eq. (75) (or an analogous equation of the Euler picture) will have the form of *canonical equations*. This is interesting, because Eq. (45) does not seem to have such a form. The variations in Eq. (75) (or those of the Euler picture) must satisfy the boundary conditions (49) [or (50)] and will, therefore, lead to trivial surface equations; the volume equations alone *have* the canonical form. For example, the would be counterpart of Eq. (45) originates from the term

$$\left. \frac{\partial \bar{H}'_m}{\partial x_3^k} \delta x^k \right|_\Sigma$$

in the variation of  $\bar{H}'_m$ . As  $\delta x^3|_\Sigma = 0$ , the only equation which is implied thereby reads

$$\left. \frac{\partial \bar{H}'_m}{\partial x_3^A} \right|_\Sigma = 0.$$

Further, a counterpart of Eq. (46) is valid for this derivative, so the above equation is equivalent to

$$z_\mu^3 T_A^\mu|_\Sigma = 0.$$

Moreover,  $z_B^3|_\Sigma = 0$ , so the equation reduces to  $T_A^3|_\Sigma = 0$ . However, these components of  $T_\nu^\mu$  vanish identically at  $\Sigma$ , because  $u^3|_\Sigma = 0$  [cf. Eq. (6)]. Thus, there is no surface field equation.

### III. FLUID SHELL

In this section, we are going to describe the dynamics of a  $\delta$ -function distribution of fluid. The matter will be coupled to the dynamical gravity from the start. We shall consider a special case: just one shell in vacuum; a generalization to more shells surrounded by a piecewise smooth matter is straightforward as long as the shells do not intersect.

#### A. Action in the volume form

The shell can be represented as a  $\delta$ -function singularity in the mole density  $h$ . The action can then be written as a volume integral of the same form as for a regular distribution of matter. This holds also for the gravitational part. We shall give a more detailed description of this volume form and then transform it to a combinations of volume and surface integrals, where no  $\delta$  functions will feature. This may be useful, because much more general choice of coordinates is then allowed. Indeed, the  $\delta$ -function method works only if the coordinates are such that the corresponding components of the four-metric are continuous. Further, the coordinate position of the shell—the three-surface in the spacetime  $M$  and the two-surface in the matter space  $Z$ —will be kept fixed. Here, everything that has already been said in Sec. II B 1 about this point can be repeated.

The matter space  $Z$  remains, therefore, three dimensional first. Let  $\Sigma_z$  be a two-dimensional surface in  $Z$  on which the matter is concentrated. Let  $z^a$  be coordinates adapted to  $\Sigma_z$  so that the equation  $z^3 = z_0^3$  determines  $\Sigma_z$ . Such coordinates are determined up to a transformation

$$\begin{aligned} z'^1 &= z'^1(z^a), & z'^2 &= z'^2(z^a), \\ z'^3 &= z'^3(z^a), & z_0'^3 &= z'^3(z^A, z_0^3), \end{aligned}$$

where  $z_0'^3$  is a constant independent of  $z^A$ . Then,

$$\frac{\partial(z^1, z^2, z^3)}{\partial(\widetilde{z}^1, \widetilde{z}^2, \widetilde{z}^3)} = \frac{\partial(z^1, z^2)}{\partial(\widetilde{z}^1, \widetilde{z}^2)} \frac{\partial z^3}{\partial \widetilde{z}^3}.$$

We decompose the molar density  $h$  in the adapted coordinates as follows:

$$h = h_s \delta(z^3 - z_0^3), \quad (76)$$

where  $h_s(z^1, z^2)$  is a two-dimensional density on  $\Sigma_z$ . If we change the adapted coordinates, we have

$$\begin{aligned} \widetilde{h} &= \frac{\partial(z^1, z^2, z^3)}{\partial(\widetilde{z}^1, \widetilde{z}^2, \widetilde{z}^3)} h = h_s \frac{\partial(z^1, z^2)}{\partial(\widetilde{z}^1, \widetilde{z}^2)} \delta(z^3 - z_0^3) \frac{\partial z^3}{\partial \widetilde{z}^3} \\ &= \widetilde{h}_s \delta(\widetilde{z}^3 - \widetilde{z}_0^3), \end{aligned}$$

so the decomposition is independent of the choice of adapted coordinates, and defines, in fact, a two-dimensional matter space  $\Sigma_z$  with a two-dimensional mole density  $h_s$ ; later, we will pass to this space.

In the spacetime  $M$  with coordinates  $x^\mu$ , the matter fields are  $z^\alpha(x^\mu)$ ; the shell occupies a three-dimensional surface  $\Sigma$ , which can be described by the embedding functions  $x^\mu = x^\mu(\xi^\alpha)$ ,  $\alpha = 0, 1, 2$ , or by means of the equation  $z^3(x^\mu) = z_0^3$ . Later, we will pass to the matter fields  $z^A = z^A(\xi^\alpha)$ , where  $z^A(\xi^\alpha) = z^A(x^\mu(\xi^\alpha))$ .

The gravitational field is described by the metric  $g_{\mu\nu}(x)$ ; we require the following conditions.

*Condition 1'*. The spacetime  $(M, g)$  is asymptotically flat and globally hyperbolic.

*Condition 2'*. There are coordinates  $x^\mu$  in a neighborhood of each point of  $\Sigma$  such that the metric  $g_{\mu\nu}(x)$  is  $C^0$  everywhere, piecewise  $C^\infty$ , so that the only discontinuity is a jump in the first derivatives at  $\Sigma$ .

The second derivatives of the metric will then have a  $\delta$ -function singularity at  $\Sigma$  so that the Einstein equations can be satisfied.

The total action for the system consisting of the shell and the gravitational field can then be written in the form

$$I = \frac{1}{16\pi G} \int_M d^4x \sqrt{|g|} R - \int_M d^4x \sqrt{|g|} n e(n). \quad (77)$$

One can use this volume form of the action to derive the equations of motion. However, there is also a ‘‘surface form’’ of the action which we will derive in the next section.

## B. The surface form of the action

The coordinates satisfying condition 2' are not uniquely determined. We can use this freedom for the derivation of the surface form; the tool will be the adapted spacetime coordinates  $x^\mu$ , defined by the property

$$x^3 = z^3(x), \quad \xi^\alpha = x^\alpha|_\Sigma.$$

Then, the induced metric  $\gamma_{\alpha\beta}$  on the shell is

$$\gamma_{\alpha\beta} = g_{\alpha\beta}|_\Sigma \quad (78)$$

and its determinant  $\gamma$  is related to the determinant  $g$  of the four-metric  $g_{\mu\nu}$  by

$$\gamma = g g^{33}. \quad (79)$$

### 1. Matter action

Formula (1) together with Eq. (76) give

$$j^\mu = h_s \delta(z^3 - z_0^3) \epsilon^{\mu\nu\rho\sigma} z_\nu^1 z_\rho^2 z_\sigma^3.$$

Hence,  $j^3 = 0$ , and as  $\delta(z^3 - z_0^3) = z_0^3 \delta(x^3 - x_0^3)$ ,  $j^\alpha$  can be written as

$$j^\alpha = j_s^\alpha \delta(x^3 - x_0^3). \quad (80)$$

The mole density  $n$  can be calculated from Eq. (3). We obtain

$$\begin{aligned} \sqrt{-g_{\mu\nu} j^\mu j^\nu} &= \sqrt{-g_{\alpha\beta} j_s^\alpha j_s^\beta} \delta(x^3 - x_0^3) \\ &= \sqrt{-\gamma_{\alpha\beta} j_s^\alpha j_s^\beta} \delta(x^3 - x_0^3), \end{aligned}$$

and Eq. (79) yields

$$n = \frac{1}{\sqrt{|\gamma|}} \sqrt{-\gamma_{\alpha\beta} j_s^\alpha j_s^\beta} \sqrt{g^{33}} \delta(x^3 - x_0^3).$$

We define the surface mole density  $n_s$  by

$$n_s = \frac{1}{\sqrt{|\gamma|}} \sqrt{-\gamma_{\alpha\beta} j_s^\alpha j_s^\beta} \quad (81)$$

so that

$$n = n_s \sqrt{g^{33}} \delta(x^3 - x_0^3), \quad (82)$$

where  $\sqrt{g^{33}} \delta(x^3 - x_0^3)$  is already a scalar with respect the reparametrizations of  $x^3$ . For the velocity  $u^\mu$ , we have the expansion

$$u^\mu = v^\alpha e_\alpha^\mu, \quad (83)$$

where

$$e_\alpha^\mu := \frac{\partial x^\mu}{\partial \xi^\alpha},$$

so that

$$\gamma_{\alpha\beta} v^\alpha v^\beta = -1$$

and

$$j_s^\alpha = \sqrt{|\gamma|} n_s v^\alpha, \quad j_s^\alpha = h_s \epsilon^{\alpha\beta\gamma} z_\beta^1 z_\gamma^2.$$

