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Positivity constraints on anomalies in supersymmetric gauge theories

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The relation between the trace and R-current anomalies in supersymmetric theories implies that the $\mathrm{U}(1)_R F^2$, $\mathrm{U}(1)_R$, and $\mathrm{U}(1)_R^3$ anomalies which are matched in studies of $N=1$ Seiberg duality satisfy positivity constraints. Some constraints are rigorous and others conjectured as four-dimensional generalizations of the Zamolodchikov c theorem. These constraints are tested in a large number of $N=1$ supersymmetric gauge theories in the non-Abelian Coulomb phase, and they are satisfied in all renormalizable models with unique anomaly-free R current, including those with accidental symmetry. Most striking is the fact that the flow of the Euler anomaly coefficient $a_{\mathrm{UV}}-a_{\mathrm{IR}}$ is always positive, as conjectured by Cardy.
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I. INTRODUCTION

The computation of chiral anomalies of the R current and conserved flavor currents is one of the important tools used to determine the nonperturbative infrared behavior of the many supersymmetric gauge theories analyzed during the last few years. The anomaly coefficients are subject to rigorous positivity constraints by virtue of their relation to two-point functions of currents and stress tensors, and to other constraints conjectured in connection with possible four-dimensional analogues of the Zamolodchikov c theorem [1]. The two-point functions have been considered [2] as central functions whose ultraviolet and infrared limits define central charges of superconformal theories at the endpoints of the renormalization group flow. The positivity conditions are reasonably well known from studies of the trace anomaly for field theories in external backgrounds. In supersymmetric theories the trace anomaly of the stress tensor and conservation anomaly of the R current are closely related, which leads [3] to positivity constraints on chiral anomalies.

Two studies of positivity constraints in the $\mathrm{SU}(N_c)$ series of supersymmetric (SUSY) gauge theories with N_f fundamental quark flavors have previously appeared. The first of these [4] analyzed the confined and free magnetic phases for $N_c<N_f<3N_c/2$, while the basic techniques for computing the flow of central charges when there is an interacting IR fixed point were developed in Ref. [3] and applied to the conformal phase for $3N_c/2<N_f<3N_c$. The most striking result of Refs. [3,4] was the positive flow $a_{\mathrm{UV}}-a_{\mathrm{IR}}>0$ of the coefficient $a[g(\mu)]$ of the Euler term in the trace anomaly in an external gravitational background, where $g(\mu)$ is the gauge coupling at renormalization group (RG) scale μ. This result agrees with the conjecture of Cardy [5] that the Euler anomaly obeys a c theorem. Positivity is also satisfied in all non-supersymmetric theories tested [5,6]. We shall refer to the inequality $a_{\mathrm{UV}}-a_{\mathrm{IR}}>0$ as the a theorem. The purpose of this paper is to present an extensive exploration of the rigorous positivity constraints and those associated with the a theorem in many supersymmetric gauge theories with interacting IR fixed points (and some IR free models). We find that the a theorem and other constraints are satisfied in all renormalizable theories we have examined, and there are other results of interest.

In Sec. II, which is largely a review of Ref. [3], the various anomalies, the theoretical basis of the positivity constraints, and the computation of central charge flows are discussed. In Sec. III we discuss some general aspects of positivity constraints and the a theorem in models with R charges uniquely fixed by classical conservation and cancellation of internal anomalies. In some models an accidental symmetry has been postulated to preserve unitarity, and the central charges must be corrected accordingly. This is discussed in Sec. IV. In Sec. V, the positivity constraints are tested in many examples of renormalizable SUSY gauge models with uniquely determined R charges. We also check the a theorem for various types of flows between conformal fixed points. The situation of some nonrenormalizable models is discussed in Sec. VI. There are other models in which the conserved, anomaly free R current is not unique. Our methods are less precise in this case, but we discuss an example in Sec. VII. Section VIII contains a discussion of results and conclusions.

II. ANOMALIES AND POSITIVITY CONSTRAINTS

The theoretical basis for the analysis of anomalies in supersymmetric theories comes from a combination of three fairly conventional ideas: namely, (A) the close relation between the trace anomaly of a four-dimensional field theory with external sources for flavor currents and stress tensor and the two point correlators $\langle J_\mu(x)J_\nu(y)\rangle$ and $\langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\rangle$ and their central charges; (B) the close relation in a supersymmetric theory between the trace anomaly $\Theta = T^{\mu}_{\mu}$ and the anomalous divergence of the R current $\partial_\mu R^\mu$; (C) the fact that anomalies of the R current can be calculated at an infrared superconformal fixed point using 't Hooft anomaly matching. This is the standard procedure, and one way to explain it is to use the all orders anomaly free S current of Kogan, Shifman, and Vainshtein [7].

