Singularities in scalar-tensor cosmologies

Nemanja Kaloper[*]

Department of Physics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

Keith A. Olive[†]

School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455

In this article, we examine the possibility that there exist special scalar-tensor theories of gravity with completely nonsingular FRW solutions. Our investigation in fact shows that while most probes living in such a universe never see the singularity, gravity waves always do. This is because they couple to both the metric and the scalar field, in a way which effectively forces them to move along null geodesics of the Einstein conformal frame. Since the metric of the Einstein conformal frame is always singular for configurations where matter satisfies the energy conditions, the gravity wave world lines are past inextendable beyond the Einstein frame singularity, and hence the geometry is still incomplete, and thus singular. We conclude that the singularity cannot be entirely removed, but only be made invisible to most, but not all, probes in the theory. [S0556-2821(97)01724-4]

PACS number(s): 04.20.Dw, 04.50.+h, 98.80.Bp

I. INTRODUCTION

One of the longest-standing challenges to our understanding of gravity has been the singularity problem. In any generic theory of gravity and matter, under reasonable assumptions about the interactions between particles and fields and about the ways of communicating these interactions (which translate into, at least classically, fairly loose energy conditions), produces solutions which contain maelstormian regions of unbounded curvature [1]. In this sense, such solutions seem to point to an intrinsic deficiency of the theory that gave them birth, because the very theory that predicted such maelstorms looses meaning in these limits. Yet, we must note that not all is ill with the fact that such strongly coupled regimes are generic in classical gravitational theories. Because singular regions generally involve very strong forces between particles and very high energies, close to a singularity much of the observed matter structure in the Universe can be created starting from arbitrary initial conditions. Further, this mechanism is built naturally into the theory, such that extrapolating present conditions backwards inevitably results in circumstances under which the present could be shaped in a generic way. Indeed, the vast body of astrophysical observation does indicate that such a dramatic event, the big bang, did take place in the past. The cosmological singularities in the past, then, seem perfectly suited to encode such cosmogonic furnaces into the theory.[1] Our task,

hence, seems to be to bring into accord the facts and the fiction by constructing a theory which could encompass all the features of the big bang while hopefully not incorporating its own demise, in the form of an uncontested singularity.

While it is not known how, and even if, all singularities may be regulated in a theory of gravity, we may be able to gather interesting information by studying the difficulties which emerge in the attempt to smooth the edges of the universe in the existing models. A start for any such investigation, of course, should be Einstein's theory of general relativity, since it is in an amazingly good agreement with experimental observations. In this theory, however, the general theorems due to Hawking and Penrose show that the configurations which are determined by the coupled Einstein-matter equations of motion, under the assumption that the above-mentioned energy conditions are valid, always contain singular regions [1]. The singularities in this context are signaled by geodesic incompleteness. This means that particle trajectories in such geometries cannot be continued past some hypersurface, because they get so strongly focused by gravity's pull that they begin to intersect and hence are not smooth any more. This phenomenon does not always imply that the curvature of the metric grows beyond bound in such limits. However, the geodesic curvatures do, since the geodesic bundles get squeezed very tightly. Hence an observer seeing such geodesic curvatures would indeed see a very strong force. With this in mind, the notion of geodesic incompleteness is indeed a good indicator of the singularity in the space-time.

---

*Electronic address: kaloper@hepvs6.physics.mcgill.ca

†Electronic address: olive@mnhep.hep.umn.edu

[1] Aside from the cosmological arguments favoring the presence of singularities in any theory of gravity, which we describe in the text, there also exist more theoretically-minded arguments, which suggest that singularities may be needed in any theory of gravity for consistency reasons. Specifically, Horowitz and Myers [2] have recently proposed that naked singularities in a theory of gravity are needed to separate positive mass black hole solutions from the negative mass ones. If the negative mass solutions were not singular, they would be more energetically favorable than the empty space, thus yielding vacuum instability in any quantum theory one might attempt to build on the original classical theory.

The presence of singularities in general relativity can be taken as a sign that at very high energies, or equivalently at very short distances, the theory fails to be a completely consistent description of nature. Because at such high energies quantum effects become significant, this suggests that general relativity should be superseded by the quantum theory of gravity, within which we should seek the answer to the conundrum of singularities. Whereas we are still lacking a complete formulation of the quantum theory of gravity, there at present is at least one very strong contender, string theory. Although the recent developments in string theory, assisted by the discovery of the power of duality [3], have greatly improved our understanding of it, the theory is still not known in a way that would enable us to ask the questions about space-time in a general manner. Instead, we have to either resort to the effective action approach which takes into account stringy phenomena in perturbation theory [4], or we could study some special classes of string solutions which can be formulated in the nonperturbative regime [5]. The latter approach is clearly more powerful in that it allows us to investigate more thoroughly the quantum dynamics of the system under scrutiny. But this is available only for some special solutions, most notably the BPS states in the string spectrum, and not for any solution we might be interested in. In particular, there still does not exist a nonperturbative formulation of generic cosmological solutions in string theory. Hence all the investigations of ''realistic'' string cosmologies have been carried out essentially in the effective action approach, which is valid for the weak to medium range of couplings and curvatures (see, for example, [6–13]). We will not dwell on the details of these investigations here, other than to mention that the departure of string-theoretic solutions away from general relativity is induced by the presence of additional degrees of freedom which arise in the massless string spectrum. These fields, the scalar dilaton field, the torsion tensor (or Kalb-Ramond) field, and others, couple to each other and to gravity nonminimally, and can influence the dynamics significantly. Thus it makes sense to ask if the spectrum of the theory may be tailored in such a way as to produce solutions which do not feature any singular behavior [14–16], even if that means abandoning string theory and just constructing certain toy models for the purpose of studying the strong coupling limit.

This approach has been taken recently in [17,18]. There a special class of scalar-tensor theories of gravity was considered, where the matter does not couple to the scalar field in some generalized Jordan-Brans-Dicke (JBD) conformal frame, but the JBD coupling of the scalar to the metric depends on the value of the scalar field. It turns out that by choosing the JBD coupling function one could construct the metric in the JBD frame which is smooth over an infinite interval of the JBD comoving time. Since the matter fields couple only to the JBD frame metric, they do not feel any singularity at all. However, in the Einstein (E) conformal frame, where the metric degrees of freedom have a canonical kinetic term, if the matter sources satisfy the energy conditions and consequently the dynamics are still subject to Hawking-Penrose theorems, the curvature singularities still plague all the solutions. The E frame singularities are invisible to matter probes which do not move along E frame geodesics. Rather, in the E frame matter couples to the scalar

field, and this coupling is adjusted precisely to even out any bumps in the metric. If there were any type of probe which couples both to the JBD scalar and the JBD metric, it would detect the original singularity. The probe would reach the singularity in a finite extent of its world line, after which it could not be extended any further. This would signal that the geometry is still incomplete, at least from the point of view of certain observers. Once the subleading interactions are properly taken into account, these observers can communicate the presence of the singularity to the remainder. Hence, to check whether a solution of some nonminimal theory of gravity is singular or not, it is not sufficient to show that curvature invariants are finite in some chosen conformal frame and that the space-time geometry is complete in it. Instead, we must investigate world lines of all the physical probes in the theory. Only if all these world lines are not inextendable can the space-time be complete and hence nonsingular. As long as there is even a single type of excitation which sees the singularity in any frame, the singularity is not absent, but lurks in the geometry waiting to exert its influence on the theory.

Our purpose here is to show that there is at least one such degree of freedom in all the scalar-tensor models studied in [17,18]. It is the graviton itself. Being generic and model independent, and not immune to the singularity, it resuscitates the singularity back into existence of any realistic observer in such theories. To demonstrate this, we only need to look at the classical theory. We will present the equations of motion of the genuine gravity waves, i.e., the tensor perturbations of the metric, and take the geometrical optics limit to show that the wave packets of gravity waves move along the E frame null geodesics. These trajectories are past inextendable because of the singularity, and the graviton wave packets reach the singular hypersurface after a finite extent of the affine parameter along their world lines. Given this, the singularity cannot be completely removed from the geometry: the gravity waves can communicate its presence to all other degrees of freedom in a finite time, ultimately making the singularity observable. The paper is organized as follows. In the next section, we will review the JBD models studied in [17,18], and establish the relationship between the JBD and E frames. Section III is devoted to the derivation of the gravity wave equations of motion, both in the JBD and E frames. The singularity is the central notion of Sec. IV. There we will carry out the geometrical optics approximation and derive the gravity wave world lines. In the last section, we will give our conclusions.

