Explicit solution of Riemann-Hilbert problems for the Ernst equation

C. Klein
Institut für Theoretische Physik, Universität Tübingen, Auf der Morgenstelle 14, 72076 Tübingen, Germany

O. Richter
Sektion Physik der Universität München, Theresienstraße 37, 80333 München, Germany

(Received 2 September 1997; published 18 December 1997)

Riemann–Hilbert problems are an important solution technique for completely integrable differential equations. They are used to introduce a free function in the solutions which can be used at least in principle to solve initial or boundary value problems. But even if the initial or boundary data can be translated into a Riemann–Hilbert problem, it is in general impossible to obtain explicit solutions. In the case of the Ernst equation, however, this is possible for a large class because the matrix problem can be shown to be gauge equivalent to a scalar one on a hyperelliptic Riemann surface that can be solved in terms of theta functions. As an example we discuss the rigidly rotating dust disk. [S0556-2821(98)01904-3]

PACS number(s): 04.20.Jb, 02.10.Rn, 02.30.Jr

The stationary axisymmetric Einstein equations in vacuum are of great physical importance since the exterior of stars and galaxies in thermodynamical equilibrium is described by solutions to these equations. The matter sources lead to boundary value problems for the vacuum equations which are equivalent to an integrable equation for a complex potential, the Ernst equation. The standard approach to solve initial or boundary value problems to integrable equations consists of two steps: translate the problem into a Riemann–Hilbert problem in the space of the so called spectral parameter, and then solve it explicitly in order to get the wanted solution to the original problem. This is however very involved in practice since there is no direct way to relate the boundary data to the ''jump data'' of the Riemann–Hilbert problem. But even if this can be done as in the case of the rigidly rotating dust disk [1], the solution to the Riemann–Hilbert problem is equivalent to the solution of a linear integral equation that cannot be given explicitly in general.

In the case of the Ernst equation, it is however possible to perform this second step in the solution process for boundary value problems in closed form for a large class of functions as will be shown in the present paper. In standard manner, we treat the Ernst equation as the integrability condition of an overdetermined linear differential system for a 2×2–matrix Φ. We formulate the Riemann–Hilbert problem for this matrix in a way that one of the components of Φ gives a solution to the Ernst equation containing a free function. Exploiting the gauge freedom of the linear system, we show that the matrix Riemann–Hilbert problem is gauge equivalent to a scalar problem on a hyperelliptic Riemann surface if the jump data entering the Riemann–Hilbert problem are polynomials. The latter can always be solved in closed form (see [2,3]) in terms of theta functions which leads to a class of solutions discussed in [4]. We demonstrate how the rigidly rotating dust disk [1] fits within this scheme.

THE RIEMANN–HILBERT PROBLEM FOR THE ERNST EQUATION

The vacuum metric in the stationary and axisymmetric case can be written in the Weyl–Lewis–Papapetrou form (see [5])

$$ds^2=-e^{2U}(dt+a\,d\phi)^2+e^{-2U}(e^{2k}(d\rho^2+d\zeta^2)+\rho^2 d\phi^2), \tag{1}$$

where ρ and ζ are Weyl's canonical coordinates and ∂_t and ∂_ϕ are the two commuting (asymptotically) timelike and spacelike Killing vectors, respectively. In this case, the field equations are equivalent to the Ernst equation for the potential f where $f=e^{2U}+ib$, and the real function b is related to the metric functions via $b_z=-(i/\rho)e^{4U}a_{,z}$. Here the complex variable z stands for $z=\rho+i\zeta$. With these settings, the Ernst equation reads

$$f_{z\bar{z}}+\frac{1}{2(z+\bar{z})}(f_{\bar{z}}+f_z)=\frac{2}{f+\bar{f}}f_z f_{\bar{z}}, \tag{2}$$

where a bar denotes complex conjugation in $\bar{\mathbb{C}}$. With a solution of the Ernst equation, the metric function U follows directly from the definition of the Ernst potential whereas a and k can be obtained from f via quadratures.