We now review these ideas briefly. More details are contained in Refs. [2,3].

A. Trace anomaly and central charges

We consider a supersymmetric gauge theory containing chiral superfields Φ_i^α in irreducible representations R_i of the gauge group G. To simplify the discussion we assume that the superpotential $W=0$, but the treatment can be generalized to include nonvanishing superpotential, and this will be done in Sec. II C below.

We consider a conserved current $J_\mu(x)$ for a nonanomalous flavor symmetry F of the theory, and we add a source $B_\mu(x)$ for the current, effectively considering a new theory with an additional gauged U(1) symmetry but without kinetic terms for B_μ. The source can be introduced as an external gauge superfield $B(x,\theta,\bar{\theta})$ so supersymmetry is preserved. We also couple the theory to an external supergravity background, contained in a superfield $H^a(x,\theta,\bar{\theta})$, but we discuss only the vierbein $e^a_\mu(x)$ and the component $V_\mu(x)$ which is the source for the R^μ current of the gauge theory.

The trace anomaly of the theory then contains several terms

$$\Theta = \frac{1}{2g^3}\,\tilde{\beta}(g)(F^a_{\mu\nu})^2 + \frac{1}{32\pi^2}\,\tilde{b}(g)B^2_{\mu\nu} + \frac{\tilde{c}(g)}{16\pi^2}\,(W_{\mu\nu\rho\sigma})^2 - \frac{a(g)}{16\pi^2}\,(\tilde{R}_{\mu\nu\rho\sigma})^2 + \frac{\tilde{c}(g)}{6\pi^2}\,V^2_{\mu\nu}, \tag{2.1}$$

where $W_{\mu\nu\rho\sigma}$ is the Weyl tensor, $\tilde{R}_{\mu\nu\rho\sigma}$ is the dual of the curvature, and $B_{\mu\nu}$ and $V_{\mu\nu}$ are the field strengths of B_μ and V_μ, respectively. All anomaly coefficients depend on the coupling $g(\mu)$ at renormalization scale μ. The first term of Eq. (2.1) is the internal trace anomaly, where $\tilde{\beta}(g)$ is the numerator of the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) beta function [8]

$$\tilde{\beta}(g(\mu)) = -\frac{g^3}{16\pi^2}\left[3T(G) - \sum_i T(R_i)[1-\gamma_i(g(\mu))]\right]. \tag{2.2}$$

Here $T(G)$ and $T(R_i)$ are the Dynkin indices of the adjoint representation of G and the representation R_i of the chiral superfield Φ_i^α, and $\gamma_i/2$ is the anomalous dimension of Φ_i^α.

The various external trace anomalies are contained in the three coefficients $\tilde{b}(g)$, $\tilde{c}(g)$, and $a(g)$. The free field (i.e., one-loop) values of $\tilde{c}$ and a have been known for many years [9]. In a free theory of N_0 real scalars, $N_{1/2}$ Majorana spinors, and N_1 gauge vectors, the results are

$$c = \frac{1}{120}\,(12N_1 + 3N_{1/2} + N_0),$$

$$a = \frac{1}{720}\,(124N_1 + 11N_{1/2} + 2N_0). \tag{2.3}$$

In a supersymmetric gauge theory with $N_v = \dim G$ gauge multiplets and N_χ chiral multiplets these values regroup as

$$c_{\rm UV} = \frac{1}{24}\,(3N_V + N_\chi), \quad a_{\rm UV} = \frac{1}{48}\,(9N_V + N_\chi). \tag{2.4}$$

If T_i^j is the flavor matrix for the current $J_\mu(x)$ which is the $\theta\bar{\theta}$ component of the superfield $\bar{\Phi}^i_\alpha T_i^j \Phi_j^\alpha$, and dim R_i is the dimension of the representation R_i, the free-field value of $\tilde{b}$ is

$$b_{\rm UV} = \sum_{i,j}\,(\dim R_i) T_i^j T_j^i. \tag{2.5}$$

The subscript UV indicates that the free-field values are reached in the ultraviolet limit of an asymptotically free theory. Clearly $\tilde{c}$ and a count degrees of freedom of the microscopic theory with different weights for the various spin fields.

The current correlation function is

$$\langle J_\mu(x) J_\nu(0)\rangle = \frac{1}{16\pi^4}\,(\Box\,\delta_{\mu\nu} - \partial_\mu\partial_\nu)\,\frac{b(g(1/x))}{x^4}. \tag{2.6}$$

It follows from reflection positivity or the Lehmann representation as used in Ref. [10] that the renormalization group invariant central function [3] $b(g(1/x))$ is strictly positive. We assume that the theory in question has UV and IR fixed points so that the following limits exist:

$$b_{\rm UV} = b(g_{\rm UV}) = \lim_{x\to 0}\, b(g(1/x)),$$

$$b_{\rm IR} = b(g_{\rm IR}) = \lim_{x\to\infty}\, b(g(1/x)). \tag{2.7}$$

These endpoint values appear as central charges in the operator product expansion of currents in the UV and IR superconformal theories at the endpoints of the RG flow.