II. HIDING THE SINGULARITY

Here we will review the models and solutions studied in [17,18] in order to set the stage for out investigation. The theories investigated in these articles were defined by the JBD actions with a variable parameter $\omega(\chi)$, given as

$$S = \int d^4x \sqrt{g} \left\{ \chi R - \frac{\omega(\chi)}{\chi} (\nabla \chi)^2 - \mathcal{L}_m(\mathcal{Y}, \nabla \mathcal{Y}, g_{\mu\nu}) \right\},$$
(2.1)

where χ is the scalar JBD field, R is the scalar curvature of the JBD frame metric $g_{\mu\nu}$, and $\mathcal{Y}$ and $\nabla \mathcal{Y}$ are any other

matter degrees of freedom and their derivatives (field strengths). Our signature conventions are $g_{\mu\nu}=(-,+,+,+)$ and $R^{\mu}{}_{\nu\lambda\sigma}=\partial_{\lambda}\Gamma^{\mu}_{\nu\sigma}-\cdots$. Note that in order to ensure that the gravitational degrees of freedom are not ghostlike (i.e., that the graviton propagator never has negative residue), we must require $\chi\geq 0$. For simplicity, we choose units such that in the E frame we have $16\pi G_N=1$. The approach taken in [17,18] was to specify the function $\omega(\chi)$ in attempting to remove curvature singularities in spatially flat FRW solutions. Note that in the JBD frame, the matter fields do not couple to the scalar χ at the tree level. To get the equations of motion, we can simply vary the action with respect to the independent degrees of freedom $g_{\mu\nu}$, χ, and $\mathcal{Y}$. After a bit of straightforward algebra, we find [19]

$$G_{\mu\nu}=\frac{1}{\chi}\nabla_{\mu}\nabla_{\nu}\chi-\frac{1}{\chi}g_{\mu\nu}\nabla^2\chi+\frac{\omega}{\chi^2}\left(\nabla_{\mu}\chi\nabla_{\nu}\chi-\frac{1}{2}g_{\mu\nu}(\nabla\chi)^2\right)$$
$$+\frac{1}{2\chi}T_{\mu\nu},$$

$$2\nabla_{\mu}\left(\frac{\omega}{\chi}\nabla^{\mu}\chi\right)+R-\frac{d}{d\chi}\left(\frac{\omega}{\chi}\right)(\nabla\chi)^2=0, \qquad (2.2)$$

$$\nabla_{\mu}T^{\mu\nu}=0, \quad T_{\mu\nu}=-2\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}-g_{\mu\nu}\mathcal{L}_m,$$

where $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor, and $T_{\mu\nu}$ is the stress-energy tensor of the matter fields $\mathcal{Y}$. The conservation of the stress energy $\nabla_{\mu}T^{\mu\nu}=0$ is equivalent to the matter equations of motion $\nabla_{\mu}(\delta\mathcal{L}_m/\delta\nabla_{\mu}\mathcal{Y})=\delta\mathcal{L}_m/\delta\mathcal{Y}$. Let us now write the explicit form of Eq. (2.2) for the spatially flat Friedmann-Robertson-Walker (FRW) cosmologies, assuming that the matter stress-energy tensor can be put in the perfect fluid form $T_{\mu\nu}=pg_{\mu\nu}+(p+\rho)u_{\mu}u_{\nu}$, where u^{μ} is the velocity of the comoving observer, $u_{\mu}u^{\mu}=-1$. The FRW metric is

$$ds^2=-d\tau^2+a^2(\tau)d\vec{x}^2 \qquad (2.3)$$

and the comoving velocity is $u^{\mu}=\mathrm{diag}(1,\vec{0})$. The equations of motion (2.2) become [19]

$$3H^2=\frac{\omega}{2}\frac{\chi'^2}{\chi^2}-3H\frac{\chi'}{\chi}+\frac{\rho}{2\chi}, \quad \rho'+3H(\rho+p)=0,$$

$$2H'+3H^2+\frac{\chi''}{\chi}+2H\frac{\chi'}{\chi}+\frac{\omega}{2}\frac{\chi'^2}{\chi^2}+\frac{p}{2\chi}=0, \qquad (2.4)$$

$$2\frac{\omega}{\chi}(\chi''+3H\chi')+\frac{d}{d\chi}\left(\frac{\omega}{\chi}\right)\chi'^2=6H'+12H^2.$$

The primes denote derivatives with respect to the JBD frame comoving time τ, and the JBD frame Hubble parameter is $H=a'/a$. For simplicity's sake, we will assume that the equation of state is $p=\gamma\rho$, with γ a constant. This is not true in the real world, since we know that γ must be a function of the temperature of the universe, and hence of time. However, since we want to investigate the possibility of constructing a classical nonsingular universe solution, this is a sufficiently good approximation to start with. We will determine the limits on γ later.

Instead of deriving the equations of motion (2.2) in the JBD frame, we could have equally well used the E conformal frame. Conformal transformations of the degrees of freedom in a specified theory are merely field redefinitions, and so they cannot change physics [20]. In this sense, they are to be regarded as changes of reference frames, which leave physical observables invariant. The important issue here then is clearly to identify the observables correctly. To do that, we must specify an observer, building it out of the physical fields in the theory. Once given, this observer does not care which reference frame we use to compute observables with. So, if we define the E frame metric and scalar field by

$$\bar{g}_{\mu\nu}=\chi g_{\mu\nu}, \quad \phi=\int d\chi\frac{\sqrt{2\omega+3}}{\sqrt{2}\chi} \qquad (2.5)$$

(using the overbar to distinguish between the two conformal frames) and, following [18], assume that the function $\omega(\chi)$ is monotonic[2] such that the functional relationship $\phi=\phi(\chi)$ is everywhere invertible, we find that in the E frame the effective action is

$$S=\int d^4x\sqrt{\bar{g}}\left\{\bar{R}-(\bar{\nabla}\phi)^2-\frac{1}{\chi^2(\phi)}\mathcal{L}_m\left(\mathcal{Y},\bar{\nabla}\mathcal{Y},\frac{1}{\chi(\phi)}\bar{g}_{\mu\nu}\right)\right\}. \qquad (2.6)$$

In this conformal frame, the metric kinetic term is canonical, i.e., just $\bar{R}$, while now the matter fields $\mathcal{Y}$ couple *both* to $\bar{g}_{\mu\nu}$ *and* ϕ. Note that in order for Eq. (2.5) to be well defined, we must require $2\omega+3>0$. This is because when $2\omega+3=0$, the scalar field is not dynamical, since a conformal transformation to the E frame removes its kinetic term. Further, for $2\omega+3<0$, the scalar would be ghostlike, since its kinetic term would be negative. Now, to find the equations of motion in this frame, we can either take Eqs. (2.2) and transform them to the E frame using Eq. (2.5), or we can vary the action (2.6). Since in the E frame the matter fields couple both to the metric and the scalar ϕ, the matter equations of motion are a little bit more complicated than in Eq. (2.2). Varying Eq. (2.6) with respect to $\mathcal{Y}$, we obtain $\bar{\nabla}_{\mu}[(1/\chi^2)\delta\mathcal{L}_m/\delta\bar{\nabla}_{\mu}\mathcal{Y}]=(1/\chi^2)\delta\mathcal{L}_m/\delta\mathcal{Y}$. Furthermore, recalling that $\delta[(1/\chi^2)\mathcal{L}_m]/\delta\phi=-\{(d\chi/d\phi)[2\mathcal{L}_m+\bar{g}_{\mu\nu}(\delta\mathcal{L}_m/\delta\bar{g}_{\mu\nu})]\}/\chi^3=(d\chi/d\phi)\bar{T}/2\chi$, where $\bar{T}$ is the trace of $\bar{T}_{\mu\nu}$, the resulting equations of motion in the E frame are [19,9]

_____

[2]We require that ω is monotonic. But since $d\phi/d\chi=\sqrt{2\omega+3}/\sqrt{2}\chi$, and we wish that all curvature invariants, not only R and $R^{\mu\nu}R_{\mu\nu}$, are smooth, we should also require that ω is analytic [18], because higher derivative invariants depend on the derivatives of ω. We will discuss this in more detail following the Eq. (2.11).

$$\overline{G}_{\mu\nu} = \overline{\nabla}_\mu \phi \overline{\nabla}_\nu \phi - \frac{1}{2}\overline{g}_{\mu\nu}(\overline{\nabla}\phi)^2 + \frac{1}{2}\overline{T}_{\mu\nu}, \quad \overline{\nabla}^2\phi = \frac{1}{4\chi}\frac{d\chi}{d\phi}\overline{T},$$

$$(2.7)$$

$$\nabla_\mu T^{\mu\nu} = -\frac{1}{2\chi}\frac{d\chi}{d\phi}\ \overline{T}\overline{\nabla}^\nu\phi,$$

$$\overline{T}_{\mu\nu} = -\frac{2}{\chi^2}\frac{\delta\mathcal{L}_m}{\delta\overline{g}^{\mu\nu}} - \frac{1}{\chi^2}\overline{g}_{\mu\nu}\mathcal{L}_m.$$

Again, we will need Eq. (2.7) restricted on the spatially flat FRW geometries with perfect fluid sources, $\overline{T}_{\mu\nu} = \overline{p}\,\overline{g}_{\mu\nu} + (\overline{\rho} + \overline{p})\overline{u}_\mu\overline{u}_\nu$, where $\overline{u}^\mu$ is now the velocity of the E frame comoving observer. The E frame metric is

$$d\overline{s}^2 = -dt^2 + \overline{a}^2(t)d\vec{x}^2 \qquad (2.8)$$

and so $\overline{u}^\mu\overline{u}_\mu = -1$, $\overline{u}^\mu = \mathrm{diag}(1,\vec{0})$. We will show that this is consistent with the JBD frame comoving velocity shortly, by establishing the transformation properties relating the JBD and E quantities. The equations of motion (2.7) then become [19,9]

$$3\overline{H}^2 = \frac{\dot{\phi}^2}{2} + \frac{\overline{\rho}}{2}, \quad \dot{\overline{\rho}} + 3\overline{H}(\overline{\rho} + \overline{p}) = \frac{\dot{\phi}}{2\chi}\frac{d\chi}{d\phi}(3\overline{p} - \overline{\rho}),$$

$$\dot{\overline{H}} + \frac{\dot{\phi}^2}{2} + \frac{\overline{\rho} + \overline{p}}{4} = 0, \qquad (2.9)$$

$$\ddot{\phi} + 3\overline{H}\dot{\phi} + \frac{1}{4\chi}\frac{d\chi}{d\phi}(3\overline{p} - \overline{\rho}) = 0.$$

Here the dot denotes derivatives with respect to the E frame time t, and the E frame Hubble parameter is $\overline{H} = \dot{\overline{a}}/\overline{a}$. Using the variational definition of the matter stress-energy tensor, we can immediately see that the conformal transformation (2.5) induces the change of the stress-energy tensor according to [19] $\overline{T}^\mu{}_\nu = T^\mu{}_\nu/\chi^2$. With this and Eq. (2.5), we can also easily show that the JBD frame equations of motion (2.2) and the E frame equations (2.7) map into each other. For the variables describing the FRW universes (2.3) and (2.8), the transformations are $d\tau = dt/\sqrt{\chi}$, $a(\tau) = \overline{a}(t)/\sqrt{\chi}$, $p = \chi^2\overline{p}$ and $\rho = \chi^2\overline{\rho}$. To find out how the comoving velocity transforms, we should look at the comoving velocity vector fields in each frame. In the JBD frame, we have $U = \partial_\tau$ and in the E frame $\overline{U} = \partial_t$, so $U = \sqrt{\chi}\overline{U}$. But this means that the components of the comoving velocities in the two frames are identical, $u^\mu = \overline{u}^\mu = \mathrm{diag}(1,\vec{0})$, as has been claimed above. This completes our survey of the conformal transformation rules for the quantities of interest here.