Complete integrability of an equation means that it can be treated as the integrability condition of an overdetermined linear differential system that contains an additional complex parameter, the so called spectral parameter, that reflects an underlying symmetry of the Ernst equation. In the case of the Ernst equation, we use the linear system for the 2×2–matrix Φ of [6]:

$$\Phi(K,\mu_0;z,\bar{z})_{,z}=\left\{\begin{pmatrix} N & 0\\ 0 & M\end{pmatrix}+\frac{K-i\bar{z}}{\mu_0}\begin{pmatrix} 0 & N\\ M & 0\end{pmatrix}\right\}\Phi(K,\mu_0;z,\bar{z}), \tag{3}$$

$$\Phi(K,\mu_0;z,\bar{z})_{,\bar{z}}=\left\{\begin{pmatrix}\bar{M} & 0\\ 0 & \bar{N}\end{pmatrix}+\frac{K+\mathrm{i}z}{\mu_0}\begin{pmatrix}0 & \bar{M}\\ \bar{N} & 0\end{pmatrix}\right\}\Phi(K,\mu_0;z,\bar{z}). \tag{4}$$

Here the spectral parameter K lives on a family of Riemann surfaces $\mathcal{L}(z,\bar{z})$ of genus zero parametrized by z and $\bar{z}$, and given by $\mu_0^2(K)=(K-\mathrm{i}\bar{z})(K+\mathrm{i}z)$. A point on $\mathcal{L}\equiv\mathcal{L}(z,\bar{z})$ is denoted by $P=(K,\mu_0(K))$ with $K\in\bar{\mathbb{C}}$. The functions M and N depend only on z and $\bar{z}$ but not on K, and have the form

$$M=\frac{f_z}{f+\bar{f}},\quad N=\frac{\bar{f}_z}{f+\bar{f}} \tag{5}$$

where f is again the Ernst potential.

It is possible to use the existence of the above linear system for the construction of solutions to the Ernst equation. To this end one investigates the singularity structure of the matrices $\Phi_z\Phi^{-1}$ and $\Phi_{\bar{z}}\Phi^{-1}$ $[\Phi\equiv\Phi(K,\mu_0;z,\bar{z})]$ with respect to the spectral parameter and infers a set of conditions for the matrix Φ that satisfies the linear system (3) and (4). This is done (see e.g. [7]) below.

Theorem 1. Let Φ be subject to the following conditions:

I. $\Phi(P)$ is holomorphic and invertible in a neigbourhood of the branch points $P_0=-\mathrm{i}z$ and $\bar{P}_0$ such that the logarithmic derivative $\Phi_z\Phi^{-1}$ diverges as $(K+\mathrm{i}z)^{-1/2}$ at P_0 and $\Phi_{\bar{z}}\Phi^{-1}$ as $(K-\mathrm{i}\bar{z})^{-1/2}$ at $\bar{P}_0$.

II. All singularities of Φ on $\mathcal{L}$ (poles, essential singularities, zeros of the determinant of Φ, branch cuts and branch points) are regular which means that the logarithmic derivatives $\Phi_z\Phi^{-1}$ and $\Phi_{\bar{z}}\Phi^{-1}$ are holomorphic there.

III. Φ is subject to the reduction condition

$$\Phi(P^\sigma)=\sigma_3\Phi(P)\gamma \tag{6}$$

where σ is the involution on $\mathcal{L}$ that interchanges the sheets, σ_3 is the third Pauli matrix, and γ is an invertible matrix independent of z and $\bar{z}$.

IV. The normalization and reality condition

$$\Phi(P=\infty^+)=\begin{pmatrix}\bar{f} & 1\\ f & -1\end{pmatrix}. \tag{7}$$

Then the function f in (7) is a solution to the Ernst equation.

A proof can be found for instance in [7]. Without loss of generality, we can choose the matrix γ to be the Pauli matrix σ_1. This implies the following.

