Entropy of the quantized spin-$\frac{1}{2}$ field in the Schwarzschild spacetime

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The entropy of the quantized massless and conformally invariant two component spinor field in Schwarzschild spacetime is constructed. It is achieved through the construction of the one parameter family of the stress-energy tensor in the Hartle-Hawking state. The relation of the present approach to the approximations of Brown and Ottewill and Frolov and Zel'nikov is briefly discussed. [S0556-2821(98)04112-5]

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The objective of this Brief Report is to construct the entropy ΔS of the quantized conformally invariant massless spinor field in Schwarzschild spacetime. We show that, assuming that the tangential component of the stress-energy tensor in the ultrastatic companion of the Schwarzschild metric may be approximated by a polynomial in $x=2M/r$ truncated at $N=6$, and utilizing the regularity conditions which the mean value of $\langle T^{\mu}_{\nu}\rangle$ in the Hartle-Hawking state should satisfy, and finally, accepting the weak thermal bath hypothesis, ΔS may be easily constructed. This could be done in spite of the fact that since there are no numerical results concerning the horizon value of the $\langle T^{\mu}_{\nu}\rangle$ the resulting approximation of the stress tensor could not be constructed completely. The obtained ΔS coincides with the expression derived earlier by Hochberg, Kephart, and York [1] within the framework of the Brown-Ottewill-Page approximation [2].

Similar method may be exploited also in the case of the conformal scalar and vector field leading to the known ΔS; however, in this case we have at our disposal not only the excellent approximations of the stress tensor in various states, but also detailed numerical calculations regarding the exact $\langle T^{\mu}_{\nu}\rangle$ [2–24]. It is a well known fact that the accuracy of existing approximations of the stress tensor of the quantized spinor field has not yet been tested against exact numerical calculations.

The idea is to construct the approximate $\langle T^{\mu}_{\nu}\rangle$ in the optical metric, and subsequently to transform it back to the physical, i.e., Schwarzschild, spacetime with the aid of a general formula derived originally by Brown and Ottewill [5] and generalized by Brown, Ottewill, and Page [2]. Equally well one may use the transformation constructed by Page [3]. It should be noted that although our constructions heavily rely on the consequences of the transformational properties of the conformally related renormalized stress tensors, our approximation is neither the Page approximation nor the one proposed by Brown, Ottewill, and Page, however, the results of the latter may be considered as a special case of our approximation.

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Under the conformal transformation the renormalized one-loop effective action of the spinor field transforms as follows [2,5]:

$$W_R[g_{\mu\nu}]=W_R[e^{-2\omega}g_{\mu\nu}]+aA[\omega;g]+bB[\omega,g], \quad (1)$$

where

$$A[\omega,g]=\int d^4x(-g)^{1/2}\left\{\left(\mathrm{Riem}^2-2\,\mathrm{Ricc}^2+\frac{1}{3}R^2\right)\omega + \frac{2}{3}[R+3(\Box\omega-\kappa)](\Box\omega-\kappa)\right\}, \quad (2)$$

$$B[\omega,g]=\int d^4x(-g)^{1/2}[(\mathrm{Riem}^2-4\,\mathrm{Ricc}^2+R^2)\omega + 4R_{\mu\nu}\omega^{;\mu}\omega^{;\nu}-2R\kappa+2\kappa^2-4\kappa\Box\omega], \quad (3)$$

and $\kappa=\omega_{;\alpha}\omega^{;\alpha}$. The coefficients a,b for the two component spinor fields are given by $18/(11520\pi^2)$ and $-11/(11520\pi^2)$, respectively.

Functionally differentiating Eq. (4), when restricting to the Ricci flat spaces, one obtains

$$\langle T^{\mu}_{\nu}\rangle=\exp(-4\omega)\tilde{T}^{\mu}_{\nu}+a(s)A^{\mu}_{\nu}+b(s)B^{\mu}_{\nu}, \quad (4)$$

where

$$A^{\mu\nu}=8R^{\alpha\mu\nu\beta}\omega_{;\alpha\beta}-\frac{4}{3}\kappa^{;\mu\nu}+2g^{\mu\nu}\left(2\omega^{;\alpha}\kappa_{;\alpha}+\kappa^2+\frac{2}{3}\Box\kappa\right) -8\kappa^{;(\mu}\omega^{;\nu)}-8\omega^{;\mu}\omega^{;\nu}\kappa \quad (5)$$

and

$$B^{\mu\nu}=8R^{\alpha\mu\nu\beta}\omega_{;\alpha\beta}+8R^{\alpha\mu\nu\beta}\omega_{;\alpha}\omega_{;\beta}-8\omega^{;\mu\alpha}\omega_{;\alpha}{}^{;\nu} -8\kappa^{;(\mu}\omega^{;\nu)}-8\kappa\omega^{;\mu}\omega^{;\nu}+4g^{\mu\nu} \times\left(\omega_{;\alpha\beta}\omega^{;\alpha\beta}+\kappa_{;\alpha}\omega^{;\alpha}+\frac{1}{2}\kappa^2\right). \quad (6)$$

