Effective field theory and matching in nonrelativistic gauge theories

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(Received 12 March 1997; published 10 November 1997)

The effective Lagrangian and power counting rules for nonrelativistic gauge theories are derived via a systematic expansion in the large c limit. It is shown that the $1/c$ expansion leads to an effective field theory which incorporates a multipole expansion. Within this theory there is no need for heuristic arguments to determine the scalings of operators. After eliminating c from the lowest order Lagrangian the states of the theory become independent of c and the scaling of an operator is given simply by its overall coefficient. We show how this power counting works in the calculation of the Lamb shift within the effective field theory formalism. [S0556-2821(98)00501-3]

PACS number(s): 11.10.Ef, 12.20.Ds, 36.10.Dr

Effective field theories are indispensable tools for studying systems with disparate scales. The idea dates back to the Euler-Heisenberg Lagrangian for QED [1] and has been utilized in the context of calculating strong interaction corrections to various processes. Recently, the application of effective field theories in heavy quark systems has led to great progress in our understanding of weak decays. In particular, heavy quark effective field theory (HQET) has been utilized to study hadrons composed of one heavy quark. The use of HQET allows us to separate the physics stemming from the two scales which are relevant to heavy-light bound states, namely the heavy mass m and the strong interaction scale Λ_{QCD}. In a seminal paper [3] Caswell and Lepage introduced a similar effective field theory to study nonrelativistic bound states. However, this theory differs from HQET in several very important ways. The description of nonrelativistic bound states is complicated by existence of the small parameter $v \sim \alpha(mv)$ in the effective theory,[1] where v is the relative velocity of the particles of mass m which compose the bound state. Furthermore, in HQET the heavy quarks are labeled by velocities which are unchanged by bound state effects at leading order. Velocity changing weak transitions are accounted for by ''integrating in'' quarks of varying velocities [4]. In heavy-heavy systems it is no longer true that the quark[2] velocity is fixed. Indeed Coulombic, velocity altering exchanges are what builds up the Schrödinger kernel. Thus, it is clear that the effective Lagrangian for the two systems should be dissimilar. Indeed, in HQET an operator scales in $1/M$ according to its dimension, whereas in nonrelativistic gauge theories this is not so unless one modifies the Lagrangian, as will be shown below.

HQET relies upon an expansion in the heavy quark mass, whereas nonrelativistic gauge effective field theories (NRGT) are expansions in the relative velocity, or equivalently, as we shall call it c, the speed of light. Though it is well known that certain identical operators of the same mass dimension may be of different orders in their respective expansion [most notably $\psi^\dagger(\mathbf{D}^2/2m)\psi$], it has not been pointed out that even for fixed dimensions the operators of the two theories will in general be different. For instance, if one uses dimensionally regulated HQET mixing can only occur between operators with the same scalings in $1/M$, this will not be true in nonrelativistic field theories unless one modifies the Lagrangian, as will be shown in this paper.

In the effective field theory formalism we write down a low energy Lagrangian which reproduces the S matrix elements (in some cases the Green's functions) of the full theory up to some chosen order in a double expansion in the coupling α and some other parameter which dictates the size of the matrix elements in the low energy theory, the heavy quark mass in the case of HQET. The difference between the full theory and the effective theory lies in the ultraviolet modes. This difference is accounted for in the low energy effective theory by the proper choice of coefficients in the Lagrangian (the ''matching'' procedure). The utility of the effective theory lies in the fact that calculations in the effective theory are much simpler now that all the short distance physics has been trivialized.

A crucial part of the matching procedure is the bookkeeping of the expansion parameters. For instance, in HQET where the expansion parameters are α_s and $1/m$, it is possible to place all the dependence on the expansion parameters into overall coefficients of operators in the Lagrangian. This is helpful for two reasons: In performing the matching, operators which are formally of higher order do not contribute to the coefficients of lower order operators. Second, the order at which matrix elements of operators enter should be determined by the coefficients of the operators. That is, the states should not depend on the expansion parameters in such a way that the power counting is jeopardized. For example, in HQET the states are independent of m since the lowest order Lagrangian is independent of m and the normalization of the states is chosen to be

$$\langle \vec{v}\,' | \vec{v} \rangle = 2v^0(2\pi)^3\delta^3(\vec{v}\,' - \vec{v}). \tag{1}$$

For the case of NRGT's the expansion parameters are α and v, the relative quark velocity. One immediately sees that things will be more difficult in this case since the expansion

[1] This complication is often said to lead to the problem of having many scales in the theory $m,\ mv,\ mv^2, \ldots$.