Now we can investigate the properties of the theory, using either of the sets of field equations (2.2)–(2.7). We should first recall the Hawking-Penrose singularity theorems [1]. To do this, we will use the E frame equations of motion (2.7), since the gravitational equations of motion are the same as in

general relativity. Consider any timelike geodesic with a unit tangent vector field ξ^μ ($\overline{g}_{\mu\nu}\xi^\mu\xi^\nu = -1$) in a globally hyperbolic space-time (i.e., space-time without any acausal pathologies such as closed timelike curves, which we will assume here). The singularity theorems then posit that these geodesics are past inextendable (i.e., incomplete) as long as the projection of the Ricci tensor on the tangent vector field is positive semidefinite, $\overline{R}_{\mu\nu}\xi^\mu\xi^\nu \geq 0$. By the equations of motion, this condition can be recast as $\overline{\Theta}_{\mu\nu}\xi^\mu\xi^\nu + \overline{\Theta} \geq 0$, where the tensor $\overline{\Theta}_{\mu\nu}$ is the total stress energy $\overline{\Theta}_{\mu\nu} = \overline{\nabla}_\mu\phi\overline{\nabla}_\nu\phi - (1/2)\overline{g}_{\mu\nu}(\overline{\nabla}\phi)^2 + \overline{T}_{\mu\nu}$. This requirement is called the strong energy condition (SEC), and is thought to be satisfied by most reasonable classical matter sources. In terms of the principal values of the stress-energy tensor $\overline{\Theta}^\mu{}_\nu = \mathrm{diag}(\hat{\rho}, \hat{p}_1, \hat{p}_2, \hat{p}_3)$, SEC translates into $\hat{\rho} + \Sigma_{i=1}^3 \hat{p}_i \geq 0$, $\hat{\rho} + \hat{p}_i \geq 0$, $i = 1,2,3$. (Here we use the hat to distinguish the principal values of $\overline{\Theta}_{\mu\nu}$ from those of $\overline{T}_{\mu\nu}$, which are both defined in the E frame.) The only feature of the JBD system we will be investigating here is the effect of the variation of ω with χ on the singularities, which still needs to be specified. Therefore, we can assume that the matter fields $\mathcal{Y}$ obey SEC, meaning that $\overline{T}_{\mu\nu}\xi^\mu\xi^\nu + \overline{T} \geq 0$ for all timelike geodesics ξ. For the homogeneous and isotropic cases which we are interested in, this tells us that $3\gamma + 1 \geq 0$, or $\gamma \geq -1/3$. When ϕ is included, its stress energy trivially satisfies SEC in the E frame. Hence, $\overline{\Theta}$ also does, being a linear combination of these two contributions. We see that in the E frame all timelike geodesics must be past inextendable, and thus all cosmological solutions are singular.

Unfortunately, this does not specify the character of the singularity. Generically, geodesic incompleteness as an indicator of the presence of a singularity signals that the metric becomes degenerate as some region of space-time is approached. To learn more about the actual properties of the singularity, we must look at concrete solutions. If we limit our attention to spatially flat FRW cosmologies, which have matter sources that satisfy SEC, we will typically find that the singularity arises because at some time, say $t = 0$, the scale factor of the universe vanishes (or diverges, such as in pole-law inflationary solutions) as some power of t, $\overline{a}(t) \propto t^\alpha$, $\alpha \neq 0$.[3] Since the curvature scalar can be expressed as

---

[3]It is clear that hypersurfaces $t = $ const, where $\overline{a}(t)$ is smooth and nonvanishing cannot be singular, since the metric there is nondegenerate. We are assuming that singularities can arise only as zeros or isolated singularities of $\overline{a}$ in the functional sense, and further that if $\overline{a}$ is unbounded for some values of t, that such singularities are not essential singularities, i.e., that $\overline{a}$ is analytic everywhere near the singularities, and that if it diverges there it admits a Laurent series expansion with a finite number of divergent terms. This is consistent, because essential singularities of $\overline{a}$ are never a part of the manifold in the sense discussed in the text. For assume $t = 0$ were an essential singularity. Then the geodesic distance from the singularity to anywhere else in the manifold satisfies $\lim_{t\to 0}\Delta\overline{\lambda} \propto \lim_{t\to 0} t\overline{a}(t)$. But since $t = 0$ is an essential singularity, $\lim_{t\to 0} t\overline{a} \to \infty$. Hence, $\Delta\overline{\lambda}$ always diverges.

$\bar{R} = 6\dot{\bar{H}} + 12\bar{H}^2$, we find that in the limit $t \to 0$, $\bar{R} = 6\alpha(2\alpha - 1)/t^2 +$ subleading terms.[4] So for $\alpha \neq 1/2$, the scalar curvature diverges at $t = 0$. If $\alpha = 1/2$, then near the singularity the universe is radiation dominated. The scalar curvature vanishes because the radiation sources are conformally invariant and hence have vanishing trace of the stress-energy tensor. However, the square of the Ricci tensor then diverges, as $\bar{R}_{\mu\nu} \bar{R}^{\mu\nu} \propto 1/t^4$. Moreover, if we look at causal geodesics, we can see that in order to move between times t and $t = 0$, they require a lapse of the affine parameter equal to

$$\Delta \bar{\lambda} = \int_0^t dt \ \bar{a} / \sqrt{v^2 + m^2 \bar{a}^2} \qquad (2.10)$$

which can be obtained by solving the causal geodesic equations for the metric (2.8), and where v^2 and m^2 arise as constants of integration; v^2 is a non-negative constant, and $m^2 = 0,1$. In the limit of small t, when $\bar{a} \propto t^\alpha$, we can approximate this expression with $\Delta \bar{\lambda} \propto t^{1+\alpha}/\sqrt{v^2 + m^2 t^{2\alpha}}$, hence noting that we can always choose v^2 and m^2 such that $\Delta \bar{\lambda}$ is finite.[5] This means that there always exist causal geodesics which reach out of the singularity to anywhere in the Universe in a finite proper time, i.e., they are past inextendable and so incomplete. As a result, all the FRW solutions with this type of behavior are indeed singular.

To see how the mechanism of conformal transformations could regulate the singularities, recall that since all FRW solutions (that is, $k = \pm 1,0$) are conformal to static geometries with maximally symmetric spatial slices, the Weyl tensor of these solutions is a constant and hence does not encode any information about the big bang singularity. The information about the singularity is completely encoded in the Ricci sector of the curvature, which changes under conformal transformations. Indeed, we can look at the leading form of the curvature in the vicinity of the singularity of every solution we have discussed above. Applying the field redefinitions (2.5) to the solutions, and using the equations of motion in the E frame (2.7), we can show that the JBD and E frame curvatures are related by

$$R = \chi \left\{ \bar{R} + \frac{3\bar{T}}{4\omega + 6} - \frac{3(\bar{\nabla}\phi)^2}{(2\omega + 3)^2} \left(2\chi \frac{d\omega}{d\chi} + 2\omega + 3 \right) \right\}. \qquad (2.11)$$

We have seen above that as the singularity is approached, the E frame Ricci scalar diverges as $\bar{R} \propto 1/t^2$. Hence if in this limit $\chi \to t^\beta$ with $\beta \geq 2$, the divergent contribution of $\bar{R}$ in Eq. (2.11) can be tamed. What about the other two terms? It is straightforward to see that in the limit $\bar{a}(t) \to t^\alpha$, the equations of motion (2.9) are approximated by

$$3\frac{\alpha^2}{t^2} = \frac{\dot{\phi}^2}{2} + \frac{\bar{\rho}}{2}, \quad \frac{\alpha}{t^2} = \frac{\dot{\phi}^2}{2} + \frac{1+\gamma}{4}\bar{\rho},$$

$$\dot{\bar{\rho}} + 3(1+\gamma)\frac{\alpha}{t}\bar{\rho} - A\dot{\phi}\bar{\rho} = 0, \quad \ddot{\phi} + 3\frac{\alpha}{t}\dot{\phi} + \frac{A}{2}\bar{\rho} = 0, \qquad (2.12)$$