Then the transcription of the matter Lagrangian density is straightforward:

$$L_m = -\sqrt{-g} n e(n) = -\sqrt{|\gamma|} n_s e_s(n_s) \delta(x^3 - x_0^3),$$

where  $e_s(n_s)$  is the energy per mole of the shell matter. We define the surface Lagrange density  $L_s$  and the surface mass density  $\rho_s$  by

$$L_m := L_s \delta(x^3 - x_0^3), \quad \rho_s := n_s e_s(n_s),$$

so that

$$L_s = -\sqrt{|\gamma|} \rho_s(n_s). \quad (84)$$

If we perform the trivial integration over  $x^3$  in  $I_m$ , the matter action becomes a surface integral

$$I_m = - \int_\Sigma d^3 \xi \sqrt{|\gamma|} n_s e_s(n_s). \quad (85)$$

This expression is invariant under the transformation of coordinates at the shell.

The action  $I_m$  can be varied with respect to the shell metric  $\gamma_{\alpha\beta}$  with the result

$$\delta I_m = \int_\Sigma d^3 \xi \left( -\frac{1}{2} \sqrt{|\gamma|} \gamma^{\alpha\beta} \rho_s - \sqrt{|\gamma|} \rho_s' \frac{\partial n_s}{\partial \gamma_{\alpha\beta}} \right) \delta \gamma_{\alpha\beta}.$$

In analogy with the formula (5), we obtain

$$\frac{\partial n_s}{\partial \gamma_{\alpha\beta}} = -\frac{1}{2} n_s (\gamma_{\alpha\beta} + v^\alpha v^\beta), \quad (86)$$

and we have

$$\delta I_m = \frac{1}{2} \int_\Sigma d^3 \xi T_s^{\alpha\beta} \delta \gamma_{\alpha\beta}, \quad (87)$$

where  $T_s^{\alpha\beta}$  is the surface stress-energy tensor,

$$T_s^{\alpha\beta} := \sqrt{|\gamma|} (\rho_s v^\alpha v^\beta - \sigma (\gamma_{\alpha\beta} + v^\alpha v^\beta)), \quad (88)$$

and

$$\sigma := n_s \rho_s' - \rho_s = -n_s^2 e_s'$$

is the surface tension (negative two-dimensional pressure). Moreover, it holds that

$$T^{\mu\nu} = \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta} T_s^{\alpha\beta} \sqrt{g^{33}} \delta(x^3 - x_0^3).$$

We obtain easily relations analogous to Eqs. (11)–(13):

$$T_s^{\alpha\beta}(\xi) = 2 \frac{\partial L_s}{\partial \gamma_{\alpha\beta}(\xi)} \quad (89)$$

[this is a form of Eq. (88)];

$$\frac{\partial L_s}{\partial z_\alpha^A} z_\beta^A + 2 \frac{\partial L_s}{\partial \gamma_{\alpha\gamma}} \gamma_{\beta\gamma} = L_s \delta_\beta^\alpha, \quad (90)$$

because  $L_s$  is a three-density on  $\Sigma$ , and the Noether identity

$$T_{s\beta}^\alpha = L_s \delta_\beta^\alpha - \frac{\partial L_s}{\partial z_\alpha^A} z_\beta^A. \quad (91)$$

## 2. Gravitation action

The next task is to rewrite the shell part of the gravitational action  $I_g$  in the surface form. The following lemma is vital.

*Lemma 3.* In the adapted coordinates  $x^\mu$  that satisfy condition 2', the  $\delta$ -function part of the gravitational Lagrange density is given by

$$\frac{1}{16\pi G} \sqrt{|g|} R = -\frac{1}{8\pi G} \sqrt{|\gamma|} [L] \delta(x^3 - x_0^3) + \dots, \quad (92)$$

where the dots represent regular terms,  $L = \gamma_{\alpha\beta} L^{\alpha\beta}$ ,  $L^{\alpha\beta}$  is the second fundamental form of  $\Sigma$  corresponding to the normal oriented outwards of  $V^-$  and the abbreviation  $[f] := f_+ - f_-$  for the jump of a quantity  $f$  across the shell is used.

The proof of this lemma is relegated to Appendix A. Equation (92) implies immediately that the gravitational action can be transformed to

$$I_g = \frac{1}{16\pi G} \int_{V^+ \cup V^-} d^4 x \sqrt{|g|} R - \frac{1}{8\pi G} \int_\Sigma d^3 \xi \sqrt{|\gamma|} [L]. \quad (93)$$

Equations (85) and (93) give the total action in the surface form

$$I = \frac{1}{16\pi G} \int_{V^-} d^4 x \sqrt{|g|} R + \frac{1}{16\pi G} \int_{V^+} d^4 x \sqrt{|g|} R - \frac{1}{8\pi G} \int_\Sigma d^3 \xi \sqrt{|\gamma|} [L] - \int_\Sigma d^3 \xi \sqrt{|\gamma|} n_s e_s(n_s). \quad (94)$$

This action functional is equivalent to that given by Eq. (77), if the coordinates satisfy condition 2'. It has, however, two advantages in comparison with Eq. (77): (i) all integrands in Eq. (94) are smooth and (ii) it is valid and can be used with more general coordinates, namely, arbitrary smooth coordinates  $x_\pm^\mu$  within  $V^\pm$  and arbitrary coordinates  $\xi^\alpha$  within  $\Sigma$ . The fields in the action (94) are the matter fields  $z^A(y)$  on  $\Sigma$  (observe that the fictitious field  $z^3$  disappeared from the action), the gravity fields  $g_{\mu\nu}(x)$  in  $V^\pm$ , and  $\gamma_{\alpha\beta}(y)$  in  $\Sigma$ . The metric has to satisfy the so-called *continuity relations*

$$\begin{aligned} \gamma_{\alpha\beta}(\xi) &= \left( g_{\mu\nu}^- \frac{\partial x_-^\mu}{\partial \xi^\alpha} \frac{\partial x_-^\nu}{\partial \xi^\beta} (x_-(\xi)) \right)^- \\ &= \left( g_{\mu\nu}^+ \frac{\partial x_+^\mu}{\partial \xi^\alpha} \frac{\partial x_+^\nu}{\partial \xi^\beta} (x_+(\xi)) \right)^+, \end{aligned} \quad (95)$$

where the symbols  $()^\pm$  denote the limits from the volumes  $V^\pm$  towards  $\Sigma$ . The role of the continuity relations (95) is to *define* the configuration space of our system as a control mode or some falloff conditions do. The embedding functions  $x_\pm^\mu(\xi)$  are fixed; their variation is zero.

We also have to specify the integration volumes; this will be done in analogy to Sec. II B 1: the volume  $V$  is chosen to be bounded by two Cauchy surfaces  $S_1$  and  $S_2$ , and by a timelike surface  $\Sigma^+$  (which will be eventually pushed to the infinity); the surface  $\Sigma$  separates  $V$  in two parts  $V^\pm$ , and the surfaces  $S_i$  into  $S_i^\pm$ ; the intersections of  $\Sigma$  with  $S_i$  will be denoted by  $\partial\Sigma_i$  and we will assume that they together form the complete boundary of  $\Sigma$ ; the intersections of  $\Sigma^+$  with  $S_i$  will be denoted by  $\partial S_i$  and we will assume that they form the complete boundary of  $\Sigma^+$ . This form of the action will be our starting point to the derivation of the field equations as well as the Hamiltonian formalism.