Corollary 1. Let $\Phi(P)$ be a matrix subject to the conditions of Theorem 1 and $C(K)$ be a 2×2 matrix that only depends on $K\in\bar{\mathbb{C}}$ with the properties

$$C(K)=\alpha_1(K)\hat{1}+\alpha_2(K)\sigma_1,$$
$$\alpha_1(\infty)=1,\quad \alpha_2(\infty)=0. \tag{8}$$

Then the matrix $\Phi'(P)=\Phi(P)C(K)$ also satisfies the conditions of Theorem 1 and $\Phi'(\infty^+)=\Phi(\infty^+)$.

In other words: matrices Φ which are related through the multiplication from the right by a matrix C of the above form lead to the same Ernst potential though their singularity structure may be vastly different (the functions α_i need not be holomorphic). Therefore this multiplication is called a gauge transformation.[1]

Theorem 1 can be used to construct solutions to the Ernst equation by determining the structure and the singularities of Φ in accordance with the conditions I to IV. In the present paper, we will only consider cuts as singularities of Φ, i.e. we concentrate on a matrix Riemann–Hilbert problem on $\mathcal{L}$ which can be formulated as follows (for different formulations of Riemann–Hilbert problems for the Ernst equation see [8,9] and references given therein): Let Γ be a set of (orientable piecewise smooth) contours $\Gamma_k\subset\mathcal{L}$ $(k=1,\dots,l)$ such that with $P\in\Gamma$ also $\bar{P}\in\Gamma$ and $P^\sigma\in\Gamma$. Let $\mathcal{G}_k(P)$ be matrices on Γ_k with Hölder-continuous components and nonvanishing determinant subject to the reality condition $\mathcal{G}_{ii}(\bar{P})=\bar{\mathcal{G}}_{ii}(P)$ for the diagonal elements, and $\mathcal{G}_{ij}(\bar{P})=-\bar{\mathcal{G}}_{ij}(P)$ for the offdiagonal elements, and $\mathcal{G}(P^\sigma)=\sigma_1\mathcal{G}(P)\sigma_1$. Both Γ and $\mathcal{G}$ have to be independent of $z,\bar{z}$. The matrix Φ has to be everywhere regular except at the contour Γ where the boundary values on both sides of the contours (denoted by $\Phi_\pm$) are related via

$$\Phi_-(P)=\Phi_+(P)\mathcal{G}_i(P)\big|_{P\in\Gamma_i}. \tag{9}$$

It may be easily checked that a matrix Φ constructed in this way satisfies the conditions of Theorem 1. Furthermore it can be seen from the Theorem that the only possible singularities of the Ernst potential can occur where the conditions are not satisfied, i.e. where Φ cannot be normalized or where P_0 coincides with one of the singularities of Φ, in our case the contour Γ. The latter makes the Riemann–Hilbert problem very useful if one wants to solve boundary value problems for the Ernst equation: choose the contour Γ in a way that $P_0\in\Gamma$ just corresponds to the contour Γ_z in the meridian $(z,\bar{z})$ plane where the boundary values are prescribed. The Ernst potential will not be continuous at this contour, but its boundary values will be bounded. Notice however that the Ernst potential will not necesssarily be singular if P_0 coincides with a singularity of Φ (it may be e.g. pure gauge). Theorem 1 merely ensures that the solution will be regular at all other points.

GAUGE TRANSFORMATION OF THE RIEMANN–HILBERT PROBLEM

The Riemann–Hilbert problem can be used to generate solutions to the Ernst equation that apparently contain four real valued free functions, the components of the matrix $\mathcal{G}$.

[1]Notice that different forms of the linear system (3), (4) (see e.g. [8]) are known which are related through gauge transformations. The choice of the linear system here, which implies the conditions of Theorem 1, does not fix the gauge uniquely. It is this remaining gauge freedom to which we refer here and in the following.