The correlator $\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle$ has the tensor decomposition [2]

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = \frac{1}{48\pi^4}\,\Pi_{\mu\nu\rho\sigma}\,\frac{c(g(1/x))}{x^4} + \Pi_{\mu\nu}\Pi_{\rho\sigma}\,\frac{f(\ln x\Lambda, g(1/x))}{x^4}, \tag{2.8}$$

where $\Pi_{\mu\nu} = (\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)$ and $\Pi_{\mu\nu\rho\sigma} = 2\Pi_{\mu\nu}\Pi_{\rho\sigma} - 3(\Pi_{\mu\rho}\Pi_{\nu\sigma} + \Pi_{\mu\sigma}\Pi_{\nu\rho})$ is the transverse traceless projector and Λ is the dynamical scale of the theory. The central function $c(g(1/x))$ is a positive RG invariant function. Its endpoint values $c_{\rm UV}$ and $c_{\rm IR}$ are also central charges. The second tensor structure in Eq. (2.8) arises because of the internal trace anomaly. It is proportional to $\tilde{\beta}(g(1/x))$ and thus vanishes at critical points.

The important point is that there is a close relation between the anomaly coefficients $\tilde{b}(g(\mu))$ and $\tilde{c}(g(\mu))$ and the central functions $b(g(\mu))$ and $c(g(\mu))$. Namely,

$\tilde{b}(g(\mu))$ and $b(g(\mu))$ differ by terms proportional to $\tilde{\beta}(g(\mu))$, so they coincide at RG fixed points. The same holds for $\tilde{c}(g(\mu))$ and $c(g(\mu))$. This means that the endpoint values of the anomaly coefficients are rigorously positive. This is evident for the free field ultraviolet values in Eqs. (2.3)–(2.5). The infrared values $b_{\rm IR}$ and $c_{\rm IR}$ must also be positive, and this is an important check on the hypothesis that the long distance dynamics of a theory is governed by an interacting fixed point.

This important relation between trace anomaly coefficients and current correlators was derived in Refs. [2,3] by an argument with the following ingredients.

(i) Since the explicit scale derivative of a renormalized correlator corresponds to the insertion of the integrated trace anomaly, the $\langle J_\mu(x)J_\nu(0)\rangle$ correlator satisfies

$$\mu\frac{\partial}{\partial\mu}\langle J_\mu(x)J_\nu(0)\rangle = \frac{1}{8\pi^2}\tilde{b}(\mu)(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)\delta^4(x) - \frac{\tilde{\beta}(g(\mu))}{2g^3}\left\langle J_\mu(x)J_\nu(0)\int d^4z(F^a_{\rho\sigma})^2\right\rangle. \tag{2.9}$$

(ii) The central function $b(g(1/x))$ satisfies a standard homogeneous renormalization group equation, but $b(g(1/x))/x^4$ requires additional regularization because it is singular at the origin. The regulated amplitude satisfies

$$\mu\frac{\partial}{\partial\mu}\left.\frac{b(g(1/x))}{x^4}\right|_{\rm reg}=\frac{1}{8\pi^2}\hat{b}(g(\mu))\delta^4(x) + \frac{\beta(g(\mu))}{g^3}\left.\frac{b(g(1/x))}{x^4}\right|_{\rm reg}, \tag{2.10}$$

where $\hat{b}(g(\mu))$ is associated with the overall divergence at $x=0$.

(iii) Using the method of differential renormalization [11] and the RG equation, one can resum a series in powers of $(\ln x\mu)^k$ to derive a nonperturbative differential equation, namely,

$$\beta(g)\frac{\partial\hat{b}(g)}{\partial g}+2\hat{b}(g)=2b(g). \tag{2.11}$$

This shows that $\hat{b}(g(\mu))$ and the central function itself, $b(g(\mu))$, coincide at fixed points. Comparing Eqs. (2.9) and (2.10) it is tempting to identify $\check{b}(g(\mu))=\hat{b}(g(\mu))$, but this also holds only at fixed points since we cannot exclude possible local $\delta^4(x)$ terms in the $\langle JJ\int F^2\rangle$ correlator. It is easy to see that contributions to $\langle JJ\int F^2\rangle$ begin at order $g(\mu)^4$. It is assumed that the local terms have no singularities which could cancel the zero of $\tilde{\beta}(g)$.