where $A = (3\gamma - 1)/\sqrt{4\omega(\chi_0) + 6} \geq 0$ when $\gamma \geq -1/3$ is a finite constant since $2\omega + 3 > 0$. On the other hand, we can see from Eq. (2.2) that we can always write down the exact solution for the fluid, since in the JBD frame the fluid couples only to the metric. The solution is $\rho = \rho_0/a^{3(1+\gamma)}$. Now, since $\bar{\rho} = \rho/\chi^2$, we can transform the energy density to the E frame: $\bar{\rho} = \rho_0 \chi^{(3\gamma - 1)/2}/\bar{a}^{3(1+\gamma)}$. In the limit $t \to 0$, as $\bar{a} \propto t^\alpha$ and $\chi \propto t^\beta$, we find $\bar{\rho} \propto t^{(3\gamma - 1)\beta/2 - 3\alpha(1+\gamma)}$. Hence in order for the ρ pole contribution to the JBD curvature to be smoothed, we must require $\beta + (3\gamma - 1)\beta/2 - 3\alpha(1+\gamma) \geq 0$. From the equations of motion near the singularity, we see that $\bar{\rho} \leq 6\alpha^2/t^2$ for all t. Therefore, for all solutions, $(3\gamma - 1)\beta/2 - 3\alpha(1+\gamma) \geq -2$, which means that the regulator field χ is no worse than in the previous case. Let us now compare this to the scalar field contributions. Assuming a strictly greater than order relation in the previous inequality, so that $\bar{\rho}$ falls off slower than $1/t^2$, we see from the constraint equation solved for $\dot{\phi}$, $\dot{\phi}^2/2 = 3\bar{H}^2 - \bar{\rho}$, that near the singularity the dominant contribution comes from $\bar{H}$. This means that the scalar field is $\dot{\phi} \propto \sqrt{6}\alpha/t$. But then, sufficiently close to $t = 0$, the $\bar{\rho}$-dependent contribution to the $\ddot{\phi}$ equation is also negligible. As a result, in this limit $\alpha \to 1/3$, and hence the solution approaches $\bar{a} \to t^{1/3}$, $\phi \to \phi_0 + \sqrt{2/3}\ln(t)$ with negligible matter field contributions, i.e., we get the scalar field-dominated cosmology [9,10]. Inspecting Eq. (2.12) near the singularity we can see that the matter sources can never dominate over the scalar field.[6] Hence near the singularity, the ϕ-dependent contributions can never be subleading to the matter stress-energy contribution, regardless of the kind of matter field.

The only remaining possibility is that the two sources remain of equal importance near the singularity. Indeed, a glance at Eq. (2.12) suggests that a solution of the asymptotic form $\dot{\phi} \propto 1/t$, $\bar{\rho} \propto 1/t^2$ near the singularity is admissible. Yet, a closer look reveals that, unless $\gamma = -1$, the four equations in Eq. (2.12) are consistent only if $\bar{\rho} = 0$ - hence again leading to the scalar field-dominated solution. If $\gamma = -1$ [which corresponds to the JBD frame cosmological constant, since $\rho = $ const from the equations of motion (2.4)], the solution which treats the sources in an egalitarian way is admissible. However, in this case the parameter β cannot be adjusted to be ≥ 2. It is fixed by $\gamma = -1$ and $3\alpha(1+\gamma) - \beta(3\gamma - 1)/2 = 2$ to be $\beta = 1$. So the JBD frame solution is still singular. For this case, it is impossible to remove the E

---

[4]Note that this is valid even for the cases when the scale factor vanishes faster than the power law, such as $\bar{a} \propto t^{2/3}[\ln(t)]^{1/2}$, which is known to be the limiting form of the solution dominated by composite hadrons [21].

[5]E.g., for $\alpha > -1$, take $m^2 = 0$ and $v^2 = 1$; then, $\Delta\bar{\lambda} \propto t^{1+\alpha}$; if $\alpha \leq -1$, choose $v^2 = 0$, $m^2 = 1$ (static observers!), so that $\Delta\bar{\lambda} \propto t$.

[6]Suppose they did; then, near the singularity the scale factor would behave as $\bar{a} \propto t^{2/3(1+\gamma)}$ and hence $\bar{\rho} \propto 1/t^2$. But this then produces the response in the scalar field according to $\dot{\phi} = P/t^{2/1+\gamma} + Q/t$, where P and $Q \neq 0$ are integration constants.

frame singularity by going to the JBD frame via the conformal transformation (2.5), regardless of the form of the coupling function $\omega(\chi)$.[7] From this discussion, we can see that all the JBD frame solutions without a singularity must have a unique behavior near the singularity in the E frame — that of the scalar field-dominated cosmology. In fact, this is the reason why a conformal transformation can be used to smooth the solution. Once the solution is dominated by the scalar field near the singularity, we can choose an almost arbitrary function of the scalar field to conformally transform the solution with — as long as the conformal factor vanishes in the limit $t \to 0$ at least as fast as t^2.

To complete the smoothing of the singularity within the context of the JBD theory, we must require that the hypersurface $t=0$ can never be reached by any particle in the JBD world. If we look at the matter sector of the theory, in which fields move along the JBD frame geodesics, we should require that no such geodesic ever reaches the $t=0$ hypersurface. When we solve the geodesic equations of the JBD metric (2.3), and express them in terms of the E frame time for convenience, we find that the JBD frame geodesic lapse between hypersurfaces t_1 and t_0 is

$$\Delta\lambda = \int_{t_0}^{t_1} \bar{a}\, dt / \sqrt{v^2 \chi^2 + m^2 \bar{a}^2 \chi}, \qquad (2.13)$$

where as before $v^2 \geq 0$ and $m^2 = 0,1$ are integration constants. To make sure that all the geodesics are complete, we must require that $\Delta\lambda$ diverges as either of the limits of integration goes to zero. Substituting the asymptotic form of the functions $\bar{a}$ and χ which we have deduced above, we find that as long as $\beta \geq \max(2, \alpha+1)$, the geodesic distance $\Delta\lambda$ between the $t=0$ hypersurface and any other $t=$const hypersurface diverges. Further, if we set $v^2 = 0$, $m^2 = 1$, we recover the integrated coordinate transformation between the JBD comoving time τ and the E comoving time t, $\tau = \int^t dt/\chi$, as we should. So we see that the hypersurface $t = 0$ as seen from the JBD frame corresponds to the infinite past (or future) of the solution, as long as $\beta \geq 2$: $\lim_{t\to 0} \tau \to \pm\infty$. Therefore, the conformal transformation to the JBD frame does push away the singularity to an infinite distance, while making the JBD curvature finite.

At this point, we must remember that while we are using the conformal transformation (2.5) to regulate the solution at $t = 0$, we must make sure that it does not behave badly elsewhere. The conditions which $\omega(\phi)$ must satisfy in order to

guarantee that the JBD solutions are nonsingular can be summarized in terms of $\Omega = 2\omega + 3$, following [18], as (1) $d^n\Omega/d\chi^n$ are smooth everywhere except in the limit $\chi \to 1$ for all $n \in \mathbf{Z}_+$, (2) as $\chi \to 1$, both Ω and its derivatives diverge as some inverse power of $(1-\chi)$, and (3) finally and most importantly, that near the singularity the solutions are dominated by the JBD scalar field ϕ (or equivalently, χ), which implies $\lim_{\chi \to 0} \Omega \geq 1/3$.

We emphasize here that whereas this condition is vacuous for a range of admissible types of matter, it is not automatically true for all solutions. In particular, we have given two simple examples (with the JBD frame matter being either the cosmological constant or a radiation fluid) where the scalar field in the E frame is not dominant near the singularity and hence the JBD frame solutions remain singular.

However, as we have indicated earlier in the discussion preceding the investigation of the conformal pictures of the physics given by Eqs. (2.1)–(2.6), even if the conformal rescaling did produce a smooth metric in the JBD frame, it may not have really removed the singularity if there still remained even a single degree of freedom which coupled to the E frame geometry. In what follows we are going to show that gravitational waves move along world lines in the JBD frame which are not geodesics. Rather, they are deformed by the dilaton force such that they are identical to the E frame geodesics. This should not be a complete surprise since it is the E frame where the graviton kinetic term takes the canonical form, being just the Ricci scalar. It has been argued by Shapere, Trivedi, and Wilczek [22] that in theories of antisymmetric p forms coupled to scalars via $f(\phi)F^2_{\mu_1 \dots \mu_{p+1}}$ the tensor field quanta move along geodesics of the metric where the term $f(\phi)$ is absorbed away by a conformal transformation. This was later proved in [23] by the present authors and collaborators to hold as the geometrical optics limit of the p-form field equations. Since gravity is a gauge theory similar to the p-form field theories, when we apply the same technique to the equations of motion of gravitational perturbations in the FRW background, we find that the gravity waves in general see a scalar force. Therefore since the E frame metric is singular, the gravity wave world lines are past inextendable, and hence the full physical arena of the theory given in either frame is still incomplete. This must be taken as a signature of a latent singularity, which simply did not go away by a conformal transformation, but only appeared invisible to the matter sector probes.

III. GRAVITY'S REDOUBT

Below we will consider small perturbations away from a fixed curved background, and look at their dynamics to the lowest order in the fluctuation. Since we want to study only the pure gravitational excitations, we will assume that the matter and the JBD scalar are unperturbed, and impose the transverse traceless gauge on the metric fluctuation. Further, since we will look for the perturbations around the FRW solutions, we will impose the stationary gauge on the background metric, which will therefore leave the background solutions identical to the ones studied in Sec. II. The resulting equations of motion give the correct dynamics of the two independent graviton polarizations, the $+$ and the $\times$ modes, in exactly the same way as discussed previously [24,25]. In

---

[7] For completeness, we should also mention another special example where frame switching fails, although this case does not violate the results of [17,18], since the scalar field is constant. Namely, if the matter is in the form of pure radiation, there exist solutions with $\phi = $const, as can be seen from either Eqs. (2.2) or (2.7). Hence the JBD and the E frame are identical, and so the singularity is not removed. Nevertheless, these solutions are not past attractors for all late-time radiation-filled universes, since as long as $\dot{\phi} \neq 0$, early enough it will dominate over radiation. So these solutions are not generic, but they illustrate that scalar field must dominate near the singularity in order for the frame switching to be successful in smoothing the solution.

the E frame, these equations will be the same as in the ordinary general relativity, because of the gauge conditions. In the JBD frame, they will contain an additional coupling to the scalar χ, which arises from the noncanonical form of the graviton propagator.