### C. The variational formula

The variation of the matter part  $I_m$  of the action (94), if we calculate in the coordinates  $\xi^\alpha$  that are adapted to the surfaces  $\partial\Sigma_i$  by  $\xi^0 = t_i$  and  $\eta^A = \xi^A$  at  $\partial\Sigma_i$ , is

$$\begin{aligned} \delta I_m = & \int_\Sigma d^3\xi \left( \frac{\partial L_s}{\partial z^A} - \partial_\alpha \frac{\partial L_s}{\partial z_\alpha^A} \right) \delta z^A + \frac{1}{2} \int_\Sigma d^3\xi T_s^{\alpha\beta} \delta\gamma_{\alpha\beta} \\ & + \int_{\partial\Sigma_2} d^2\eta \frac{\partial L_s}{\partial z_0^A} \delta z^A - \int_{\partial\Sigma_1} d^2\eta \frac{\partial L_s}{\partial z_0^A} \delta z^A. \end{aligned}$$

We define the matter momenta  $p_A$  by

$$p_A := \frac{\partial L_s}{\partial z_0^A}.$$

As  $p_A$  is a well-defined two-surface density [cf. the discussion below Eq. (21)], we obtain the covariant formula

$$\begin{aligned} \delta I_m = & \int_\Sigma d^3\xi \left( \frac{\partial L_s}{\partial z^A} - \partial_\alpha \frac{\partial L_s}{\partial z_\alpha^A} \right) \delta z^A + \frac{1}{2} \int_\Sigma d^3\xi T_s^{\alpha\beta} \delta\gamma_{\alpha\beta} \\ & + \int_{\partial\Sigma_2} d^2\eta p_A \delta z^A - \int_{\partial\Sigma_1} d^2\eta p_A \delta z^A. \end{aligned} \quad (96)$$

To calculate the variation of the gravitational part  $I_g$  of the action (94), we first rewrite the surface integral in  $I_g$  with the help of the trace part of Eq. (54):

$$Q = 2\sqrt{|\gamma|}L,$$

as

$$-\frac{1}{8\pi G} \int_\Sigma d^3\xi \sqrt{|\gamma|} [L] = -\frac{1}{16\pi G} \int_\Sigma d^3\xi \gamma_{\alpha\beta} [Q^{\alpha\beta}], \quad (97)$$

so that

$$\begin{aligned} \delta \left( -\frac{1}{8\pi G} \int_\Sigma d^3\xi \sqrt{|\gamma|} [L] \right) = & -\frac{1}{16\pi G} \int_\Sigma d^3\xi (\delta\gamma_{\alpha\beta} [Q^{\alpha\beta}] \\ & + \gamma_{\alpha\beta} \delta [Q^{\alpha\beta}]). \end{aligned} \quad (98)$$

Then, we apply lemma 1 [Eqs. (52) and (54)] and Eq. (98) with the result

$$\begin{aligned} \delta I_g = & -\frac{1}{16\pi G} \int_{V^-} d^4x G^{\mu\nu} \delta g_{\mu\nu} - \frac{1}{16\pi G} \int_{V^+} d^4x G^{\mu\nu} \delta g_{\mu\nu} \\ & - \frac{1}{16\pi G} \int_\Sigma d^3\xi [Q^{\alpha\beta}] \delta\gamma_{\alpha\beta} - \frac{1}{16\pi G} \int_{S_2} d^3y q_{kl} \delta\pi^{kl} \\ & + \frac{1}{16\pi G} \int_{S_1} d^3y q_{kl} \delta\pi^{kl} - \frac{1}{8\pi G} \int_{\partial\Sigma_2} d^2\eta \lambda \delta[\alpha] \\ & + \frac{1}{8\pi G} \int_{\partial\Sigma_1} d^2\eta \lambda \delta[\alpha] + \frac{1}{8\pi G} \int_{\partial\Sigma_2} d^2\eta \lambda \delta\alpha \\ & - \frac{1}{8\pi G} \int_{\partial\Sigma_1} d^2\eta \lambda \delta\alpha - \frac{1}{16\pi G} \int_\Sigma d^3\xi \gamma_{\alpha\beta} \delta Q^{\alpha\beta}. \end{aligned} \quad (99)$$

Here,  $q_{kl}(y)$  and  $\delta q_{kl}(y)$  are continuous along  $S_i$ , but  $\pi^{kl}(y)$  and  $\delta\pi^{kl}(y)$  have a jump at  $\partial\Sigma_i$ . Equations (96) and (99) imply the following formula for the total action:

$$\begin{aligned} \delta I = & -\frac{1}{16\pi G} \int_{V^-} d^4x G^{\mu\nu} \delta g_{\mu\nu} - \frac{1}{16\pi G} \int_{V^+} d^4x G^{\mu\nu} \delta g_{\mu\nu} \\ & + \int_\Sigma d^3\xi \left( \frac{\partial L_s}{\partial z^A} - \partial_\alpha \frac{\partial L_s}{\partial z_\alpha^A} \right) \delta z^A + \frac{1}{16\pi G} \int_\Sigma d^3\xi (8\pi G T_s^{\alpha\beta} \\ & - [Q^{\alpha\beta}]) \delta\gamma_{\alpha\beta} - \frac{1}{16\pi G} \int_{S_2} d^3y q_{kl} \delta\pi^{kl} \\ & + \frac{1}{16\pi G} \int_{S_1} d^3y q_{kl} \delta\pi^{kl} - \frac{1}{8\pi G} \int_{\partial\Sigma_2} d^2\eta \sqrt{\lambda} \delta[\alpha] \\ & + \frac{1}{8\pi G} \int_{\partial\Sigma_1} d^2\eta \sqrt{\lambda} \delta[\alpha] + \int_{\partial\Sigma_2} d^2\eta p_A \delta z^A \\ & - \int_{\partial\Sigma_1} d^2\eta p_A \delta z^A + \frac{1}{8\pi G} \int_{\partial\Sigma_2} d^2\eta \sqrt{\lambda} \delta\alpha \\ & - \frac{1}{8\pi G} \int_{\partial\Sigma_1} d^2\eta \sqrt{\lambda} \delta\alpha - \frac{1}{16\pi G} \int_\Sigma d^3\xi \gamma_{\alpha\beta} \delta Q^{\alpha\beta}. \end{aligned} \quad (100)$$

From the variational formula (100), we can read off the field equations. In  $V^-$  and  $V^+$ ,

$$G^{\mu\nu} = 0 \quad (101)$$

and at  $\Sigma$ , we obtain

$$\frac{\partial L_s}{\partial z^A} - \partial_\alpha \frac{\partial L_s}{\partial z_\alpha^A} = 0, \quad (102)$$

$$[Q^{\alpha\beta}] = 8\pi G T_s^{\alpha\beta}. \quad (103)$$

Equation (103) is the well-known dynamical equation for thin shells [11]; we shall refer to it as the ‘‘Israel’s equation’’. It may be considered as the singular part of Einstein equations, corresponding to  $\delta$ -like sources. Equation (102) can be

interpreted as a three-dimensional stress-energy conservation: we can show in a way analogous to Sec. II A 2 that

$$\nabla_\alpha T_{s\beta}^\alpha = \left( \frac{\partial L_s}{\partial z^A} - \partial_\alpha \frac{\partial L_s}{\partial z_\alpha^A} \right) z_\beta^A,$$

where  $\nabla_\alpha$  is a covariant derivative associated with the metric  $\gamma_{\alpha\beta}$ . This identity implies that Eq. (102) is equivalent to

$$\nabla_\alpha T_{s\beta}^\alpha = 0$$

(which comprises only two independent equations).

Equation (100) implies a generating formula for the field equations analogous to Eq. (22). In order to derive this formula, we first have to introduce a foliation of the integration volume in Eq. (100). This is an arbitrary smooth family of spacelike surfaces  $S_t$  such that  $S_t = S_{t'}$ ; we allow for the surfaces  $S_t$  having a cusp at  $\partial\Sigma_t = \Sigma \cap S_t$  so that the normal  $n^\mu$  can have a step discontinuity there. This leads to jumps in  $\pi^{kl}$  and  $\alpha$  across  $\partial\Sigma_t$ .

We also have to introduce adapted coordinates  $x^\mu$  so that the surfaces  $S_t$  are given by  $x^0 = t$ ,  $\Sigma$  by  $x^3 = 0$ , and  $\Sigma^+$  by  $x^3 = r^+$ ; further,  $y^k = x^k|_{S_t}$ ,  $\xi^\alpha = x^\alpha|_\Sigma$ ,  $\xi^\alpha = x^\alpha|_{\Sigma^+}$ ,  $\eta^A = x^A|_{\partial\Sigma_t}$ , and  $\eta^A = x^A|_{\partial S_t}$ . Observe that the full four-metric need not be continuous across  $\Sigma$  with respect to these coordinates. Observe that  $t = \text{const}$  is a continuous surface intersecting  $\Sigma$  and  $\Sigma^+$ , and  $\partial/\partial t$  is a continuous vector field everywhere.