[2] We will refer to the bound state constituents as quarks though they may be electrons as well. Furthermore, we call the gauge particles gluons to generalize to the non-Abelian case.

parameter v is dimensionless. This complication led to velocity scaling rules, derived via heuristic arguments, which assigned powers of v to fields, operators and derivatives [5]. A simpler bookkeeping method was presented in [6] where the authors rescale fields and coordinates by v in such a way as to make explicit the powers of v in the Lagrangian. Here we introduce a slightly different approach which follows simply via an expansion in the now dimensionful parameter $1/c$. To the extent that the physical system is truly nonrelativistic, operators with velocity dimension n are of magnitude $\sim(v/c)^n$, with v a *dynamically* generated scale. This is analogous to HQET where operators of mass dimension n are of magnitude $(\Lambda_{\rm QCD}/m)^n$. Moreover the $1/c$ expansion forces one to modify the Lagrangian.

Let us consider the large c limit of a non-Abelian gauge theory. In this limit the Lagrangian density is given by [2],

$$\mathcal{L}=\frac{1}{2}\left(\partial_i A_0^a-\frac{1}{c}\partial_0 A_i^a-\frac{g}{c}f_{abc}A_i^b A_0^c\right)^2-\frac{1}{4}\left(\partial_i A_j^a-\partial_j A_i^a-\frac{g}{c}f_{abc}A_i^b A_j^c\right)^2+\psi^\dagger\left(iD_0+\frac{\mathbf{D}^2}{2m}\right)\psi+\frac{c_F}{2mc}\psi^\dagger\boldsymbol{\sigma}\cdot\mathbf{B}\psi$$
$$+\frac{1}{8m^3c^2}\psi^\dagger\mathbf{D}^4\psi+\frac{c_D g}{8m^2c^2}\psi^\dagger(\mathbf{D}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{D})\psi+\frac{c_S}{8m^2c^2}\psi^\dagger\boldsymbol{\sigma}\cdot(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D})\psi+O(1/c^3), \tag{2}$$

where

$$D_0=\frac{\partial}{\partial t}-gA_0;\quad \mathbf{D}=\boldsymbol{\nabla}-\frac{g}{c}\mathbf{A}, \tag{3}$$

and ψ is a nonrelativistic 2-spinor describing the heavy quark. In addition we have rescaled the fermion field by a factor of $\sqrt{c}$. For simplicity we have omitted a 2-spinor describing the heavy antiquark. The constants c_F, c_D and c_S are determined by matching onto the full theory.

The explicit powers of the dimensionful parameter $1/c$ in Eq. (2) now makes the power counting simple. The lowest order Lagrangian is

$$\mathcal{L}_0=\frac{1}{2}\left(\partial_i A_0^a-\frac{1}{c}\partial_0 A_i^a\right)^2-\frac{1}{4}(\partial_i A_j^a-\partial_j A_i^a)^2$$
$$+\psi^\dagger\left(iD_0+\frac{\nabla^2}{2m}\right)\psi. \tag{4}$$

Notice that we have retained the $1/c$ term in the kinetic energy of the transverse gluons in the lowest order Lagrangian. That this is necessary is easily seen by considering the Hamiltonian, in which the coefficient of the kinetic energy is c^2. The eigenstates of this lowest order Hamiltonian constitute the states of the effective theory.

At this point we still have not accomplished what we set out to do, namely, trivialize the c dependence of the Lagrangian. Equation (4), as it stands, will lead to a transverse gluon propagator with nontrivial c dependence which can, and as we shall see below does, jeopardize the power counting in $1/c$. To fix this problem we have a choice to either rescale the time or spatial coordinates of the gauge field by c. However, rescaling the time coordinate of the gauge field is unacceptable as it will destroy the initial value problem because, the Hamiltonian for the gauge field and fermion fields would then depend on different time coordinates. Thus, we make the rescaling

$$\widetilde{A}_i(\vec{y}=\vec{x}/c,t)=\sqrt{c}A_i(\vec{x},t), \tag{5}$$

leaving the Coulomb gauge Lagrangian

$$L_0=\int d^3y\frac{1}{2}[(\partial_0\widetilde{A}_i^a)^2-(\partial_i\widetilde{A}_j^a)^2]+\int d^3x\left[\psi^\dagger\left(iD_0+\frac{\boldsymbol{\partial}^2}{2m}\right)\psi\right.$$
$$\left.+\frac{1}{2}(\partial_i A_0^a)^2\right]. \tag{6}$$

We choose to work in the Coulomb gauge for the rest of the paper since it is the most natural choice in a nonrelativistic theory. Moreover, it allows for the clean separation of powers of $1/c$ as is clear from Eq. (6). Note that the states of the effective theory are the eigenstates of the lowest order Hamiltonian derived from this Lagrangian and, as such, are unconfined Coulombic bound states. While these states are independent of c they are not independent of g. The Coulomb gluons are leading order and are not treated perturbatively. We will return to the issue of confining effects at the end of the paper.