In the E frame, therefore, we will use the following ansatz for the metric and matter fields:

$$\hat{\bar{g}}_{\mu\nu} = \bar{g}_{\mu\nu} + \bar{h}_{\mu\nu}, \quad \delta\phi = \delta\mathcal{Y} = 0. \quad (3.1)$$

The genuine gravitational degrees of freedom correspond to two graviton polarizations defined by imposing the transverse traceless conditions on the metric perturbation. In the momentum space, this means that we require the polarization tensor of the graviton $\bar{\epsilon}_{\mu\nu}$ to be traceless and orthogonal to the direction of motion, i.e., $\bar{\epsilon}^\mu{}_\mu = k^\mu \bar{\epsilon}_{\mu\nu} = 0$. In terms of the perturbation $\bar{h}_{\mu\nu}$, these conditions can be written down as $\bar{h}^\mu{}_\mu = \nabla_\mu \bar{h}^\mu{}_\nu = 0$. Here the covariant derivatives and raising and lowering of indices are taken with respect to the background metrics $\bar{g}_{\mu\nu}$. The inverse metric, to linear order in perturbation $\bar{h}$ is $\hat{\bar{g}}^{\mu\nu} = \bar{g}^{\mu\nu} - \bar{h}^{\mu\nu}$. Notice here that the determinant of the metric is not perturbed: $\hat{\bar{g}} = \det(\hat{\bar{g}}_{\mu\nu}) = \bar{g} + \bar{h}^\mu{}_\mu = \bar{g}$. A straightforward calculation shows that the Christoffel symbols in the perturbed background are $\hat{\Gamma}^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} + \bar{\gamma}^\mu_{\nu\lambda}$, where $\bar{\gamma}^\mu_{\nu\lambda} = (1/2)\bar{g}^{\mu\rho}(\nabla_\lambda \bar{h}_{\nu\rho} + \nabla_\nu \bar{h}_{\rho\lambda} - \nabla_\rho \bar{h}_{\nu\lambda})$. The Ricci tensor, expanded to linear order in $\bar{h}_{\mu\nu}$, is, in the transverse traceless gauge,

$$\mathcal{R}_{\mu\nu} = \bar{R}_{\mu\nu} + \frac{1}{2}\nabla_\lambda\nabla_\mu \bar{h}^\lambda{}_\nu + \frac{1}{2}\nabla_\lambda\nabla_\nu \bar{h}^\lambda{}_\mu - \frac{1}{2}\bar{\nabla}^2 \bar{h}_{\mu\nu}. \quad (3.2)$$

Note that the perturbation in $R_{\mu\nu}$ is traceless because of the radiation gauge conditions, as expected. Tracing out the gravitational equations of motion in Eq. (2.7) and subtracting the Ricci scalar-dependent term, we see that $\bar{R}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi + \bar{T}_{\mu\nu} - (1/2)\bar{g}_{\mu\nu}\bar{T}$. The perturbed equations of motion are identical to these. In order to find the equations of motion of the perturbations, then, we have to expand the equations to linear order in $\bar{h}_{\mu\nu}$ and cancel the lowest order terms since the background is a solution to Eq. (2.7). We also need to demonstrate that the perturbation is consistent with the matter sector of the theory, in that it is not exciting any perturbations there. At this time, it is convenient to refer to the explicit form of the background solution, and introduce the gauge conditions for it. We will work with the synchronous gauge, where the metric is exactly the same as in (2.8), $d\bar{s}^2 = -dt^2 + \bar{a}^2(t)d\vec{x}^2$, and the perturbation satisfies $\bar{h}_{00} = \bar{h}_{0k} = 0$. Then, on all cosmological backgrounds of the previous section, $\bar{h}^{\mu\nu}\bar{\nabla}_\nu\phi = \dot{\phi}\bar{h}^{\mu 0} = 0$, and so, the perturbed d'Alembertian is identical to the unperturbed one:

$$\hat{\bar{\nabla}}^2\phi = \frac{1}{\sqrt{\hat{\bar{g}}}}\partial_\mu\left(\sqrt{\hat{\bar{g}}}\,\hat{\bar{g}}^{\mu\nu}\partial_\nu\phi\right) = \bar{\nabla}^2\phi - \frac{1}{\sqrt{\bar{g}}}\partial_\mu\left(\sqrt{\bar{g}}\,\bar{h}^{\mu\nu}\partial_\nu\phi\right)$$

$$= \bar{\nabla}^2\phi. \quad (3.3)$$

Next, since we are working in the synchronous gauge, the comoving velocity is unchanged: $\hat{\bar{u}}^\mu = \bar{u}^\mu = \mathrm{diag}(1,\vec{0})$, and so is the trace of the stress energy tensor. The perturbation of the stress-energy tensor for fluid sources is

$$\delta\mathcal{T}_{\mu\nu} = \bar{\mathcal{T}}_{\mu\nu} - \frac{1}{2}\hat{\bar{g}}_{\mu\nu}\bar{T} - \bar{T}_{\mu\nu} + \frac{1}{2}\bar{g}_{\mu\nu}\bar{T} = \bar{p}\,\bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}_{\mu\nu}\bar{T}$$

$$= \bar{h}_{\mu\nu}\frac{\bar{\rho} - \bar{p}}{2}. \quad (3.4)$$

Also, the stress-energy conservation equation is identical to the case without the perturbation because of the gauge conditions; specifically, $\hat{\bar{\nabla}}_\mu\bar{T}^{\mu\nu} = \bar{\nabla}_\mu\bar{T}^{\mu\nu}$. Substituting these conditions and the Eq. (3.2) in Eq. (2.7), we obtain the covariant form of the equations of motion for tensor perturbations:

$$\nabla_\lambda\nabla_\mu \bar{h}^\lambda{}_\nu + \nabla_\lambda\nabla_\nu \bar{h}^\lambda{}_\mu = \bar{\nabla}^2 \bar{h}_{\mu\nu} + \frac{\bar{\rho} - \bar{p}}{2}\bar{h}_{\mu\nu}. \quad (3.5)$$

We now want to rewrite this equation in terms of the mixed perturbation tensor $\bar{h}^\mu{}_\nu$, which is the natural variable to use because of the gauge conditions. Using the explicit form of the background metric (2.8), we can verify that in the synchronous gauge the following conditions hold identically (see, e.g., [24,25]):

$$\nabla_\nu \bar{h}_{00} = \nabla_0 \bar{h}_{0\nu} = 0, \quad \nabla_j \bar{h}_{k0} = -\bar{H}\bar{h}_{jk},$$

$$\nabla_0 \bar{h}_{jk} = \dot{\bar{h}}_{jk} - 2\bar{H}\bar{h}_{jk}, \quad \nabla_i \bar{h}_{jk} = \partial_i \bar{h}_{jk}. \quad (3.6)$$

Since these equations are covariant with respect to the assumed background, we can raise the indices using the background metric, and after some straightforward algebra, we can also verify the following conditions on the second covariant derivatives of the perturbation [24,25]:

$$\nabla^2 \bar{h}^0{}_0 = \nabla^2 \bar{h}^0{}_j = \nabla_\mu\nabla^0 \bar{h}^\mu{}_0 = \nabla_\mu\nabla^0 \bar{h}^\mu{}_j = \nabla_\mu\nabla^j \bar{h}^\mu{}_0 = 0,$$

$$\nabla_\mu\nabla^j \bar{h}^\mu{}_k = (\dot{\bar{H}} + 4\bar{H}^2)\bar{h}^j{}_k, \quad (3.7)$$

$$\nabla^2 \bar{h}^j{}_k = \frac{1}{a^2}\vec{\nabla}^2 \bar{h}^j{}_k - \ddot{\bar{h}}^j{}_k - 3\bar{H}\dot{\bar{h}}^j{}_k + 2\bar{H}^2 \bar{h}^j{}_k.$$

In these equations, $\vec{\nabla}^2$ is just the three-dimensional flat space Laplacian $\vec{\nabla}^2 = \Sigma_{j=1}^3 \partial_j^2$. Upon substituting these expressions in Eq. (3.5), we find

$$\ddot{\bar{h}}^j{}_k + 3\bar{H}\dot{\bar{h}}^j{}_k - \frac{1}{a^2}\vec{\nabla}^2 \bar{h}^j{}_k + \left(2\dot{\bar{H}} + 6\bar{H}^2 + \frac{\bar{p} - \bar{\rho}}{2}\right)\bar{h}^j{}_k = 0. \quad (3.8)$$

The last term looks similar to the environment-induced mass; however, by the E frame FRW equations of motion (2.9), this term is identically zero. Hence, finally, the equations of motion for the transverse traceless perturbations of the metric are

$$\ddot{\overline{h}}{}^j{}_k + 3\overline{H}\,\dot{\overline{h}}{}^j{}_k - \frac{1}{a^2}\vec{\nabla}^2\,\overline{h}{}^j{}_k = 0, \tag{3.9}$$

i.e., the propagation equations for a set of minimally coupled scalars. The index structure of the perturbations $\overline{h}{}^j{}_k$ can be easily accounted for by going to the mode expansion $\overline{h}{}^j{}_k = \overline{\epsilon}{}^j{}_k(t,\vec{p})\exp(i\vec{p}\cdot\vec{x})$. The gauge conditions $\overline{\epsilon}{}^j{}_j = p_j\,\overline{\epsilon}{}^j{}_k = 0$ can be easily solved as follows. We orient the spatial reference frame s.t. the z axis is along the direction of propagation of the wave, and then we find that two linearly independent polarization tensors are given by two Pauli matrices $\epsilon_+ = \sigma_3$ and $\epsilon_\times = \sigma_1$. Any other perturbation is their linear combination: $\overline{\epsilon}{}^j{}_k = f_+\epsilon_+ + f_\times\epsilon_\times$, where f_k, $k=(+,\times)$ are the mode functions. Since we will see later that the mixed index perturbations $\overline{h}{}^\mu{}_\nu$ are conformally invariant, and f_k are modes of these degrees of freedom, $\overline{f}_k = f_k$ and from now on we will omit the bars from f_k's. In a different coordinate system, the basis polarizations are given by $\epsilon_k = R^{-1}(\vec{p})\sigma_k R(\vec{p})$, where $R(\vec{p})$ is the rotation matrix which orients $\vec{p}$ along the z axis. Then, as we mentioned above, the equations of motion (3.9) can be rewritten in terms of the mode functions $f_+, f_\times$ as the Klein-Gordon equations for a set of minimally coupled massless scalar fields, exactly as in general relativity:

$$\overline{\nabla}^2 f_k = 0. \tag{3.10}$$

This equation suggests very strongly that the metric perturbations choose to propagate along geodesics in the Einstein conformal frame. We will demonstrate that this is indeed true in the next section.