The remarks on the gauge freedom of the matrix Φ indicate however that two of them are related to gauge transformations. It seems plausible that one can choose a gauge in which $\mathcal{G}$ has only two independent components. For instance one can show that $\mathcal{G}$ is gauge equivalent to $\mathcal{G}'$ which has the form

$$\mathcal{G}' = \begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix} \tag{10}$$

for the jump matrices on the contours in the upper sheet which is the form used by Neugebauer and Meinel (see [10]). The fact that $\mathcal{G}$ can—for any solvable Riemann–Hilbert problem—be brought into the form (10) follows directly from the properties of the gauge transformation (8). We have to show that the Riemann–Hilbert problem at the contour Γ,

$$\mathcal{G}C_- = C_+\mathcal{G}', \tag{11}$$

with $\mathcal{G}'$ of the form (10) is solvable which is however the case (not necessarily uniquely). Thus it is possible to reduce the freedom in the Riemann–Hilbert problem to two real valued functions without changing the singularity structure of Φ which is everywhere regular except at the contour Γ. The obvious disadvantage of this formulation of the matrix Riemann–Hilbert problem is however that such problems cannot be solved explicitly in general. For $\beta \neq 0$, this problem is equivalent to a linear integral equation (see e.g. [3]). Only for $\beta = 0$, an explicit solution can be given. In this case one is led to the static solutions of the Weyl class.

However, for the purposes of the Ernst equation, one may go one step further for a large class of Riemann–Hilbert problems if one drops the condition that the gauge transformed matrix Φ' has the same singularity structure as the original matrix in (9). We will prove this next.

Theorem 2. Let the conditions for (9) hold and let the projection of the contour Γ into the complex plane consist of one simple smooth arc. The components of the matrix $\mathcal{G}$ shall be quotients of holomorphic functions. Then the matrix Riemann–Hilbert problem (9) on $\mathcal{L}$ is gauge equivalent to a problem with the diagonal matrix $\mathcal{G}' = \mathrm{diag}(G,1)$ (on the contours in the copies of the upper sheet) on a two sheeted covering $\hat{\mathcal{L}}$ of $\mathcal{L}$, where G is a Hölder-continuous function on Γ.

Proof. The proof uses again the explicit construction of the gauge transformation which takes the form

$$(\mathcal{G}_{11}+\mathcal{G}_{12}+\mathcal{G}_{21}+\mathcal{G}_{22})(\alpha_1+\alpha_2)^- = (G+1)(\alpha_1+\alpha_2)^+, \tag{12}$$

$$(\mathcal{G}_{11}-\mathcal{G}_{12}-\mathcal{G}_{21}+\mathcal{G}_{22})(\alpha_1-\alpha_2)^- = (G+1)(\alpha_1-\alpha_2)^+, \tag{13}$$

$$(\mathcal{G}_{11}-\mathcal{G}_{12}+\mathcal{G}_{21}-\mathcal{G}_{22})(\alpha_1-\alpha_2)^- = (G-1)(\alpha_1+\alpha_2)^+, \tag{14}$$

$$(\mathcal{G}_{11}+\mathcal{G}_{12}-\mathcal{G}_{21}-\mathcal{G}_{22})(\alpha_1+\alpha_2)^- = (G-1)(\alpha_1-\alpha_2)^+. \tag{15}$$

As already mentioned, this system will in general not have a solution if the α_i are holomorphic except at Γ. We therefore make the ansatz

$$\frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2} = \lambda\ \exp\left(\frac{1}{2\pi i}\int_\Gamma \frac{dK'}{K-K'}\ln\frac{\mathcal{G}_{11}+\mathcal{G}_{12}+\mathcal{G}_{21}+\mathcal{G}_{22}}{\mathcal{G}_{11}-\mathcal{G}_{12}-\mathcal{G}_{21}+\mathcal{G}_{22}}\right) \tag{16}$$

where λ is a possibly multivalued function of $K \in \overline{\mathrm{C}}$ alone. With this ansatz, we can solve the above system and determine G and λ:

$$\lambda = \sqrt{\frac{(\mathcal{G}_{11}-\mathcal{G}_{12}+\mathcal{G}_{21}-\mathcal{G}_{22})(\mathcal{G}_{11}-\mathcal{G}_{12}-\mathcal{G}_{21}+\mathcal{G}_{22})}{(\mathcal{G}_{11}+\mathcal{G}_{12}-\mathcal{G}_{21}-\mathcal{G}_{22})(\mathcal{G}_{11}+\mathcal{G}_{12}+\mathcal{G}_{21}+\mathcal{G}_{22})}}, \tag{17}$$