Now if we follow through with the consequences of the $1/c$ expansion we will be forced to incorporate the multipole expansion. To see this, let us now consider the $1/c$ corrections in Eq. (2), concentrating for the moment on the Abelian pieces (the extension to the non-Abelian case follows trivially). The leading $1/c$ corrected fermionic bilinear Lagrangian is given by

$$L_{c^{-1}}=\int d^3x\,\psi^\dagger(t,\vec{x})\left[\frac{e}{mc^{3/2}}\widetilde{A}_i(t,\vec{x}/c)\frac{\partial}{\partial x^i}\right.$$
$$\left.+\frac{c_F}{2mc^{5/2}}\boldsymbol{\sigma}\cdot\widetilde{\mathbf{B}}(t,\vec{x}/c)\right]\psi(t,\vec{x}). \tag{7}$$

Expanding in $1/c$ leads to the multipole expansion

$$\mathcal{L}_{\rm mp}=\frac{e}{mc^{3/2}}\psi^\dagger(t,\vec{x})\left(\widetilde{\mathbf{A}}(t,\vec{0})+\frac{x_i}{c}\nabla\cdot\widetilde{\mathbf{A}}(t,\vec{0})\frac{\boldsymbol{\partial}}{\partial x^i}+\cdots\right)$$
$$\times\psi(t,\vec{x})+\frac{c_F}{2mc^{5/2}}\psi^\dagger(t,\vec{x})\boldsymbol{\sigma}\cdot\widetilde{\mathbf{B}}(t,\vec{0})\psi(t,\vec{x})+\cdots, \tag{8}$$

where

$$\widetilde{\mathbf{B}}=\epsilon_{ijk}\frac{\partial}{\partial(y^j)}\widetilde{A}_k(t,\vec{y}). \tag{9}$$

The expansion breaks translational as well as gauge invariance which are symmetries that are restored at each order in $1/c$, and the coefficients in front of each operator are therefore fixed.

Now let us consider the matching procedure for this theory. This will elucidate the power counting scheme as well as the problems one runs into if the multipole expansion is not performed. Using standard diagrammatic techniques it is easy to show that in the full theory the one particle irreducible amputated N point functions with L loops has an overall factor of[3]

$$\Gamma^N \propto (\alpha)^{L-1+N/2} c^{-N/2}. \tag{10}$$

Use of dimensional regularization leads to the functional dependence $\Gamma^N(q_i\cdot q_j/(mc)^2,\mu^2/(mc)^2,\epsilon)$. On shell these corrections will be both IR as well as UV divergent. The UV divergences are taken care of by renormalizing the full theory, while the IR divergences will cancel with those in the effective theory, since both theories behave the same in the infrared. To match onto the effective theory we then expand this full theory result in a power series in $1/c$. The coefficients c_i are then chosen so that the above expansion is reproduced by the effective theory, which we now discuss.

At tree level the matching is trivial. Beyond the terms in Eq. (7) the effective theory Hamiltonian will also contain spatially nonlocal, instantaneous four quark operators which scale as $1/c^2$. These operators arise as a consequence of taking the leading term in the c expansion of full theory diagrams with transverse gluon exchange between quarks.

Let us now study the one loop correction in the effective theory. We will first perform the calculation using Eq. (2) and show that operators which are supposedly of higher order in $1/c$ will generate lower order operators, even within dimensional regularization. We will then show that using Eq. (7) no such mixing occurs, even within dimensional regularization. Let us consider the calculation of the correction to the two point function coming from two insertions of the magnetic moment operator, $\sigma\cdot\mathbf{B}$. If one wishes to keep the power counting such that a given operator should scale as a fixed power in $1/c$ (as dictated by its overall coefficient), the higher order operators will not contribute to the renormalization of the lowest order Lagrangian. That this is so in heavy quark effective theory is easy to see simply on dimensional grounds.