Now we turn our attention to the description of the wave propagation from the point of view of the JBD frame. We could proceed in precisely the same way as in the derivation of the wave equation in the E frame. Starting with a fixed JBD background, we add a perturbation to the metric as $\hat{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, and expand the equations of motion (2.2) to first order in the fluctuation. Here, however, we do not have to repeat all the steps of the derivation in the E frame, because most of the details are the same, with barred quantities (E frame) replaced with the unbarred ones. We will therefore outline only the main points and establish a correspondence between the two pictures. First, note that since the field redefinitions (2.5) imply that $h_{\mu\nu} = \overline{h}_{\mu\nu}/\chi$, the mixed index perturbations are conformally invariant: $h^\mu{}_\nu = \overline{h}{}^\mu{}_\nu$. Next, the transverse traceless conditions in the JBD synchronous gauge are also conformally invariant: clearly, the equality of the mixed index tensors implies that if one is traceless, so is the other. Further, the constraints of the JBD synchronous gauge on $h^\mu{}_\nu$ are also identical: $h^0{}_0 = h^0{}_j = 0$. Finally, since

$$\nabla_\mu h^\mu{}_\nu = \overline{\nabla}_\mu \overline{h}{}^\mu{}_\nu + \gamma^\mu{}_{\mu\rho}\overline{h}{}^\rho{}_\nu - \gamma^\rho{}_{\mu\nu}\overline{h}{}^\mu{}_\rho = \overline{\nabla}_\mu \overline{h}{}^\mu{}_\nu - 2\,\overline{h}{}^0{}_\nu\frac{\dot{\chi}}{\chi}$$

$$+ \overline{h}{}^\mu{}_\mu\frac{\partial_\nu\chi}{2\chi} = \overline{\nabla}_\mu \overline{h}{}^\mu{}_\nu \tag{3.11}$$

the transversality condition is also the same: $\nabla_\mu h^\mu{}_\nu = 0$. Hence the perturbation of the Ricci tensor in the JBD frame is of the same form as Eq. (3.2), except that the quantities are all unbarred. There however are additional source terms, since now we have to take into account the contributions of the second derivatives of the field χ as presented in Eq. (2.2). These terms will be important for showing that the geometrical optics limit of the wave dynamics picks the E frame evolution. Using the conformal correspondence we have established so far, we can write down the equations of motion for the perturbations in the JBD frame:

$$\nabla_\lambda\nabla_\mu h^\lambda{}_\nu + \nabla_\lambda\nabla_\nu h^\lambda{}_\mu = \nabla^2 h_{\mu\nu} + \frac{\rho-p}{2}h_{\mu\nu} + h_{\mu\nu}\frac{\nabla^2\chi}{\chi}$$

$$+ \frac{\nabla_\rho\chi}{\chi}(\nabla^\rho h_{\mu\nu} - \nabla_\mu h^\rho{}_\nu - \nabla_\nu h^\rho{}_\mu). \tag{3.12}$$

In the FRW background, after some straightforward algebra, we can rewrite these as

$$h''{}^j{}_k + 3Hh'{}^j{}_k - \frac{1}{a^2}\vec{\nabla}^2 h^j{}_k + \frac{\chi'}{\chi}h'{}^j{}_k$$

$$+ \left(2H' + 6H^2 + \frac{\chi'' + 5H\chi'}{\chi} + \frac{p-\rho}{2\chi}\right)h^j{}_k = 0. \tag{3.13}$$

Again, the environment-induced mass term vanishes, now by way of the JBD frame FRW equations of motion (2.4). The final set of the equations of motion for the transverse traceless perturbations of the metric are

$$h''{}^j{}_k + 3Hh'{}^j{}_k - \frac{1}{a^2}\vec{\nabla}^2 h^j{}_k + \frac{\chi'}{\chi}h'{}^j{}_k = 0. \tag{3.14}$$

These equations contain the term proportional to χ'/χ, since in the JBD frame the tensor perturbations also couple to the scalar χ. To see how this term arises in the form above, recall that because Eqs. (3.9) and (3.14) map into each other under conformal transformations (2.5), the χ'-dependent term must arise from the effect of the conformal map on the comoving time. Now, it is evident that the same polarization basis as that used in the E frame analysis above can be employed to represent an arbitrary perturbation. In fact, we could just take any solution of Eq. (3.9), and conformally transform it to the JBD frame according to Eq. (2.5), and it will be guaranteed to solve Eq. (3.14) as well. In terms of the mode functions, then, the JBD frame equation of motion for gravity waves (3.14) can be written as

$$\nabla_\mu(\chi\nabla^\mu f_k) = 0, \tag{3.15}$$

where f_k are the mode functions, with $k \in \{+,\times\}$. In this case, the modes propagate under the influence of an additional conformal coupling to χ. This equation is conformal to Eq. (3.10) under Eq. (2.5). These two equations, (3.10) and (3.15), will be our starting point of derivation of the geometrical optics approximation in the next section.

IV. SINGULARITY REVEALED

We are finally ready to show that smoothing the JBD frame is not sufficient to remove the singularity from FRW cosmological solutions we have discussed in Sec. II. As we have indicated in the introduction, this is because the gravitons, when considered as probes of the geometry in the classical limit, move along null geodesics of the E frame, and not JBD. Since the E frame is manifestly singular, the graviton world lines are incomplete and so they reintroduce the singularity's effects even in the JBD frame.

Let us first derive the geodesic equations and then consider their implications. As a warm up, we will first find the geodesics in the E frame, since there the mode equation is very simple — it is just the Klein-Gordon equation for a minimally coupled massless scalar field for each polarization mode of the graviton sector $\bar{\nabla}^2 f_k = 0$. Now, the geometrical optics limit corresponds to setting $f_k = \exp(iS)$, and identifying the phase S as the action of the pointlike probe which replaces the wave packet [23]. The field equation in terms of S becomes $i\bar{\nabla}^2 S - (\bar{\nabla}S)^2 = 0$, and so, assuming that S is real and separating the real and imaginary parts of the equation we find

$$(\bar{\nabla}S)^2 = 0, \quad \bar{\nabla}^2 S = 0. \tag{4.1}$$

Now, after S is integrated, it must be a function of the space-time coordinates only, since it is the phase of f_k. Hence, if we use $S = S(x^\mu)$, the action must be representable as a path integral along a geodesic which the wave packet is following: $S = \int dx^\mu \bar{\nabla}_\mu S$. Let us now introduce $V_\mu = \bar{\nabla}_\mu S$ and recall that $dx^\mu = \dot{x}^\mu d\bar{\lambda}$ along a geodesic, where $\bar{\lambda}$ is the affine parameter. Hence [23],

$$S = \int V_\mu \dot{x}^\mu d\bar{\lambda}. \tag{4.2}$$

The equations of motion for S (4.1) become, in terms of the field V_μ, the following two constraints: $\bar{g}^{\mu\nu} V_\mu V_\nu = 0$, $\bar{\nabla}_\mu \bar{g}^{\mu\nu} V_\nu = 0$. The first constraint is local and hence can be easily enforced at the level of the particle action with the help of a Lagrange multiplier. It just tells us that V_μ is a null vector. The second constraint is not local, and in fact in the geometrical optics limit is always a very small quantity compared to the first one, and is usually ignored. Here we will retain it, and use it to determine the Lagrange multiplier. The constrained particle action is then $S = \int d\bar{\lambda}(V_\mu \dot{x}^\mu + \bar{\eta}\bar{g}^{\mu\nu} V_\mu V_\nu)$. In the geometrical optics approximation, because of destructive interference of waves, only those trajectories for which S is extremized survive. Treating V_μ as an independent variable, and varying S with respect to it, we find $\bar{g}^{\mu\nu} V_\nu = -\dot{x}^\mu/2\bar{\eta}$. This simply means that V_μ is tangent to the wave packets world lines, which by $\bar{g}^{\mu\nu} V_\mu V_\nu = 0$ must be null. The differential (second) constraint translates into the condition $\bar{\nabla}_\mu (\dot{x}^\mu/\bar{\eta}) = 0$. Now, in the flat space limit, the covariant derivative would have become an ordinary derivative, and we would have been able to use $\partial_\mu \dot{x}^\mu = d(\partial_\mu x^\mu)/d\bar{\lambda} = 0$ to assert that $\dot{x}^\mu \bar{\partial}_\mu (1/\bar{\eta}) = d(1/\bar{\eta})/d\bar{\lambda} = 0$, i.e., that $\bar{\eta}$ is a constant along trajectories. Then, with an