Equation (94) and

$$I = \int_{t_1}^{t_2} dt \mathcal{L}$$

imply for  $\mathcal{L}$

$$\begin{aligned} \mathcal{L} = & \frac{1}{16\pi G} \int_{S_t^-} d^3y \sqrt{|g|} R + \frac{1}{16\pi G} \int_{S_t^+} d^3y \sqrt{|g|} R \\ & - \frac{1}{8\pi G} \int_{\partial S_t^-} d^2\eta \sqrt{|\gamma|} [L] - \int_{\partial S_t^-} d^2\eta \sqrt{|\gamma|} n_s e_s(n_s). \end{aligned} \quad (104)$$

If we rewrite Eq. (100) in the form

$$\delta I = \int_{t_1}^{t_2} dt \delta \mathcal{L},$$

we obtain the *variation formula* for our system:

$$\begin{aligned} \delta \mathcal{L} = & - \frac{1}{16\pi G} \int_{S_t} d^3y (q_{kl} \delta \pi^{kl}) \cdot \\ & - \frac{1}{8\pi G} \int_{\partial\Sigma_t} d^2\eta (\sqrt{\lambda} \delta[\alpha]) \cdot + \int_{\partial\Sigma_t} d^2\eta (p_A \delta z^A) \cdot \\ & + \frac{1}{8\pi G} \int_{\partial S_t} d^2\eta (\sqrt{\lambda} \delta\alpha) \cdot - \frac{1}{16\pi G} \int_{\partial S_t} d^2\eta \gamma_{\alpha\beta} \delta Q^{\alpha\beta}. \end{aligned} \quad (105)$$

Performing the variation in Eq. (104) and comparing the result with the RHS of Eq. (105) recovers the definition of momenta and the field equations.

#### D. The Legendre transformation

Let us define the Hamiltonian in way analogous to Sec. II B 3:

$$\begin{aligned} \check{\mathcal{H}} = & -\mathcal{L} - \frac{1}{16\pi G} \int_S d^3y q_{kl} \dot{\pi}^{kl} - \frac{1}{8\pi G} \int_{S \cap \Sigma} d^2\eta \sqrt{\lambda} [\dot{\alpha}] \\ & + \frac{1}{8\pi G} \int_{S \cap \Sigma^+} d^2\eta \sqrt{\lambda} \dot{\alpha} + \int_{S \cap \Sigma} d^2\eta p_A \dot{z}^A, \end{aligned} \quad (106)$$

where  $[\alpha] := -\alpha_+ - \alpha_-$  at  $\Sigma$ ,  $\alpha_+$  is defined by the normal to  $\Sigma$  that is outward to  $V^+$  and the future normal to  $S^-$ ,  $\alpha_-$  is defined by the normal to  $\Sigma$  that is outward to  $V^-$  and the future normal to  $S^+$ .

To calculate the variation of  $\check{\mathcal{H}}$ , we have to regroup terms in Eq. (105):

$$\begin{aligned} \delta \mathcal{L} = & \frac{1}{16\pi G} \left\{ - \int_{S^-} d^3y (q_{kl} \delta \pi^{kl}) \cdot \right. \\ & \left. + 2 \int_{S^- \cap \Sigma} d^2\eta (\sqrt{\lambda} \delta \alpha_-) \cdot \right\} + \frac{1}{16\pi G} \\ & \times \left\{ - \int_{S^+} d^3y (q_{kl} \delta \pi^{kl}) \cdot + 2 \int_{S^+ \cap \Sigma} d^2\eta (\sqrt{\lambda} \delta \alpha_+) \cdot \right. \\ & \left. + 2 \int_{S^+ \cap \Sigma^+} d^2\eta (\sqrt{\lambda} \delta \alpha) \cdot \right\} + \int_{S \cap \Sigma} d^2\eta (p_A \delta z^A) \cdot \\ & - \frac{1}{16\pi G} \int_{\partial S} d^2\eta \gamma_{\alpha\beta} \delta Q^{\alpha\beta}. \end{aligned}$$

Then we vary Eq. (106), substitute for  $\delta \mathcal{L}$  the regrouped expression, and apply lemma 1 to each of the two volumes  $V^\pm$ ; the result is

$$\begin{aligned} \delta \check{\mathcal{H}} = & - \frac{1}{16\pi G} \int_S d^3y (\dot{\pi}^{kl} \delta q_{kl} - \dot{q}_{kl} \delta \pi^{kl}) \\ & + \frac{1}{16\pi G} \int_{S \cap \Sigma} d^2\eta \sqrt{\lambda} \left( \frac{\dot{\lambda}}{\lambda} \delta[\alpha] - [\dot{\alpha}] \frac{\delta \lambda}{\lambda} \right) \\ & - \int_{S \cap \Sigma} d^2\eta (\dot{p}_A \delta z^A - \dot{z}^A \delta p_A) \\ & - \frac{1}{16\pi G} \int_{S \cap \Sigma^+} d^2\eta \sqrt{\lambda} \left( \frac{\dot{\lambda}}{\lambda} \delta \alpha - \dot{\alpha} \frac{\delta \lambda}{\lambda} \right) \\ & + \frac{1}{16\pi G} \int_{S \cap \Sigma^+} d^2\eta \gamma_{\alpha\beta} \delta Q^{\alpha\beta}. \end{aligned} \quad (107)$$

This is the Hamiltonian variation formula for the field equations.

Let us compute the value of the Hamiltonian. To this aim, we insert from Eqs. (94) and (97) into Eq. (106) and apply lemma 2. A simple calculation leads to



$$\begin{aligned}
\check{\mathcal{H}} = & \frac{1}{8\pi\mathbf{G}} \int_{S^-} d^3y G_0^0 + \frac{1}{8\pi\mathbf{G}} \int_{S^+} d^3y G_0^0 \\
& + \frac{1}{8\pi\mathbf{G}} \int_{S \cap \Sigma} d^2\eta [Q_0^0] - \int_{S \cap \Sigma} d^2\eta T_{s0}^0 \\
& + \frac{1}{8\pi\mathbf{G}} \int_{S \cap \Sigma^+} d^2\eta L_0^0. \tag{108}
\end{aligned}$$

The term  $-(1/8\pi\mathbf{G})[Q_0^0]$  has an interesting interpretation. It has been obtained as the sum

$$-\frac{1}{8\pi\mathbf{G}}[Q_0^0] = -\frac{1}{16\pi\mathbf{G}}[Q] + \frac{1}{8\pi\mathbf{G}}[L_0^0].$$

The first summand is the  $\delta$ -function term in the gravitational Lagrangian density in the volume form

$$\frac{1}{16\pi\mathbf{G}} \sqrt{|g|} R,$$

and the meaning of the second is given by the following lemma.

*Lemma 4.* If the four-metric is continuous in the adapted coordinates, then we have at  $\Sigma$

$$-\frac{1}{8\pi\mathbf{G}} \sqrt{|g|} R_0^0 = -\frac{1}{8\pi\mathbf{G}} [L_0^0] \delta(x^3 - x_0^3) + \dots, \tag{109}$$

where the dots represent regular terms.

Thus, the term is the  $\delta$ -function part of the expression

$$-\frac{1}{8\pi\mathbf{G}} G_0^0$$

[cf. Eq. (53)]. If the assumptions of lemma 4 are satisfied, then the three first integrals on the RHS of Eq. (107) can be written in volume form just as

$$-\frac{1}{8\pi\mathbf{G}} \int_S d^3y G_0^0.$$

The Legendre transformation at  $\Sigma^+$  similar to that in Sec. II B 3 can be performed exactly as was done there. The transformation to the Euler picture in the matter part of the shell Hamiltonian (106) is much simpler than the analogous transformation of the step Hamiltonian (66), because all formulas of Sec. II A 5 remain valid, they must only be rewritten in three spacetime and two matter space dimensions.

### E. The form of the Hamiltonian

In this section, the Hamiltonian (108) will be expressed as a functional of the canonical variables  $N$ ,  $N_k$ ,  $q_{kl}$ ,  $\pi^{kl}$ ,  $\lambda$ , and  $\alpha$ .

Observe that the formulas (107) and (108) are valid in any coordinates that are adapted to the foliation and that make the embedding formulas for  $\Sigma$  and  $\Sigma^+$  time independent.