$$\frac{G+1}{G-1} = \sqrt{\frac{(\mathcal{G}_{11}-\mathcal{G}_{12}-\mathcal{G}_{21}+\mathcal{G}_{22})(\mathcal{G}_{11}+\mathcal{G}_{12}+\mathcal{G}_{21}+\mathcal{G}_{22})}{(\mathcal{G}_{11}-\mathcal{G}_{12}+\mathcal{G}_{21}-\mathcal{G}_{22})(\mathcal{G}_{11}+\mathcal{G}_{12}-\mathcal{G}_{21}-\mathcal{G}_{22})}}. \tag{18}$$

In an abuse of notation, we have denoted here the analytic continuation of the $\mathcal{G}_{ij}$ (which is obvious since the functions are assumed to be quotients of holomorphic functions) with the same symbol as the functions that were originally only defined on Γ (this is still the case for the function G). Writing λ in the form $\lambda^2 = F/H$ where F and H are holomorphic functions (which is possible by assumption), one can recognize that the whole system has to be considered on the Riemann surface $\hat{\mathcal{L}}$ given by

$$\hat{\mu}^2(K) = F(K)H(K). \tag{19}$$

This is a two–sheeted covering of the two–sheeted surface $\mathcal{L}$ on which the spectral parameter varies, and thus a four–sheeted covering of the complex plane. It is on this surface that the gauge transformed matrix Φ' and the function λ are single valued.

In other words, it is always possible to transform the Riemann–Hilbert problem with ‘‘holomorphic’’ jump data to diagonal form. The price one has to pay for this is the introduction of a four-sheeted Riemann surface $\hat{\mathcal{L}}$ since otherwise the gauge transformation would be multivalued. The condition that the projection of Γ into the complex plane consists of only one arc can be replaced by the condition that the analytic continuations of the $\mathcal{G}_{ij}$ coincide on all contours Γ_k.

EXPLICIT SOLUTION OF THE RIEMANN–HILBERT PROBLEM IN TERMS OF HYPERELLIPTIC THETA FUNCTIONS

In the following, we will restrict ourselves to the case in which the Riemann surface $\hat{\mathcal{L}}$ given by (19) has a finite number $2g$ of branch points (which implies that it is compact), the defining equation for the surface $\hat{\mathcal{L}}$ being

$$\hat{\mu}^2 = \prod_{i=1}^{g} (K-E_i)(K-F_i). \tag{20}$$

As can be seen from (17), this will be the case if the components of $\mathcal{G}$ are quotients of polynomials. Since any holomorphic function can be approximated by polynomials, the resulting solutions will lie dense in the topological space of solutions to the Ernst equation with holomorphic boundary data at Γ_z. We will show below that the class of solutions one may construct using this additional assumption is very rich and contains e.g. the recently obtained solution for the rigidly rotating disk of dust.

The reality condition on the $\mathcal{G}_{ij}$ implies $E_i=\overline{F_i}$. We will concentrate on regular Riemann surfaces which means that all branch points have multiplicity 1 since coinciding branch points can be treated as a limiting case. This is the well known soliton limit that can also be obtained with the help of Bäcklund transformations (see e.g. [11]), and leads to the so called multi-black-hole solutions which contain the Kerr solution.