appropriate normalization ($\bar{\eta} = -1/4$) we would have found $S = \int d\bar{\lambda} \dot{x}^\mu \dot{x}_\mu$ with $\dot{x}^\mu \dot{x}_\mu = 0$, i.e., just the standard action of a relativistic massless particle. In the curved space, the tangent vector field of a congruence of geodesics need not be divergenceless in general, i.e. it need not be $\bar{\nabla}_\mu \dot{x}^\mu = 0$. However, since this quantity is the second constraint in Eq. (4.1), because of destructive interference, its contribution to wave propagation is negligible in the geometrical optics approximation. Moreover, if we go to the Riemann normal coordinates, defined at a point $\mathbf{x_0}$ by the condition $\bar{\Gamma}^\mu_{\nu\lambda}(\mathbf{x_0}) = 0$, we will indeed find that to the lowest order $\bar{\nabla}_\mu \dot{x}^\mu$ vanishes. So, we will set $\bar{\nabla}_\mu \dot{x}^\mu = 0$, as a part of the geometrical optics approximation. This then tells us that $\dot{x}^\mu \bar{\nabla}_\mu (1/\bar{\eta}) = d(1/\bar{\eta})/d\bar{\lambda} = 0$ — just as in the flat space limit, $\bar{\eta}$ is a constant along geodesics. Therefore, the particle action in the E frame becomes, after again choosing $\bar{\eta} = -1/4$ [23]

$$S = \int d\bar{\lambda} \, \bar{g}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \quad \bar{g}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0, \tag{4.3}$$

which defines dynamics of the E frame massless minimally coupled particles, i.e., the null geodesics. The geodesic equations can be derived straightforwardly by varying this action and imposing $\bar{g}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$. The result is

$$\ddot{x}^\mu + \bar{\Gamma}^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda = 0, \quad \bar{g}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0. \tag{4.4}$$

Hence as claimed, in the geometrical optics limit the quanta of gravitational perturbations move along the E frame null geodesics, and not along trajectories in the JBD frame.

We can now derive the same result using the JBD form of the equation of motion, in order to check how the dilaton force deforms the JBD graviton trajectories into the E frame null geodesics. To show that the result is a conformal transform of Eq. (4.4), we will derive it from first principles, rather than just apply the field redefinition (2.5). Bearing in mind that we are following the motion of the same mode ($h^\mu_\nu = \bar{h}^\mu_\nu$), but in a different reference frame, we again use $f_k = \exp(iS)$, but now we substitute it in the JBD frame equation $\nabla_\mu(\chi \nabla^\mu f_k) = 0$ from Eq. (3.15). When S is real, we again get two equations

$$(\nabla S)^2 = 0, \quad \nabla_\mu(\chi \nabla^\mu S) = 0. \tag{4.5}$$

The first of these two equations is identical, up to an overall factor of χ, to the first equation in Eq. (4.1). Thus it also must correspond to the null condition, $g^{\mu\nu} V_\mu V_\nu = 0$ where we still have $V_\mu = \nabla_\mu S$, albeit in the JBD frame. The second equation is a little bit more complicated than in Eq. (4.1). To see the effect of the coupling χ, let us rederive the equations of motion for graviton probes along the same lines we followed in the E frame calculation above. As we have shown, S is a function of the space-time coordinates only, and when we use $S = S(x^\mu)$, S again is just the (same) integral $S = \int dx^\mu \nabla_\mu S$. This of course must be identical, since $\nabla_\mu S = \partial_\mu S = \bar{\nabla}_\mu S$. With $V_\mu = \nabla_\mu S$ and $dx^\mu = \dot{x}^\mu d\bar{\lambda}$ along a trajectory,

$$S = \int V_\mu \dot{x}^\mu d\overline{\lambda}. \qquad (4.6)$$

The JBD frame equations of motion for S then are $g^{\mu\nu}V_\mu V_\nu = 0$, $\nabla_\mu(\chi g^{\mu\nu}V_\nu) = 0$. Enforcing the first constraint at the level of the action with the help of a Lagrange multiplier, we find $S = \int d\overline{\lambda}(V_\mu \dot{x}^\mu + \eta g^{\mu\nu}V_\mu V_\nu)$. Note that here we are contracting the indices with the JBD frame metric $g_{\mu\nu}$, and so $\eta \neq \overline{\eta}$. Varying this action with respect to V_μ, we find $g^{\mu\nu}V_\nu = -\dot{x}^\mu/2\eta$, and so again, V_μ is a null vector tangent to the graviton trajectories in the JBD frame. However, the differential constraint is then $\nabla_\mu(\chi \dot{x}^\mu/\eta) = 0$. We still implement $\nabla_\mu \dot{x}^\mu = 0$ as a part of the geometrical optics approximation, and find $\dot{x}^\mu \overline{\nabla}_\mu(\chi/\eta) = d(\chi/\eta)/d\overline{\lambda} = 0$, implying that $\chi/\eta = c$ is constant along each particle path. Choosing $c = -4$, we can write the JBD frame particle action as

$$S = \int d\overline{\lambda} \frac{1}{\chi} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \qquad g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0. \qquad (4.7)$$

The particle trajectories are those paths which extremize this action. By varying this action, and taking the null constraint into account, we find [23]

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda - \frac{\nabla_\nu \chi}{\chi} \dot{x}^\nu \dot{x}^\mu = 0, \qquad g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0. \qquad (4.8)$$

Hence in the JBD frame gravitons do not move along geodesics, but rather along null trajectories determined by the additional force proportional to the four gradient of the scalar field χ. Of course, this is what we have expected all along, since we can see that in the JBD frame action (2.1) the χR coupling implies that the JBD ''gravitons'' have χ field charge, and hence must couple to χ's field strength. Yet, when we use the conformal transformation (2.5) which transforms the JBD action (2.1) to the E frame action (2.6), rendering the graviton kinetic term canonical, i.e., just $\overline{R}$, we see that this same redefinition removes completely the χ-dependent force from the Eq. (4.8). Under the conformal transformation (2.5), the connexion changes according to $\Gamma^\mu_{\nu\lambda} = \overline{\Gamma}^\mu_{\nu\lambda} + (1/2\chi)(\delta^\mu_\nu \nabla_\lambda \chi + \delta^\mu_\lambda \nabla_\nu \chi - \overline{g}^{\mu\rho} \overline{g}_{\nu\lambda} \nabla_\rho \chi)$ and when contracted with $\dot{x}^\nu \dot{x}^\lambda$, the difference precisely cancels the χ-dependent force in Eq. (4.8). Hence, the Eqs. (4.4) and (4.8) are conformal images of each other, and they imply that the gravity wave packets, to the lowest order, move along null geodesics of the Einstein frame metric $\overline{g}_{\mu\nu}$.

Having proven this, we conclude that the conformal removal of the singularity was only partially successful. By the construction of the theory as given in the action (2.1), the matter fields $\mathcal{Y}$ coupled only to the JBD frame metric $g_{\mu\nu}$. This metric was made smooth by the choice of the coupling function $\omega(\chi)$, whose role was to push the E frame singularity to the asymptotic timelike infinity of the geometry. Due to this effect, the pointlike matter probes were unable to ever reach the singular region. Since they follow the JBD frame geodesics, these geodesics are all complete. However, in this case, geodesic completeness is insufficient to conclude that the solution is nonsingular. The gravity waves

couple to both the metric and the scalar field in the JBD frame, and the net effect of these couplings is to deform the graviton trajectories back to the E frame null geodesics. These world lines are incomplete, because the E frame metric has a singularity a finite affine distance from any other place in the manifold. As a result, we cannot arbitrarily extend the history of one such universe. Sooner or later, we will reach the E frame singularity, where we will have to deal with the problem of defining the proper initial conditions for gravity waves. Because the graviton sector there is ill-defined, we simply would not be able to unambiguously set the initial conditions. Gravity waves could then communicate the presence of the singularity to the other degrees of freedom in the theory, at higher order. To see this, note that because the matter fields couple to the metric, they would also couple to the tensor perturbations at higher order. Since to linear order the tensor perturbations were the only degrees of freedom which detected the singularity, none of the other modes could be adjusted to completely cancel the influence of the singularity on the gravitons. Therefore the gravitons would render the hidden singularity again visible to all the fields in the theory. In particular, the relic gravitons from a very early era of the universe that survived recombination and that comprise the present graviton background would still encode information about the maelstrom they came from.

V. CONCLUSION

In this article, we have shown that conformal transformations alone cannot completely remove the initial singularity from a cosmological solution in the scalar-tensor models, even if we allow the scalar-gravity coupling strength to depend on the scalar field. The matter sector in the models we have considered consists only of modes whose stress-energy tensor satisfies the strong energy condition. Clearly, if the matter sources violate the SEC, then evading singularity may yet be possible — but this cannot be accomplished by a simple conformal transformation involving the JBD scalar. Such conformal transformations merely hide the singularity from observers which propagate along geodesics of the JBD frame metric, that can be arranged to be smooth by adjusting the coupling $\omega(\phi)$. However, once different observers are allowed, which couple to both the JBD frame metric and scalar, because of the additional scalar force they do not move along the JBD frame geodesics, but along those of a different conformal frame. In particular, gravitons move along the Einstein frame null geodesics, which are always incomplete, because the E frame solutions are all singular. Hence gravity waves see the singularity in all scalar-tensor models, regardless of the specifics of the model in question. We can in fact see that a similar property should hold in any nonminimal effective theory of gravity. The plain vanilla field redefinitions cannot remove singularities because they do not couple universally to all the fields in the theory, and so there will always be a mode which will discern the presence of the singularity in the manifold.