We distinguished the traceless stress tensor in the conformal space with a tilde.

Taking the conformal factor in the form $\omega = 1/2 \ln|g_{00}|$, the sum of the last two terms on the right-hand side (RHS) of Eq. (4), which we denote by S^{μ}_{ν}, for a spinor field in the Schwarzschild spacetime is

$$S^{t}_{t} = 3T\frac{x^6(240-384x+161x^2)}{8(1-x)^2}, \tag{7}$$

$$S^{r}_{r} = T\frac{x^6(16-48x+15x^2)}{8(1-x)^2}, \tag{8}$$

and

$$S^{\theta}_{\theta} = T\frac{x^6(-32-72x+87x^2)}{8(1-x)^2}, \tag{9}$$

where $T=1/(458^4 M^4 \pi^2)$.

The conservation equations in conformal space give [13]

$$\frac{\partial}{\partial x}\tilde{T}^{r}_{r} - \frac{2-3x}{x(1-x)}\tilde{T}^{r}_{r} + \frac{2-3x}{x(1-x)}\tilde{T}^{\theta}_{\theta} = 0 \tag{10}$$

and

$$\tilde{T}^{r}_{t} = -x^2(1-x)^2\frac{k}{4M^4}, \tag{11}$$

where k is the integration constant connected to the luminosity. Since in the Hartle-Hawking state the net fluxes are absent we put $k=0$.

We assume that the tangential component of the stress tensor may be approximated by the following polynomial:

$$\tilde{T}^{\theta}_{\theta} = T'\left(1+\sum_{i=1}^{N} a_i x^i\right). \tag{12}$$

Studying the asymptotic behavior of $\langle T^{\mu}_{\nu}\rangle$ as $x\to 0$ gives $T' = \frac{7}{8}T$.

Now, counting the number of available informations one concludes that the simplest choice is $N=6$. Transforming back the tangential component to the physical space, one finds that the regularity conditions [25] are satisfied provided

$$a_3 = \frac{1}{14}(50-28a_1-21a_2-7a_4+7a_6) \tag{13}$$

and

$$a_5 = a_1 + \frac{1}{14}(-30+7a_2-7a_4-21a_6). \tag{14}$$

Substituting Eqs. (13) and (14) into $\langle T^{\theta}_{\theta}\rangle$ and subsequently solving the conservation equation one obtains the radial component

$$\tilde{T}^{r}_{r} = \frac{7}{8}T\Big[1+2a_1x-\frac{2}{7}(4c-5+7a_1)x^2+\frac{1}{7}(8c-60+28a_1 + 14a_2+7a_4-7a_6)x^3+\frac{1}{28}(150-70a_1-35a_2-49a_4 + 21a_6)x^4+\frac{1}{28}(90-42a_1-21a_2+21a_4+35a_6)x^5 - x^6a_6-(1-x)x^2(a_1+2a_2)\ln(x)\Big], \tag{15}$$

where c is an integration constant. Note that the logarithmic term survives even if the regularity conditions of the tangential component are satisfied. Now, transforming the stress tensor $\tilde{T}^{\mu}_{\nu}$ back to the physical spacetime and imposing the regularity conditions [25] one obtains

$$a_2 = -\frac{a_1}{2} \tag{16}$$

and

$$a_6 = 10-\frac{32}{7}c+\frac{49}{2}a_1+a_4. \tag{17}$$

Further, making use of the weak thermal bath hypothesis [24], that in our context states that the stress-energy tensor should be of the form

$$\langle T^{\mu}_{\nu}\rangle = \frac{7}{8}T(1+2x+3x^2)\mathrm{diag}[-3,1,1,1]^{\mu}_{\nu}+O(x^3) \tag{18}$$

yields

$$a_1 = 0 \tag{19}$$

and

$$c = \frac{5}{4}. \tag{20}$$

This hypothesis is usually motivated by the observation that since the curvature is proportional to M/r^3, the curvature corrections are expected to be of order x^3.