More specifically, the coordinate  $t$  must be constant along the surfaces  $S$ , the embedding formulas for  $\Sigma$  and  $\Sigma^+$  must read

$$t = \xi^0, \quad y_{\pm}^k = y_{\pm}^k(\xi^K),$$

and the embedding formulas for  $S \cap \Sigma$  in  $\Sigma$  is

$$\xi^0 = \text{const}, \quad \xi^K = \eta^K.$$

Let us recall that one important point of our method is that the boundaries are time independent in the above sense and their variations are zero.

The 2 + 1 decomposition of the metric  $\gamma_{\alpha\beta}$  at  $\Sigma$  and  $\Sigma^+$  is analogous to that of  $g_{\mu\nu}$ . In particular, we define the (surface) lapse  $\nu$  and the (surface) shift  $\nu_K$  by

$$\gamma^{\alpha\beta} = \begin{pmatrix} -\frac{1}{\nu^2}, & \frac{\nu^L}{\nu^2} \\ \frac{\nu^K}{\nu^2}, & \lambda^{KL} - \frac{\nu^K \nu^L}{\nu^2} \end{pmatrix}$$

so that

$$\gamma_{00} = -\nu^2 + \lambda^{KL} \nu_K \nu_L;$$

the 2 + 1 decomposition of the continuity relations (95) reads

$$\nu = \sqrt{N_{\pm}^2 - (N_{\pm}^{\perp})^2}, \tag{110}$$

$$\nu_K = N_k^{\pm} e_{\pm K}^k, \tag{111}$$

$$\lambda_{KL} = q_{kl}^{\pm} e_{\pm K}^k e_{\pm L}^l, \tag{112}$$

where

$$N_{\pm}^{\perp} = N_k^{\pm} m_{\pm}^k, \quad e_{\pm K}^k = \frac{\partial y_{\pm}^k}{\partial \xi^K},$$

and  $m_{\pm}^k$  is the unit normal vector to  $\Sigma \cap S$  tangent to  $S$  and oriented from  $S^-$  to  $S^+$ . From this definition, it follows that

$$\delta m_{\pm}^k = \frac{1}{2} m_{\pm}^{\pm} m_{\pm}^r m_{\pm}^s \delta q_{rs}^{\pm}.$$

Using the decomposition (70), we can write for the integrand of the volume terms in Eq. (108)

$$G_0^0 = -\sqrt{q} \left( N \frac{G^{\perp\perp}}{\sqrt{|g|}} + N^k \frac{G_k^{\perp}}{\sqrt{|g|}} \right);$$

observe that the RHS is invariant with respect to transformations of coordinates  $y^k$ . The form of  $G^{\perp\perp}$  and  $G_k^{\perp}$  is well known [9]:

$$\frac{\sqrt{q}}{\sqrt{|g|}} G^{\perp\perp} = \frac{2\pi^{kl}\pi_{kl} - \pi^2}{4\sqrt{q}} - \frac{\sqrt{q}}{2} R^{(3)},$$

$$\frac{\sqrt{q}}{\sqrt{|g|}} G_k^\perp = -\pi_{k|l}^\perp,$$

where  $R^{(3)}$  is the curvature scalar of the metric  $q_{kl}$ .

Within  $\Sigma$  and  $\Sigma^+$ , an analogous decomposition yields for the surface terms

$$T_{s0}^0 = -\frac{\sqrt{\lambda}}{\sqrt{|\gamma|}} (\nu T_s^{\perp\perp} + \nu^K T_{sK}^\perp)$$

and

$$[Q_0^0] = -\frac{\sqrt{\lambda}}{\sqrt{|\gamma|}} (\nu [Q^{\perp\perp}] + \nu^K [Q_K^\perp]).$$

$T_{s0}^0$  can be expressed by means of the canonical variables  $z^A$ ,  $p_A$ ,  $\lambda_{KL}$ ,  $\nu$ , and  $\nu_K$  in a way parallel to Sec. II A 4: the formulas are independent of the dimension of spacetime. In general, the form is only implicit. However, the dependence of  $T_{s0}^0$  on  $\lambda_{KL}$ ,  $\nu$ , and  $\nu_K$  can be inferred from the relation

$$T_s^{\alpha\beta} = -2 \frac{\partial(-T_{s0}^0)}{\partial\gamma_{\alpha\beta}} \quad (113)$$

analogous to Eq. (27). It follows that

$$\frac{\partial T_{s0}^0}{\partial\nu} = -\frac{\sqrt{\lambda}}{\sqrt{|\gamma|}} T_s^{\perp\perp}, \quad (114)$$

$$\frac{\partial T_{s0}^0}{\partial\nu^K} = -\frac{\sqrt{\lambda}}{\sqrt{|\gamma|}} T_{sK}^\perp, \quad (115)$$

and

$$\frac{\partial T_{s0}^0}{\partial\lambda_{KL}} = \frac{1}{2} T_s^{KL}. \quad (116)$$

In particular,  $(\sqrt{\lambda}/\sqrt{|\gamma|})T_s^{\perp\perp}$  and  $(\sqrt{\lambda}/\sqrt{|\gamma|})T_{sK}^\perp$  are both independent of  $\nu$  and  $\nu_K$ .

In a way analogous to Sec. II A 4 we obtain easily

$$\frac{1}{n_s^2} \left( \frac{j_s^0}{\sqrt{\lambda}} \right)^2 = 1 + \frac{\lambda^{KL} z_K^A z_L^B p_A p_B}{(j_s^0)^2 \rho_s'^2}$$

and

$$T_{s0}^0 = -\nu \left( \frac{\sqrt{\lambda} n_s}{\rho_s' (j_s^0)^2} \lambda^{KL} z_K^A z_L^B p_A p_B + \sqrt{\lambda} \rho_s \right) - \nu^K z_K^A p_A.$$

For a dust shell, we have

$$\frac{1}{n_s^2} \frac{j_s^0}{\sqrt{\lambda}} = 1 + \frac{\lambda^{KL} z_K^A z_L^B p_A p_B}{\mu^2 (j_s^0)^2}$$

and

$$T_{s0}^0 = -\nu \sqrt{\mu^2 (j_s^0)^2 + \lambda^{KL} z_K^A z_L^B p_A p_B} - \nu^K z_K^A p_A.$$

Equation (116) then yields

$$T_{sKL} = \nu \frac{p_A p_B z_K^A z_L^B}{\sqrt{\mu^2 (j_s^0)^2 + \lambda^{KL} z_K^A z_L^B p_A p_B}}.$$

The form of  $[Q_0^0]$  can be given explicitly. Let us observe [14] that the normals  $\tilde{m}$ ,  $n$ , and  $m$  are related by

$$\tilde{m}^\mu = n^\mu \sinh \alpha + m^\mu \cosh \alpha,$$

$$\tilde{n}^\mu = n^\mu \cosh \alpha + m^\mu \sinh \alpha;$$

recall that  $n$  is the normal to  $S$  in  $M$ ,  $m$  is the normal to  $S \cap \Sigma$  in  $S$ , ( $m$  is orthogonal to  $n$ ),  $\tilde{n}$  is the normal to  $S \cap \Sigma$  in  $\Sigma$ , and  $\tilde{m}$  is the normal to  $\Sigma$  in  $M$  ( $\tilde{n}$  is orthogonal to  $\tilde{m}$ ). A simple calculation then confirms that the corresponding second fundamental forms  $L_{\alpha\beta}$ ,  $K_{kl}$ , and  $l_{KL}$  of  $\Sigma$  in  $M$ ,  $S$  in  $M$  and  $\Sigma \cap S$  in  $S$ , respectively, satisfy the relations

$$L_{\alpha\beta} e_K^\alpha e_L^\beta = -K_{kl} e_K^k e_L^l \sinh \alpha + l_{KL} \cosh \alpha,$$

$$L_{\alpha\beta} \tilde{n}^\alpha e_L^\beta - K_{kl} m^k e_L^l = -\alpha_{,L}.$$

It follows that

$$\frac{1}{\sqrt{|\gamma|}} Q^{\perp\perp} = \frac{1}{\sqrt{q}} \pi^{\perp\perp} \sinh \alpha - l \cosh \alpha,$$

$$\frac{1}{\sqrt{|\gamma|}} Q_K^\perp - \frac{1}{\sqrt{q}} \pi_K^\perp = \alpha_{,K}.$$