Using the Feynman rules derived from Eq. (2) we have

$$i\Gamma^{(2)}=\frac{c_F^2C(R)e^2}{4m^2c^2}\int\frac{d^nk}{(2\pi)^n}\times\frac{(\vec{k}\times\vec{\sigma})\cdot(\vec{k}\times\vec{\sigma})}{(k_0^2/c^2-\vec{k}^2+i\epsilon)(E+k_0-(\vec{p}+\vec{k})^2/2m+i\epsilon)}. \tag{11}$$

We integrate over k_0 and choose to close the contour in the upper half plane, picking up only the negative energy pole. Integrating over the magnitude of $\vec{k}$ leaves

$$\Gamma^{(2)}=\frac{c_F^2C(R)\alpha}{\pi}\Gamma(2\epsilon)\left[2\left(E-\frac{\vec{p}^2}{2m}\right)+4mc^2+\frac{4}{3}\frac{\vec{p}^2}{2m}\right]+\text{finite}, \tag{12}$$

where $C(R)$ is 1 and 4/3 in QED and QCD, respectively. We see that using Eq. (2) leads to the mixing of operators of different orders in $1/c$. This can be avoided by expanding Eq. (11) in powers of $\vec{k}$. In previous matching calculations [8,9,7] this is in fact what has been done, and it is justified as a bona-fide approximation, as an expansion in small $\vec{k}$ [8]. This amounts to dropping the $\vec{k}$ dependent terms in the denominators. Such an expansion makes working with dimensional regularization particularly simple, since we now have a scaleless integral which vanishes in this scheme.

However, we emphasize that, once we choose to amend the Feynman rules we must also amend the low energy theory. Not to do so would destroy the power counting scheme. In Ref. [6], the authors point out that the transverse gluons can lead to enhancements in c (their $1/v$) in the low energy theory. This is only true if one insists upon calculating the low energy matrix elements using Eq. (2). Instead one must calculate using an amended Lagrangian (8) which reproduces the Feynman rules utilized in the matching calculations. Indeed it is simple to show that Feynman rules in Eq. (8) leads to the necessary expansion.

Using Eq. (8) we may now calculate anew the contribution to the two point function from two insertions of $\sigma\cdot B$,

$$i\Gamma^{(2)}=\frac{c_F^2C(R)e^2}{2m^2c^5}\int\frac{d^nk}{(2\pi)^n}\times\frac{\vec{k}^2}{(k_0^2-\vec{k}^2+i\epsilon)(E-\vec{p}^2/2m+k_0+i\epsilon)}. \tag{13}$$

Notice that all the c dependence is now explicit, and the variable $\vec{k}$ has units of energy. A simple calculation leads to the result

$$\Gamma^{(2)}=\frac{c_F^2C(R)\alpha}{2\pi}\Gamma(2\epsilon)\frac{(E-\vec{p}^2/2m)^3}{m^2c^4}+\text{finite}. \tag{14}$$

This correction renormalizes some higher order operator which vanishes by the equations of motion. It is clear that once we use the correct effective, theory the corrections re-

[3]Since $\alpha=e^2/c$ we could equally write the expansion purely in terms of $1/c$ taking $e\approx 1$. We have chosen to write the expansion in terms of α in analogy with the expansion in HQET.

sulting from the insertion of higher order operators will never feed down into the matching for lower order operators.

Just as the $1/c$ expansion dictates the proper matching procedure, it also greatly simplifies the power counting rules in the low energy theory. In HQET the magnitude of a matrix element is dictated by the explicit power of $1/m$ in its coefficient. Thus, an operator of mass dimension d with a factor of m^{-n} in its coefficient is of order $\Lambda_{QCD}^{n+4-d}/m^n$. In NRGT's, as formulated here, the power counting is just as simple. All the powers of $1/c$ have been made explicit, and we can read off the size of a matrix element simply by counting powers of c and doing dimensional analysis.