In string theory, however, this may merely mean that the effective action approach must break down close to the singularity. Since all the states in string theory are comprised of strings (and D-branes, as we have seen recently in some of

the models), close to the singularity, gravity's pull on low energy pointlike degrees of freedom is so strong that they effectively decompactify. Thus near the singularity instead of a gas of interacting highly energetic particles we find a gas of interacting highly energetic strings. But strings couple naturally to the string frame metric, which therefore seems to be the frame we must choose to study the effects of finite size of probes (or higher order α' terms in the derivative expansion). However, strings also couple to the dilaton, and we must consider its effect on dynamics too, since it represents the string coupling constant and so controls the validity of the semiclassical approximation. Therefore in string theory, if we are to remove the singularity, we must regulate both the metric and the dilaton, which is reminiscent of the situation in the scalar-tensor models we have considered here. This also implies that if we manage to cure the solution in one frame, as long as we take only reasonable field redefinitions (i.e., those which do not alter the global properties of the space-time) we find that the solution is free of singularities in any frame. This, of course, is consistent with the fact that field redefinitions do not change physics, but just alter the language we describe physics with.

Finding whether the singularities could be removed from the effective theory still awaits. What we have learned to date seems to indicate that to regulate the singularities we must impose severe alterations on the theory. Simple tweaks of the classical (or semiclassical) effective action do not seem to work. To illustrate this point, we may recall the graceful exit problem in the pre-big-bang scenario [14]. The idea, briefly, was to use the multiplicity of solutions which arise because of duality, and paste them together in such a way that the result is a smooth Universe of an infinite life span and with a region of very large coupling *and* curvature, mimicking the Big Bang. The difficulty with this proposal is that the continuity of the solutions and the equations of motion, in the effective potential approximation (i.e., the effective action truncated to second order in derivatives) makes the smooth matching of the two branches with the proper asymptotic behavior impossible unless very exotic conditions develop at very high energies [16]. The matching is supposed to take place very near the singularity — but in the semiclassical approximation, the singularity does all the steering of the dynamics near it, and hence the solutions cannot evade it. The branch changing thus is not possible in the semiclassical limit, with the matter sources satisfying an even weaker version of the energy conditions than in general relativity. A necessary condition (but not sufficient) for branch changing has been derived recently by Brustein and Madden (in the last of Ref. [16]), who showed that unless the null energy condition (i.e., the requirement $\rho + p \geqslant 0$ for fluid sources) is violated, the branch change cannot occur. This still does not guarantee the evasion of the singularity via branch changing, but it does tell us that unless we violate the null energy condition in some way, in a region of high curvature, we cannot even hope to avoid the singularity. Since the conventional matter (pointlike or stringlike) degrees of freedom generally do not violate the null energy condition, this requires the presence of exotic matter sources (''string phase'' [26]) or nonperturbative phenomena (such as discussed by [12,15,5]) to account for singularity smoothing. At this moment, we know very little about this type of matter. The recent rapid development of string theory, however, gives hope that we may learn more in foreseeable future.

[1] See, e.g., S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).

[2] G.T. Horowitz and R.C. Myers, Gen. Relativ. Gravit. **27**, 915 (1995).

[3] E. Witten, Nucl. Phys. **B443**, 85 (1995).

[4] C. Lovelace, Phys. Lett. **135B**, 75 (1984); E.S. Fradkin and A.A. Tseytlin, *ibid.* **158B**, 316 (1985); C.G. Callan, D. Freidan, E.J. Martinec, and M.J. Perry, Nucl. Phys. **B262**, 593 (1985); A. Sen, Phys. Rev. D **32**, 2102 (1985); Phys. Rev. Lett. **55**, 1846 (1985); C.G. Callan, I.R. Klebanov, and M.J. Perry, Nucl. Phys. **B278**, 78 (1986); S. Deser and N. Redlich, Phys. Lett. B **176**, 350 (1986); D. Luest, S. Theisen, and G. Zoupanos, Nucl. Phys. **B296**, 800 (1988); J. Lauer, D. Luest, and S. Theisen, *ibid.* **B304**, 236 (1988); D. Gross and J. Sloan, *ibid.* **B291**, 41 (1987).

[5] E. Kiritsis and C. Kounnas, Phys. Lett. B **331**, 51 (1994); e-print hep-th/9407005; e-print gr-qc/9509017.

[6] D.G. Boulware and S. Deser, Phys. Lett. B **175**, 409 (1986); S. Kalara, C. Kounnas, and K.A. Olive, *ibid.* **215**, 265 (1988); S. Kalara and K.A. Olive, *ibid.* **218**, 148 (1989).

[7] R.C. Myers, Phys. Lett. B **199**, 371 (1987); M. Mueller, Nucl. Phys. **B337**, 37 (1990).

[8] I. Antoniadis, C. Bachas, J. Ellis, and D.V. Nanopoulos, Phys. Lett. B **211**, 393 (1988); Nucl. Phys. **B328**, 117 (1989).

[9] N. Kaloper and K.A. Olive, Astropart. Phys. **1**, 185 (1993).

[10] A.A. Tseytlin, Mod. Phys. Lett. A **6**, 1721 (1991); Class. Quantum Grav. **9**, 979 (1992); Int. J. Mod. Phys. D **1**, 223 (1992).

[11] P. Binetruy and M.-K. Gaillard, Phys. Rev. D **34**, 3069 (1986); I. Antoniadis and C. Kounnas, Nucl. Phys. **B175**, 729 (1987); N.R. Stewart, Class. Quantum Grav. **8**, 1701 (1991); B.A. Campbell, A.D. Linde, and K.A. Olive, Nucl. Phys. **B355**, 146 (1991); G. Veneziano, Phys. Lett. B **265**, 287 (1991); B.A. Campbell, N. Kaloper, and K.A. Olive, *ibid.* **277**, 265 (1992); D. Goldwirth and M. Perry, Phys. Rev. D **49**, 5019 (1994); E.J. Copeland, A. Lahiri, and D. Wands, *ibid.* **50**, 4868 (1994); **51**, 1569 (1995).

[12] R.H. Brandenberger and C. Vafa, Nucl. Phys. **B316**, 391 (1989); A.A. Tseytlin and C. Vafa, *ibid.* **B372**, 443 (1992).

[13] K. Behrndt and S. Forste, Phys. Lett. B **320**, 253 (1993); Nucl. Phys. **B430**, 441 (1994); A. Lukas, B. Ovrut, and D. Waldram,

Phys. Lett. B **393**, 65 (1997); Nucl. Phys. **B495**, 365 (1997); e-print hep-th/9611204; N. Kaloper, Phys. Rev. D **55**, 3394 (1997); H. Lu *et al.*, Phys. Rev. D **55**, 7926 (1997); e-print hep-th/9612224; e-print hep-th/9707182; R. Poppe and S. Schwager, Phys. Lett. B **393**, 51 (1997); F. Larsen and F. Wilczek, Phys. Rev. D **55**, 4591 (1997); J. Maharana, S. Mukherji, and S. Panda, Mod. Phys. Lett. A **12**, 447 (1997).

[14] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993); R. Brustein and G. Veneziano, Phys. Lett. B **329**, 429 (1994).

[15] I. Antoniadis, J. Rizos, and K. Tamvakis, Nucl. Phys. **B415**, 497 (1994); J. Rizos and K. Tamvakis, Phys. Lett. B **326**, 57 (1994); R. Easther and K. Maeda, Phys. Rev. D **54**, 7252 (1996).

[16] N. Kaloper, R. Madden, and K.A. Olive, Nucl. Phys. **B452**, 677 (1995); Phys. Lett. B **371**, 34 (1996); R. Easther, K. Maeda, and D. Wands, Phys. Rev. D **53**, 4247 (1996); S.J. Rey, Phys. Rev. Lett. **77**, 1929 (1996); in *Supersymmetry '96, Theoretical Perspectives and Experimental Outlook*, *Proceedings of the International Conference*, College Park, Maryland, edited by R. Mohapatra and A. Rasin [Nucl. Phys. B (Proc. Suppl.) **52A**, 344 (1997)], e-print hep-th/9607148; e-print hep-th/9609115; R. Brustein and R. Madden, e-print hep-th/9702043.

[17] J.D. Barrow, Phys. Rev. D **48**, 3592 (1993).

[18] S.K. Rama, Phys. Rev. Lett. **78**, 1620 (1997); Phys. Rev. D **56**, 6230 (1997); Phys. Lett. B **408**, 91 (1997); e-print hep-th/9704047.

[19] C. Wetterich, Nucl. Phys. **B302**, 645 (1988); **B302**, 302 (1988).

[20] R.R. Metsaev and A.A. Tseytlin, Nucl. Phys. **B293**, 385 (1987); **B298**, 109 (1988); C.M. Hull and P.K. Townsend, Phys. Lett. B **191**, 115 (1987); I. Jack and D.R.T. Jones, *ibid.* **193**, 449 (1987); K.A. Meissner, J. Pawełczyk, and S. Pokorski, Phys. Rev. D **38**, 1144 (1988); A.A. Tseytlin, Phys. Lett. B **317**, 559 (1993).

[21] K. Huang and S. Weinberg, Phys. Rev. Lett. **25**, 895 (1970); R. Hagedorn, Astron. Astrophys. **5**, 184 (1970); also see S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), Chap. 15, Sec. 11.

[22] A. Shapere, S. Trivedi, and F. Wilczek, Mod. Phys. Lett. A **6**, 2677 (1991).

[23] B.A. Campbell, N. Kaloper, R. Madden, and K.A. Olive, Nucl. Phys. **B399**, 137 (1993).

[24] B. Allen and S. Koranda, Phys. Rev. D **52**, 1902 (1995).

[25] M. Gasperini, Phys. Rev. D **56**, 4815 (1997).

[26] M. Gasperini, M. Maggiore, and G. Veneziano, Nucl. Phys. **B494**, 315 (1997).