With these assumptions, the above results can be used for the construction of explicit solutions to the Ernst equation. The procedure for solving the Riemann–Hilbert problem on $\hat{\mathcal{L}}$ follows basically the construction of finite gap solutions for the sine–Gordon equation (see e.g. [12]) that was used in [7] to get the corresponding solutions to the Ernst equation. To construct a single-valued solution to the Riemann–Hilbert problem on the surface $\hat{\mathcal{L}}$, we introduce an auxiliary (hyperelliptic) Riemann surface $\mathcal{L}_H=\hat{\mathcal{L}}/\sigma$ obtained by factorizing $\hat{\mathcal{L}}$ with respect to the involution σ on $\mathcal{L}$ that has a natural lift to $\hat{\mathcal{L}}$. This hyperelliptic surface of genus g is given by

$$\mu_H^2=(K-\mathrm{i}\overline{z})(K+\mathrm{i}z)\prod_{i=1}^{g}(K-E_i)(K-F_i) \tag{21}$$

since the fixed points of the involution become branch points on the factorized surface. To work with the surface $\mathcal{L}_H$ has the advantage that all quantities can be expressed there in terms of explicit integrals since one can use the powerful calculus of hyperelliptic Riemann surfaces, see [2,13]. This means that we are working with the three surfaces $\mathcal{L}$, $\hat{\mathcal{L}}$ and $\mathcal{L}_H$ and denote the points on the surfaces by $P=(K,\mu_0(K))$, $\hat{P}=(K,\mu_0(K),\hat{\mu}(K))$ and $P_H=(K,\mu_H(K))$ respectively.

On $\mathcal{L}_H$, we introduce the standard quantities associated with a Riemann surface, namely with a canonical cut system (see [13]) the g normalized differentials of the first kind $\mathrm{d}\omega_i$ defined by $\oint_{a_i}\mathrm{d}\omega_j=2\pi\mathrm{i}\delta_{ij}$, and the Abel map $\omega_i(P)=\int_{P_0}^{P}\mathrm{d}\omega_i$. Furthermore, we define the Riemann matrix Π with the elements $\pi_{ij}=\oint_{b_i}\mathrm{d}\omega_j$, and the theta function $\Theta[^{\alpha}_{\beta}](z)=\sum_{N\in Z^g}\exp\{\frac{1}{2}\langle\Pi(N+\alpha/2),(N+\alpha/2)\rangle+\langle(z+\pi i\beta),(N+\alpha/2)\rangle\}$ with half integer characteristic $[^{\alpha}_{\beta}]$ and $\alpha_i,\beta_i=0,1$ ($\langle N,z\rangle=\sum_{i=1}^{g}N_iz_i$). The normalized (all a-periods zero) Cauchy analogue with poles in P and P_0 will be denoted by $\mathrm{d}\omega_{PP_0}$. Let D be the divisor of zeros of the function H in (19), u be the vector with the components $u_i=1/(2\pi\mathrm{i})\int_\Gamma \ln G d\omega_i$ and $\mathcal{K}$ be the Riemann vector. Then the function ψ given by

$$\psi(P_H)=\psi_0\frac{\Theta(\omega(P_H)+u-\omega(D)-\mathcal{K})}{\Theta(\omega(P_H)-\omega(D)-\mathcal{K})}\times\exp\left(\frac{1}{2\pi\mathrm{i}}\int_\Gamma \ln G d\omega_{P_HP_0}\right) \tag{22}$$

is single-valued on $\mathcal{L}_H$, has g poles at the points of the divisor D and a discontinuity at the contour Γ of the form (see [2])

$$\psi_-=\psi_+G|_\Gamma. \tag{23}$$

Here the path of integration between P_0 and P has to be the same for all integrals, and ψ_0 is a normalization constant.

It can be easily seen that $\psi(P_H)$ is a single-valued function on $\mathcal{L}_H$: the difference of two paths of integration between P_0 and P can be represented as a linear combination of the a- and b-cuts since they are a basis of the homology. Therefore, by a change of the integration path, the theta quotient will be multiplied by $\exp(-\langle N,u\rangle)$, but this term is just compensated by the integral in the exponent. The analytic properties of ψ follow from the definition of the Abelian differentials and the properties of the theta function. By definition, the divisor D is non-special which means that $\Theta(\omega(P_H)-\omega(D)-\mathcal{K})$ does not vanish identically. The latter cannot be said in the general case of $\Theta(\omega(P_H)+u-\omega(D)-\mathcal{K})$. If this function is identically zero for given z, $\overline{z}$, the theta function has to be replaced by its first non–vanishing partial derivative (see [2]).