Hence,

$$Q_0^0 = -\nu (\sqrt{\lambda} \tilde{\pi}^{\perp\perp} \sinh \alpha - \sqrt{\lambda} l \cosh \alpha) - \nu^K (\sqrt{\lambda} \tilde{\pi}_K^\perp + \sqrt{\lambda} \alpha_{,K}), \quad (117)$$

where

$$l = \lambda^{KL} m_{k|l} e_K^k e_L^l = q^{kl} m_{k|l}$$

depends only on  $q_{kl}$  and its first derivatives. We use also the abbreviations

$$\tilde{\pi}^{\perp\perp} = \frac{\pi^{kl}}{\sqrt{q}} m_k m_l, \quad \tilde{\pi}_K^\perp = \frac{\pi^{kl}}{\sqrt{q}} q_{lr} m_k e_r^K, \quad \tilde{\pi}_{KL} = \frac{\pi^{kl}}{\sqrt{q}} e_K^k e_L^l,$$

where

$$e_K^k := \frac{\partial y^k}{\partial \eta^K}.$$

Finally, the complete Hamiltonian reads

$$\begin{aligned} \tilde{\mathcal{H}} = & \frac{1}{16\pi G} \int_{S^-} d^3y \left\{ N \left( \frac{2\pi^{kl}\pi_{kl} - \pi^2}{2\sqrt{q}} - \sqrt{q}R^{(3)} \right) \right. \\ & \left. + N^k(-2\pi'_{k|l}) \right\} + \frac{1}{16\pi G} \int_{S^+} d^3y \left\{ N \left( \frac{2\pi^{kl}\pi_{kl} - \pi^2}{2\sqrt{q}} \right. \right. \\ & \left. \left. - \sqrt{q}R^{(3)} \right) + N^k(-2\pi'_{k|l}) \right\} \\ & - \frac{1}{8\pi G} \int_{S \cap \Sigma} d^2\eta \sqrt{\lambda} (\nu[\tilde{\pi}^{\perp\perp} \sinh \alpha - l \cosh \alpha] \\ & + \nu^K[\tilde{\pi}_K^{\perp} + \alpha_{,K}]) - \int_{S \cap \Sigma} d^2\eta T_{s0}^0 + \frac{1}{8\pi G} \int_{S \cap \Sigma^+} d^2\eta L_0^0. \end{aligned} \quad (118)$$

The surface term at  $\Sigma^+$  is left unchanged; it has to be transformed according to the control mode used and/or shifted to infinity.

The surface super-Hamiltonian  $\mathcal{H}_s$  and the surface super-momentum  $\mathcal{H}_{sK}$  at the shell are given by

$$\mathcal{H}_s = -\frac{1}{8\pi G} [\tilde{\pi}^{\perp\perp} \sinh \alpha - l \cosh \alpha] + \tilde{T}_s^{\perp\perp},$$

$$\mathcal{H}_{sK} = -\frac{1}{8\pi G} [\tilde{\pi}_K^{\perp} + \alpha_{,K}] + \tilde{T}_{sK}^{\perp};$$

here

$$\tilde{T}_s^{\perp\perp} = \frac{1}{\sqrt{|\gamma|}} T_s^{\alpha\beta} \tilde{n}_\alpha \tilde{n}_\beta,$$

$$\tilde{T}_{sK}^{\perp} = \frac{1}{\sqrt{|\gamma|}} T_s^{\alpha\beta} \tilde{n}_\alpha e_{\beta K}.$$

The geometric meaning of the gravitational part of  $\mathcal{H}_s$  and  $\mathcal{H}_{sK}$  can be inferred from Eq. (54):

$$\frac{Q^{\perp\perp}}{\sqrt{|\gamma|}} = -L_{\alpha\beta} e_K^\alpha e_L^\beta \gamma^{KL},$$

$$\frac{Q_K^\perp}{\sqrt{|\gamma|}} = -L_{\alpha\beta} \tilde{n}^\alpha e_K^\beta.$$

In particular,  $L_{\alpha\beta} e_K^\alpha e_L^\beta$  is the second fundamental form of the two-surface  $S \cap \Sigma$  corresponding to the normal  $\tilde{m}$  (each two-surface has two independent second fundamental forms in the spacetime); hence, the gravity part of the surface super-Hamiltonian is the jump in the (two-)trace of this form.

### F. Equations of motion

In this subsection, we calculate the variation of the Hamiltonian (118) explicitly. In this way, we can check if our method leads to the well-known equations of motion; more-

over, we can study the structure of the canonical equations and constraints at the surface  $\Sigma$ . In varying the Hamiltonian, we must carefully deal with boundary terms.

The variation of the volume integrands can be given in the form

$$\delta(2G_0^0) = C_k \delta N^k + C \delta N + a^{kl} \delta q_{kl} + b_{kl} \delta \pi^{kl} + \sqrt{q} B_{|k}^k,$$

where

$$C_k = -2\pi'_{k|l},$$

$$C = \frac{1}{\sqrt{q}} \left( \pi^{kl} \pi_{kl} - \frac{1}{2} \pi^2 \right) - \sqrt{q} R^{(3)},$$

$$\begin{aligned} a^{kl} = & \frac{N}{\sqrt{q}} \left( 2\pi_m^k \pi^{lm} - \pi \pi^{kl} - \frac{1}{2} \pi^{mn} \pi_{mn} q^{kl} + \frac{1}{4} \pi^2 q^{kl} \right) \\ & + N \sqrt{q} \left( R^{(3)kl} - \frac{1}{2} R^{(3)} q^{kl} \right) + \sqrt{q} (N_{|m}^m q^{kl} - N_{|kl}) \\ & - \mathcal{L}_{\tilde{N}} \pi^{kl}, \end{aligned}$$

$$b_{kl} = \frac{N}{\sqrt{q}} (2\pi_{kl} - \pi q_{kl}) - \mathcal{L}_{\tilde{N}} q_{kl},$$

and

$$\begin{aligned} B^r = & -\frac{1}{\sqrt{q}} (N^k \pi^{lr} + N^l \pi^{kr} - N^r \pi^{kl}) \delta q_{kl} \\ & - \frac{2}{\sqrt{q}} N_l \delta \pi^{lr} - N (q^{kl} \delta \Gamma_{kl}^r - q^{kr} \delta \Gamma_{kl}^l) \\ & + \frac{1}{2} N_{,s} (q^{rk} q^{sl} + q^{rl} q^{sk} - 2q^{rs} q^{kl}) \delta q_{kl}. \end{aligned}$$

Here,  $\mathcal{L}_{\tilde{X}}$  is the Lie derivative with respect to the vector field  $\tilde{X}$ . The comparison with the volume term of Eq. (107) yields the well-known canonical form of Einstein equations [9]

$$C_k = 0, \quad C = 0,$$

$$\dot{\pi}^{kl} = -a^{kl}, \quad \dot{q}_{kl} = b_{kl}. \quad (119)$$

The divergence term contributes to the variation of the surface term at  $S \cap \Sigma$  by

$$\frac{1}{16\pi G} \int_{S \cap \Sigma} d^2\eta \sqrt{\lambda} (-B_+^k m_k^+ + B_-^k m_k^-).$$

The following identity can be easily derived:

$$q^{kl} m_r (\delta \Gamma_{kl}^r - \delta_k^r \Gamma_{ls}^s) = 2\delta l + 2l^{KL} \delta \lambda_{KL} - \lambda^{KL} (m^k e_L^l \delta q_{kl})_{|K}, \quad (120)$$

where the parallel symbol denotes the covariant derivative associated with the metric  $\lambda_{KL}$  on  $S \cap \Sigma$ . Using Eq. (120) and the continuity relations (111) and (112), we can rewrite the surface term in the 2+1 form

$$\begin{aligned}
\sqrt{\lambda} B^k m_k &= -\sqrt{\lambda} (2 \tilde{\pi}^{\perp\perp} N^\perp + \tilde{\pi}^{\perp K} \nu_K) (m^k m^l \delta q_{kl}) \\
&\quad - 2 \sqrt{\lambda} \tilde{\pi}^{\perp\perp} \nu^K (e_k^k m^l \delta q_{kl}) - \sqrt{\lambda} (\tilde{\pi}^{\perp L} \nu^K + \tilde{\pi}^{\perp K} \nu^L \\
&\quad - N^\perp \tilde{\pi}^{KL} + \tilde{\pi}^{\perp\perp} N^\perp \lambda^{KL} + \tilde{\pi}^{\perp M} \nu_M \lambda^{KL} \\
&\quad + N_{,k} m^k \lambda^{KL} - N l^{KL}) \delta \lambda_{KL} + 2 \sqrt{\lambda} N \delta l \\
&\quad - 2 \sqrt{\lambda} (N^\perp m_k m_l + \nu_K e_k^K m_l) \delta \tilde{\pi}^{kl}. \tag{121}
\end{aligned}$$