As an example of the power counting procedure let us consider the pedagogical example of the relativistic corrections to the bound state energy of hydrogen [10]. Given that α and v/c are of the same order, we must calculate the matching corrections to the appropriate order in α for the accuracy we wish to attain. The leading corrections come from pure $(v/c)^2$ corrections stemming from the effective Lagrangian. The first such term is the correction to the kinetic energy $\psi^\dagger \boldsymbol{\partial}^4/(8m^3c^2)\psi$. By dimensional analysis, its matrix element is $mv^2(v/c)^2$, which yields a relative contribution of order $(v/c)^2$. There are no α corrections in the matching to this operator. Next there are $(v/c)^2$ corrections coming from the Darwin term and the spin orbit coupling which are again of relative order $(v/c)^2$, since we may pick out the Coulombic piece of the electric field. Both the Darwin term and the spin orbit term will get matching corrections at order α and will thus give a contribution to the energy shift at relative order $(v/c)^3$ as well. There are no further corrections at order $(v/c)^2$, assuming the proton to be infinitely heavy so that the magnetic interactions become irrelevant. There is a correction of relative order $(v/c)^3$ coming from corrections to the Coulomb potential due to pair creation which are accounted for in the matching and lead to a term in the effective Lagrangian given by

$$O_U = c_U \alpha \frac{\partial_i A_0 \partial_j^2 \partial_i A_0}{m^2 c^2}. \tag{15}$$

Finally, we come to the energy shift due to transverse photon propagation in the bound state. This is a self-energy correction of the electron propagating in the Coulomb background and is of relative size $(v/c)^3$ due to a factor of $c^{-3/2}$ coming from each transverse photon vertex. This agrees with the well known result for the Lamb shift.

Let us now return to the issue of confinement in the non-Abelian theory. While it may be surprising that the non-Abelian couplings are subleading, it is clear that the non-Abelian nature of the theory should be irrelevant to the details of the bound state as its size is reduced. The confining effects in a Coulombic bound state should be suppressed by powers of Λ_{QCD}/m. Setting $r \sim 1/mv$ in the virial theorem

$$\frac{\alpha_s(1/r)}{r} \sim \frac{mv^2}{c}, \tag{16}$$

gives $v/c \sim \alpha_s(mv)$, which leads to the conclusion that, at small v, Λ_{QCD}/m scales like[4] v/c. The question then becomes how do we properly take into account the effects of confinement in this effective field theory? Since confinement will not arise in perturbation theory we must insist that the zeroth order states contain the confining potential. The omission of the nonperturbative effects will lead to the breakdown of the $1/c$ expansion as we will now show.

If we include the full non-Abelian gluondynamics in the zeroth order effective theory, then the Coulombic potential will now be modified by the linear rise due to confinement. At first this may seem a bit worrisome since the non-Abelian piece has explicit factors of $1/c$ in front of them and thus the states of the theory will depend upon c, which is exactly what we were trying to avoid. However, this dependence on c will not destroy the systematics. The analytic dependence now implicit in the matrix elements will clearly not upset the systematics of the $1/c$ expansion. The nonanalytic dependence on c will be introduced through factors such as

$$e^{\kappa c/g^2} = \left(\frac{\Lambda_{QCD}}{m}\right)^P, \tag{17}$$

where P is some positive power. Thus, the c dependence of the matrix elements due to nonperturbative effects can only introduce higher order corrections. If we chose to use the Coulombic states, we would never see these higher order effects, and since $\Lambda_{QCD}/m \sim v/c$, we would possibly miss effects of the same order we wish to keep.

We have shown that nonrelativistic gauge theory (NRGT) is conveniently organized in temrs of an expansion in the dimensionful parameter $1/c$. Furthermore, if dimensional regularization is used, the organization of the expansion is very simple, there is no operator mixing across different orders in the expansion and the order of the operators in the expansion can be read off directly from their dimensions without resort to heuristic arguments. We emphasize, however, that dimensional regularization is not mandatory. Indeed one may formulate NRGT's with, for example, a momentum cut-off Λ. If $\Lambda \ll m$, with m the quark mass, then the multipole expansion is automatic. However, one then has a triple expansion, in α, $1/c$ and Λ/m. It is not necesary to choose $\Lambda < m$; with $\Lambda \gg m$ one can work with a double expansion, in α and $1/c$ only.[5] In either case operators of different orders in the $1/c$ expansion mix. The organization by $1/c$ is still useful because once the matching procedure has been completed the order at which an operator enters can still be trivially obtained from its velocity dimension.

We thank Mark Wise for many discussions and for emphasizing that we should explore the full consequences of the $1/c$ expansion. We also thank M. Luke, A. Falk and S. Sharpe for helpful comments. This work was supported in part by the Department of Energy under Grant No. DOE-FG03-90ER40546.

[4] We will take Λ_{QCD} to have the units of mass.

[5] In fact, it is unnecessary to perform a multipole expansion, if one is willing to fine tune the coefficients in the expansion order by order in perturbation theory.

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