However, the function $\mu_0(P_H)\doteq\mu_0\mathrm{pr}_1(P_H)=\mu_0(K)$ is not single-valued on $\mathcal{L}_H$ since it changes the sign at every a-cut there. The same holds for the function

$$\chi(P_H)=\chi_0\frac{\Theta(\omega(P_H)+u-\omega(\overline{P_0})-\omega(D)-\mathcal{K})}{\Theta(\omega(P_H)-\omega(D)-\mathcal{K})}\times\exp\left(\frac{1}{2\pi\mathrm{i}}\int_\Gamma \ln G d\omega_{P_HP_0}\right), \tag{24}$$

where χ_0 is again a normalization constant. This can be seen in the proof of the single-valuedness of ψ. Except for the behavior at the a-cuts, the analytic properties of χ are identical to those of ψ. Both functions μ_0 and χ are however single-valued on $\hat{\mathcal{L}}$ which can be viewed as two copies of $\mathcal{L}_H$ cut along $[P_0,\overline{P_0}]$ and glued together along this cut. We define a vector X on $\hat{\mathcal{L}}$ by fixing the sign in front of χ in the vicinity of the points $P_0^{\pm}\in\hat{\mathcal{L}}$,

$$X(\hat{P})=\begin{pmatrix}\psi(\hat{P})\\ \pm\chi(\hat{P})\end{pmatrix},\quad \hat{P}\to P_0^{\pm}. \tag{25}$$

With the help of this vector, we can construct a matrix Φ on $\mathcal{L}$ via

$$\Phi(P)=(X(K,\mu_0(K),+\hat{\mu}(K)),X(K,\mu_0(K),-\hat{\mu}(K))), \tag{26}$$

where the signs are again fixed in the vicinity of $P_0^{\pm}$. Let us show that $\Phi(P)$ satisfies the conditions of Theorem 1. First, this ansatz is in accordance with the reduction condition (6) [this is in fact the reason why one has to define the function χ in the way (24)]. The behavior at the singularities is as required in condition II. For the contour Γ, this is obvious. At the branch points E_i and F_i, one gets the following behavior: at points P_i of the divisor D, the components of Φ have a simple pole, and the determinant diverges as (K

$-P_i)^{-1/2}$. At the remaining branch points, the components are regular but the determinant vanishes as $(K-P_i)^{1/2}$. Since Φ in (26) is only a function of P, it will not be regular at the cuts $[E_i,F_i]$. At the a–cuts, we get $\Phi_-=\Phi_+\sigma_1|_{a_i}$. The logarithmic derivatives with respect to z and $\overline{z}$ are however holomorphic at all these points. Normalizing ψ and χ (if possible) in a way that $\psi(\infty_H^-)=1$ and $\chi(\infty_H^-)=-1$ and observing that the reality condition is satisfied due to the reality properties of $\hat{\mathcal{L}}$ and the Riemann–Hilbert problem, we find that Φ is indeed in accordance with all conditions of Theorem 1. We may summarize these results.

Theorem 3. Let $[{\epsilon \atop \epsilon'}]=\omega(\overline{P_0})+\omega(D)+\mathcal{K}$ and $\Theta[{\epsilon \atop \epsilon'}](\omega(\infty^-)+u)\neq 0$. Then the Riemann–Hilbert problem as formulated in (9) where the components of the jump matrix $\mathcal{G}$ are quotients of polynomials, and where the zeros respectively poles of the linear combinations of these components are of first order, leads to the solution of the Ernst equation

$$f(z,\overline{z})=\frac{\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\omega(\infty^+)+u)}{\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\omega(\infty^-)+u)}\exp\left(\frac{1}{2\pi\mathrm{i}}\int_\Gamma \ln G\, d\omega_{\infty^+\infty^-}\right); \tag{27}$$

here the paths between $[P_0,\infty^-]$ and $[P_0,\infty^+]$ are the same in all integrals and have the same projection into the complex plane (i.e. one is the involuted of the other).