Substitution of the thus obtained parameters into Eq. (4) results in the one-parameter family of the stress-energy tensor

$$\langle T^{t}_{t}\rangle = -\frac{7}{8}T\left[3+6x+9x^2+12x^3+\frac{1}{14}(330+14a_4)x^4 + \frac{186}{7}x^5-69x^6\right], \tag{21}$$

$$\langle T^{r}_{r}\rangle = \frac{7}{8}T\left[1+2x+3x^2-\frac{52}{7}x^3-\frac{1}{14}(130+14a_4)x^4-\frac{18}{7}x^5 + \frac{15}{7}x^6\right], \tag{22}$$

and

$$\langle T^{\theta}_{\theta}\rangle = \frac{7}{8}T\left[1+2x+3x^2+\frac{68}{7}x^3+\frac{1}{14}(230+14a_4)x^4 + \frac{102}{7}x^5+\frac{87}{7}x^6\right]. \quad (23)$$

Note that the obtained $\langle T^{\mu}_{\nu}\rangle$ does not satisfy the strong thermal bath hypothesis, in which one assumes that the curvature corrections should be of order x^6 [24]. An equivalent method of obtaining Eqs. (21)–(23) is to assume that both $\tilde{T}^{\theta}_{\theta}$ and $\tilde{T}^{r}_{r}$ may be expressed as the polynomials truncated at $N=6$. Then, making use of the conservation equation in the ultrastatic metric, solving resulting equations for unknown parameters to reduce their number, and finally making use of the regularity conditions, one arrives at Eqs. (21)–(23).

The unknown parameter a_4 is to be determined assuming that on the event horizon

$$\langle T^{\mu}_{\nu}\rangle = T^{\mu}_{\nu}, \quad (24)$$

where T^{μ}_{ν} is (unknown as yet) the exact value of the stress tensor on the event horizon.

Although we are left with the one unspecified constant, the difference $\langle T^{t}_{t}\rangle - \langle T^{r}_{r}\rangle$ is, by construction, independent of a_4 and the entropy of the quantized spinor field may be easily constructed. Indeed, following the steps of Ref. [25] or making use of the elegant formula derived by Zaslavskii [26],

$$\Delta S = 32\pi^2 M\int_{2M}^{r} dr' r'^2\left[\langle T^{r}_{r}\rangle - \langle T^{t}_{t}\rangle - \langle T^{\mu}_{\mu}\rangle \ln\left(\frac{r}{r'}\right)\right], \quad (25)$$

one obtains

$$\Delta S = \frac{7}{8}\frac{8\pi}{K}\left[\frac{8}{9}x^{-3}+\frac{8}{3}x^{-2}+8x^{-1}-\frac{16}{9}-\frac{200}{21}x-8x^2 + \frac{488}{63}x^3+\frac{128}{7}\ln(x)\right], \quad (26)$$

where the numerical factor $K=3840\pi$ has been singled out for convenience. As seen in Eq. (22) the arbitrary integration constant has been fixed by demand that ΔS vanishes on the event horizon. It is interesting to note that one obtains the same ΔS regardless of the value of the stress tensor on the event horizon, i.e., for any a_4. The entropy given by Eq. (26) coincides with ΔS constructed earlier by Hochberg, Kephart, and York [1], within the framework of the Brown-Ottewill-Page approximation, and has all the properties that the entropy of the quantized, massless, and conformally invariant field in the Schwarzschild spacetime is expected to possess.

The freedom of choosing the temperature of the thermal state in the Brown and Ottewill approach must not be confused with the freedom in choosing the parameter a_4. We remark here that in order to recover the stress tensor constructed within the framework of the Page-Brown-Ottewill approximation it suffices to take $a_4=-30/7$. On the other hand, the Frolov-Zel'nikov approximation [8] depends on one free parameter, say, d. It should be noted that since the difference $\langle T^{t}_{t}\rangle - \langle T^{\mu}_{\mu}\rangle$ depends on d, so does the entropy, however, with $d=5/(2880\pi^2)$ one obtains Eq. (26) [27]. Finally, we note here that a similar method, with different asymptotics and the Christensen-Fulling conditions yields the approximate one-parameter $\langle T^{\mu}_{\nu}\rangle$ in the Unruh state.

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