The variation of the surface term  $[Q_0^0]$  can be written in the following way:

$$\delta Q_0^0 = \delta \nu \left( -\frac{\sqrt{\lambda}}{\sqrt{|\gamma|}} Q^{\perp\perp} \right) - \delta \nu^K \left( \frac{\sqrt{\lambda}}{\sqrt{|\gamma|}} Q_K^\perp \right) + \text{rest}. \tag{122}$$

Similarly, using Eqs. (114), (115) and (116), we obtain that

$$\begin{aligned}
\delta T_{s0}^0 &= \delta \nu \left( -\frac{\sqrt{\lambda}}{\sqrt{|\gamma|}} T_s^{\perp\perp} \right) - \delta \nu^K \left( \frac{\sqrt{\lambda}}{\sqrt{|\gamma|}} T_{sK}^\perp \right) + \frac{1}{2} T_s^{KL} \delta \lambda_{KL} \\
&\quad + \left( \frac{\partial T_{s0}^0}{\partial z^A} - \frac{\partial}{\partial \eta^M} \frac{\partial T_{s0}^0}{\partial z_M^A} \right) \delta z^A + \frac{\partial T_{s0}^0}{\partial p_A} \delta p_A. \tag{123}
\end{aligned}$$

Comparing the first two terms in Eqs. (121) and (123), we obtain the first three Israel equations:

$$[Q^{\perp\perp}] = 8 \pi G T_s^{\perp\perp}, \quad [Q_K^\perp] = 8 \pi G T_{sK}^\perp. \tag{124}$$

The last two terms in Eq. (123), if compared with the corresponding surface term in Eq. (107) yield the dynamical equations for matter inside the three-dimensional spacetime of the shell surface  $\Sigma$ :

$$\frac{\partial T_{s0}^0}{\partial z^A} - \frac{\partial}{\partial \eta^M} \frac{\partial T_{s0}^0}{\partial z_M^A} = \dot{p}_A, \tag{125}$$

$$\frac{\partial T_{s0}^0}{\partial p_A} \delta p_A = -\dot{z}^A. \tag{126}$$

What remains from Eq. (107) can be written as follows:

$$\begin{aligned}
[2 \text{rest} - \sqrt{\lambda} B^k m_k] &= (8 \pi G T_s^{KL} - \sqrt{\lambda} [\dot{\alpha}] \lambda^{KL}) \delta \lambda_{KL} \\
&\quad + \sqrt{\lambda} \lambda^{KL} \dot{\lambda}_{KL} \delta [\alpha]. \tag{127}
\end{aligned}$$

A somewhat lengthy calculation starting with Eqs. (121) and (117) gives

$$\begin{aligned}
2 \text{rest} - \sqrt{\lambda} B^k m_k &= -2 \sqrt{\lambda} \tilde{\pi}^{\perp\perp} (\nu \sinh \alpha - N^\perp) m^k m^l \delta q_{kl} \\
&\quad - \sqrt{\lambda} \{ \tilde{\pi}^{\perp\perp} \lambda^{KL} (\nu \sinh \alpha - N^\perp) + \tilde{\pi}^{KL} N^\perp \\
&\quad + l^{KL} N - \nu l \lambda^{KL} \cosh \alpha + \nu^M \lambda^{KL} \alpha_{,M} \\
&\quad - N_{,k} m^k \lambda^{KL} \} \delta \lambda_{KL} + 2 \sqrt{\lambda} (\nu \cosh \alpha \\
&\quad - N) \delta l - 2 \sqrt{\lambda} (\nu \sinh \alpha - N^\perp) m_k m_l \delta \tilde{\pi}^{kl} \\
&\quad + 2 \sqrt{\lambda} (-\nu \tilde{\pi}^{\perp\perp} \cosh \alpha + \nu l \sinh \alpha \\
&\quad + \nu_{\parallel K}^K) \delta \alpha.
\end{aligned}$$

Substituting this into Eq. (127), we immediately obtain

$$\nu \sinh \alpha_\pm = N_\pm^\perp, \quad \nu \cosh \alpha_\pm = N_\pm, \tag{128}$$

$$[-\tilde{\pi}^{\perp\perp} \cosh \alpha + l \sinh \alpha] = 0. \tag{129}$$

The remaining equations, simplified by Eqs. (128) and (129), read

$$\begin{aligned}
\sqrt{\lambda} [-\tilde{\pi}^{KL} N^\perp - l^{KL} N + N l \lambda^{KL} + N_{,k} m^k \lambda^{KL} + \nu^M \lambda^{KL} \alpha_{,M} \\
+ \dot{\alpha} \lambda^{KL}] = 8 \pi G T_s^{KL}, \tag{130}
\end{aligned}$$

and

$$\dot{\lambda} = -2 \lambda (N \tilde{\pi}^{\perp\perp} - N^\perp l - \nu_{\parallel K}^K). \tag{131}$$

From the point of view of physical (or geometrical) content, Eq. (128) just reproduces the definition of  $\alpha$  and is compatible with the continuity relation (110). The three Eqs. (130) are equivalent to the remaining three Israel equations. Finally, Eqs. (129) and (131) follow from the continuity relations (110)–(112) and the equation of motion (119). Indeed, taking the limit of the second Eq. (119) from both sides towards the shell and projecting the result by  $e_K^k e_L^l$ , we obtain

$$\dot{\lambda}_{KL} = \left\{ 2N \left( \tilde{\pi}_{KL} - \frac{1}{2} \tilde{\pi} \lambda_{KL} \right) + N_{K\parallel L} + N_{L\parallel K} + 2l_{KL} N^\perp \right\}_\pm; \tag{132}$$

the expression in the brackets on the RHS must be continuous, hence

$$[-N (\tilde{\pi} \lambda_{KL} - 2 \tilde{\pi}_{KL}) + 2l_{KL} N^\perp] = 0. \tag{133}$$

The trace of Eq. (133) is Eq. (129) and the trace of Eq. (132) is Eq. (131).

Eqs. (124), (133), and the trace-free part of Eq. (130) can be written in a more symmetric form:

$$[-K_{KL} \sinh \alpha + l_{KL} \cosh \alpha] = -8 \pi G \left( \tilde{T}_{sKL} - \frac{1}{2} \tilde{T} \lambda_{KL} \right), \tag{134}$$

$$[K_{KL} \cosh \alpha - l_{KL} \sinh \alpha] = 0, \tag{135}$$

$$[\tilde{\pi}_K^\perp + \alpha_{,K}] = -8 \pi G \tilde{T}_{sK}^\perp, \tag{136}$$

where  $K_{KL} = -\tilde{\pi}_{KL} + (1/2) \tilde{\pi} \lambda_{KL}$ . They give the jumps of the two independent second fundamental forms of the shell two-surface in the spacetime, one corresponding to the normal  $\tilde{n}$  in the direction of the shell motion (continuous), the other to  $\tilde{m}$ , which is perpendicular to the direction of motion.

From the point of view of the theory of constraint systems, Eqs. (124), (129), and the trace-free part of Eq. (130) are constraints. The trace of Eq. (130),

$$\begin{aligned}
[\dot{\alpha}] &= 4 \pi G \nu \tilde{T}_s^{KL} \lambda_{KL} - \frac{1}{2} [-N^\perp \tilde{\pi}^{KL} \lambda_{KL} + N l + 2N_{,k} m^k \\
&\quad + 2 \nu^K \alpha_{,K}], \tag{137}
\end{aligned}$$

Eq. (131), and Eqs. (125), (126) are canonical equations. Finally, Eqs. (110)–(112) and (128) are defining equations of the Hamiltonian system, analogous to falloff conditions or control conditions.

It seems that some of the constraints are second class. For example, Equation (129) follows from the variation with respect to  $\bar{\alpha} = (1/2)(\alpha_+ + \alpha_-)$ , which is a Lagrange multiplier. Equation (129) contains this Lagrange multiplier; thus, its Poisson bracket with  $\pi_{\bar{\alpha}}$ , which is the momentum conjugate to  $\bar{\alpha}$ , and which is also constrained to vanish, is not zero (if we extend the system by this momentum).