The above forms a subclass of solutions of the class discussed in [4]. It is suggested by the construction that the Ernst potential is regular at the branch points if they do not lie on the contour Γ which was proven in [4] by an analysis of the theta functions at these points. Moreover, it was shown that the only possible singularities of the Ernst potentials (27) except for Γ_z can occur where $\Theta[{\epsilon \atop \epsilon'}](\omega(\infty^-)+u)=0$. At the contour Γ_z, the limiting values of the Ernst potential and its derivatives are analytic if $E_i\notin\Gamma$. In case $E_i\in\Gamma$, the limiting value of the Ernst potential at the contour Γ_z will only be Hölder–continuous. In [4], equatorially symmetric solutions were identified within (27). From the relations given there, one can recover the respective conditions for the jump data α and β of the matrix Riemann–Hilbert problem: $\alpha(-K)=\alpha(K)$, $\beta(-K)=-\beta(K)$.

As an example, we want to consider the rigidly rotating dust disk of radius ρ_0 and dust parameter ν, which is a combination of the angular velocity, the radius ρ_0 and the central redshift of the disk. Neugebauer and Meinel showed that the corresponding solution to the Ernst equation can be obtained on a hyperelliptic Riemann surface of genus 2 with branch points $E_1=-\sqrt{(\mathrm{i}-\nu)/\nu}$, $E_2=-F_1$ and $F_i=\overline{E_i}$. The characteristic has the form $[{0\ 0 \atop 1\ 1}]$, Γ is the covering of the imaginary axis between $-\mathrm{i}\rho_0,\mathrm{i}\rho_0$ in the $+$-sheet, and G is given by $G=[\sqrt{1+\nu^2(1+X^2)^2}+\nu(1+X^2)]^2$ in dimensionless coordinates ρ/ρ_0 and ζ/ζ_0. Inverting the relations (17) and (18), we get the coefficients α and β of the original matrix Riemann–Hilbert problem Neugebauer and Meinel were able to obtain from the boundary value problem for the rigidly rotating dust disk,

$$\alpha=\frac{1}{1+2\nu^2\left(\sqrt{1+\dfrac{1}{\nu^2}}+1\right)(K^2+1)},$$

$$\beta=2\sqrt{2}\mathrm{i}\nu^2K(K^2+1)\alpha\sqrt{\sqrt{1+\frac{1}{\nu^2}}+1}. \tag{28}$$

OUTLOOK

In this paper, we were able to show that a large class of Riemann–Hilbert problems for the Ernst equation can be solved in closed form in terms of hyperelliptic theta functions. Thus the task of solving a boundary value problem with analytic boundary data for the Ernst equation is essentially reduced to the identification of the jump data $\mathcal{G}$ in (9). The explicit form of the obtained solutions offers moreover the possibility for a new approach to the solution of boundary value problems: one can enter the boundary conditions directly with solutions of the form (27), and has to identify the function G and the branch points E_i from the resulting transcendental equation. Whether this is actually possible is however an open question.

In the limit $g\to\infty$ (i.e. if the components of $\mathcal{G}$ are quotients of non-polynomial holomorphic function), the resulting surface $\hat{\mathcal{L}}$ will no longer be compact. A generalization to this case is not straightforward. It can be seen, however, that the hyperelliptic solutions are dense in the topological space of solutions that are generated via Riemann–Hilbert problems with ''holomorphic'' jump data. The most interesting question will be of course to identify the boundary value problems that have a solution in terms of hyperelliptic functions of finite genus. It seems plausible that the limit $g\to\infty$ corresponds to the solutions constructed by Woodhouse and Mason [9] in non–Hausdorffian Twistor spaces (the relation to these will have to be investigated in the future). The advantage of the theta functions for finite genus is, however, that the expression for the Ernst potential can be evaluated numerically without problems. Moreover it is possible for a given solution (as was shown in [4]) to identify physically interesting regions in a spacetime like ergospheres.

ACKNOWLEDGMENT

We thank J. Frauendiener and H. Pfister for carefully reading the manuscript.

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