Another important observation is that the LHS of Eq. (129) can be smeared only by a function of two variables, because the domain of definition of the LHS is the shell surface. On the other hand, Eq. (129) contains so-called volume quantities, namely,  $\pi^{kl}$  and  $l^{KL}$ ; a derivative with respect to these variables and the Poisson brackets of these variables result in three-dimensional  $\delta$  functions. Thus, the LHS of Eq. (129) *cannot* be smeared so that it becomes a differentiable function on the phase space. We call such constraints *singular*. The best way of tackling this constraint may be to solve it for  $\bar{\alpha}$  and insert the solution back into the action [18]. A similar procedure exists hopefully for the two constraints which result from the tracefree part of Eq. (130):

$$\begin{aligned} & 8\pi G \left( \tilde{T}_s^{KL} - \frac{1}{2} \tilde{T}_s^{MN} \lambda_{MN} \lambda^{KL} \right) \\ &= \left[ - \left( \tilde{\pi}^{KL} - \frac{1}{2} \tilde{\pi}^{MN} \lambda_{MN} \lambda^{KL} \right) \cosh \alpha \right. \\ & \quad \left. - \left( l^{KL} - \frac{1}{2} l^{MN} \lambda_{MN} \lambda^{KL} \right) \sinh \alpha \right]; \end{aligned}$$

they together with Eq. (129) exhaust the singular constraints of our system. It is interesting to observe that the total Hamiltonian *is* a differentiable function. Although the surface integrals in the Hamiltonian represent two-dimensional smearing of some volume quantities, the presence of volume terms and the continuity relations between the surface  $(\nu, \nu^K)$  and volume  $(N, N^K)$  smearing functions guarantee an effectively three-dimensional smearing of all volume quantities.

These difficulties and the related problem of the Bergmann-Dirac analysis [17] seem to be nontrivial; we will try to tackle them in a future paper.

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#### APPENDIX A: PROOF OF LEMMA 3

Let us calculate the  $\delta$ -function terms in the expression  $L_g = h^{\mu\nu} R_{\mu\nu}$ , where  $h^{\mu\nu}$  is defined by

$$h^{\mu\nu} := \frac{1}{16\pi G} |g|^{1/2} g^{\mu\nu}. \quad (\text{A1})$$

For this aim, we have to isolate the second derivative terms. A simple calculation gives

$$h^{\mu\nu} R_{\mu\nu} = \partial_\rho (h^{\mu\nu} A_{\mu\nu}^\rho) + \dots,$$

where  $A_{\mu\nu}^\rho$  is defined by

$$A_{\mu\nu}^\lambda := \Gamma_{\mu\nu}^\lambda - \delta_{(\mu}^\lambda \Gamma_{\nu)\kappa}^\kappa \quad (\text{A2})$$

and the ellipsis represent regular terms. Thus, the  $\delta$  function term in  $h^{\mu\nu} R_{\mu\nu}$  is given by

$$h^{\mu\nu} [A_{\mu\nu}^3] \delta(x^3 - x_0^3).$$

Thus, we prove the identity

$$h^{\mu\nu} [A_{\mu\nu}^3] = - \frac{1}{8\pi G} \sqrt{|\gamma|} [L], \quad (\text{A3})$$

where  $L_{\alpha\beta}$  is the second fundamental form of  $\Sigma$  and  $L = \gamma^{\alpha\beta} L_{\alpha\beta}$ . Observe that the RHS of Eq. (A3) is written in a three-covariant form. From the definition of Eq. (A2) of the quantity  $A_{\mu\nu}^\rho$ , it follows that

$$A_{33}^3 = -\Gamma_{3\alpha}^\alpha, \quad (\text{A4})$$

$$A_{\alpha 3}^3 = \frac{1}{2} (\Gamma_{3\alpha}^3 - \Gamma_{\beta\alpha}^\beta), \quad (\text{A5})$$

$$A_{\alpha\beta}^3 = \Gamma_{\alpha\beta}^3. \quad (\text{A6})$$

Let  $m_\mu$  denote the unit normal vector to  $\Sigma$  oriented in the direction of increasing  $x^3$ , that is outward with respect to  $V^-$ :

$$m_\mu = \frac{1}{\sqrt{g^{33}}} \delta_\mu^3. \quad (\text{A7})$$

Then,

$$L_{\alpha\beta} = m_{\alpha;\beta} = - \frac{\Gamma_{\alpha\beta}^3}{\sqrt{g^{33}}}, \quad (\text{A8})$$

so

$$A_{\alpha\beta}^3 = -\sqrt{g^{33}} L_{\alpha\beta}. \quad (\text{A9})$$

$h^{\mu\nu}$  is a tensor density, hence

$$\nabla_\kappa h^{\mu\nu} = h_{,\kappa}^{\mu\nu} + \Gamma_{\kappa\lambda}^\mu h^{\lambda\nu} + \Gamma_{\kappa\lambda}^\nu h^{\mu\lambda} - \Gamma_{\kappa\lambda}^\lambda h^{\mu\nu},$$

but it is a tensor density formed from the components of the metric tensor, thus  $\nabla_\kappa h^{\mu\nu} = 0$  for all  $\kappa, \mu, \nu$ . Setting  $\mu = 3$ ,  $\nu = \kappa = \alpha$  in this equation and using Eqs. (A4), (A8) and (A9), we obtain that

$$A_{33}^3 = \frac{h_{,\alpha}^{3\alpha} - \sqrt{g^{33}} L_{\alpha\beta} h^{\alpha\beta}}{h^{33}}. \quad (\text{A10})$$

Similarly

$$A_{\alpha 3}^3 = - \frac{h_{,\alpha}^{33} - 2\sqrt{g^{33}} L_{\alpha\beta} h^{3\beta}}{2h^{33}}.$$

The metric  $g_{\mu\nu}(x)$  is continuous across the shell, as are  $h^{\mu\nu}$  and the tangential derivatives of  $h^{\mu\nu}$ . It follows that

$$[A_{33}^3] = - \frac{g^{\alpha\beta}}{\sqrt{g^{33}}} [L_{\alpha\beta}], \quad (\text{A11})$$

$$[A_{\alpha 3}^3] = \frac{g^{3\beta}}{\sqrt{g^{33}}} [L_{\alpha\beta}], \quad (\text{A12})$$

where we also have substituted for  $h^{\mu\nu}$  from Eq. (A1). Moreover, using Eq. (A9), we have that

$$[A_{\alpha\beta}^3] = - \frac{g^{33}}{\sqrt{g^{33}}} [L_{\alpha\beta}]. \quad (\text{A13})$$

Equations (A11)–(A13) imply

$$h^{\mu\nu} [A_{\mu\nu}^3] = - \frac{1}{8\pi G} \sqrt{|g|} g^{33} \left( g^{\alpha\beta} - \frac{g^{3\alpha} g^{3\beta}}{g^{33}} \right) [L_{\alpha\beta}].$$

However, the following well-known relation holds:

$$\gamma^{\alpha\beta} = g^{\alpha\beta} - \frac{g^{3\alpha} g^{3\beta}}{g^{33}},$$

from which, if Eq. (79) is used, the identity (A3) follows immediately.

#### APPENDIX B: PROOF OF LEMMA 4

If we rewrite  $R_0^0$  in terms of the connection,

$$R_0^0 = g^{0\mu} (\partial_\rho \Gamma_{\mu 0}^\rho - \partial_0 \Gamma_{\mu\rho}^\rho + \dots),$$

where the ellipsis represent terms that do not contain second derivatives of the metric, we obtain immediately that

$$R_0^0 = g^{0\mu} [\Gamma_{\mu 0}^3] \delta(x^3 - x_0^3) + \dots. \quad (\text{B1})$$

We easily find

$$g^{0\mu} [\Gamma_{0\mu}^3] = -g^{33} \gamma^{0\alpha} [g_{0\alpha,3}].$$

The following two equations are easily verified:

$$g_{\alpha\beta,3} = \frac{1}{g^{33}} L_{\alpha\beta},$$

$$L_{\alpha\beta} = \frac{1}{|\gamma|} \left( \frac{1}{2} Q \gamma_{\alpha\beta} - Q_{\alpha\beta} \right).$$

Then, some computation leads to

$$g^{0\mu} [\Gamma_{0\mu}^3] = \frac{\sqrt{g^{33}}}{\sqrt{|\gamma|}} \left( [Q_0^0] - \frac{1}{2} [Q] \right).$$

Finally, using Eqs. (79) and (B1), we obtain Eq. (109) immediately.

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