The anomaly coefficient $a(g(\mu))$ is related to three-point correlators of the stress tensor [12] rather than to $\langle T_{\mu\nu}T_{\rho\sigma}\rangle$. However, it is clear that $a(g(\mu))$ is significant, and that the fixed point values $a_{\rm UV}$, $b_{\rm UV}$, $c_{\rm UV}$ and $a_{\rm IR}$, $b_{\rm IR}$, $c_{\rm IR}$ are important quantities which characterize the superconformal theories at the fixed points of the RG flow.

c theorems. In two dimensions Zamolodchikov established the c theorem by constructing a function $C(g(\mu))$ as a linear combination of (suitably scaled) $\langle T_{zz}T_{zz}\rangle$, $\langle T_{zz}\Theta\rangle$, and $\langle\Theta\Theta\rangle$ correlators which satisfies

$$\mu\frac{\partial}{\partial\mu}C(g(\mu))\geqslant 0,$$

$$\frac{\partial}{\partial g}C(g(\mu))\Big|_{g=g*}=0,$$

$$C(g^*)=c^*, \tag{2.12}$$

where c^* is the Virasoro central charge of the critical theory at the fixed point $g=g^*$ or, equivalently, the fixed point value of the external trace anomaly coefficient

$$\Theta=\frac{1}{24\pi}c^*R, \tag{2.13}$$

where R is the scalar curvature. The properties (2.12) imply $c_{\rm UV}-c_{\rm IR}>0$ which is the form in which the c theorem is usually tested [13]. The ingredients of Zamolodchikov's proof of these properties are conservation Ward identities, rotational symmetry, reflection positivity, and Wilsonian renormalizability. There is a similar proof [10] of a k theorem for the central charges of conserved currents, which leads to $b_{\rm UV}-b_{\rm IR}\geqslant 0$ in our notation. There are alternative proofs [6,10] of the c and k theorems in two dimensions based on the Lehmann representation for the invariant amplitudes in the decomposition of $\langle T_{\mu\nu}(p)T_{\rho\sigma}(-p)\rangle$ and $\langle J_\mu(p)J_\nu(-p)\rangle$.

The techniques used in the two-dimensional case cannot be extended to four dimensions [5,6], and it has not so far been possible to construct any C function for four-dimensional theories which satisfies Eq. (2.11). The best thing one now has is Cardy's conjecture [5] that there is a universal c theorem based on the Euler anomaly, so that $a_{\rm UV}-a_{\rm IR}>0$ in all theories. There is theoretical support for this conjecture [12], and empirical support by direct test in models where the infrared dynamics is understood. The a theorem is true in all models so far tested which include the following.

(i) SU(N_c) QCD with N_c^2-1 gluons and N_fN_c quarks [5]. An infrared realization as a confined theory with chiral symmetry breaking and N_f^2-1 decoupled Goldstone bosons is assumed.

(ii) QCD at large N_c with $N_f=11N_c/2-k$ near the asymptotic freedom limit. The infrared limit is computable in perturbation theory because of the well known close two-loop fixed point [14]. Actually $a_{\rm UV}-a_{\rm IR}=0$ to order $1/N_c^2$ for reasons we discuss below.

(iii) SU(N_c) $N=1$ SUSY QCD in the confined and free magnetic phase for $N_c\leqslant N_f\leqslant 3N_c/2$ [4].

(iv) SU(N_c) $N=1$ SUSY QCD in the non-Abelian Coulomb phase for $3N_c/2<N_f<3N_c$ [3].

One may take a more general empirical approach and test whether other c-theorem candidates such as the total flow

$b_{\rm UV}-b_{\rm IR}$ and $c_{\rm UV}-c_{\rm IR}$ (or possible linear combinations with $a_{\rm UV}-a_{\rm IR}$) are positive in the models above. It is known that $c_{\rm UV}-c_{\rm IR}$ is positive in the situations (i) [6] and (iii) [4] above, but negative in situation (ii) [6] and changes sign from positive to negative as N_f increases in the theories of (iv). Thus a universal ''c theorem'' is ruled out. In the Appendix below we present brief calculations to show that a b theorem cannot hold in situations (i)–(iii) above, and it is known [3] not to hold in situation (iv).

Thus the a theorem $a_{\rm UV}-a_{\rm IR}>0$ emerges as the only surviving candidate for a universal theorem in four dimensions. The desired physical interpretation requires the existence of an A function $A(g(\mu))$ which decreases monotonically from $a_{\rm UV}$ to $a_{\rm IR}$ and counts effective degrees of freedom at a given scale. Thus the relation $a_{\rm UV}-a_{\rm IR}>0$ would make little physical sense unless $a_{\rm IR}$ is positive. Indeed it has been argued [15] that $a(g(\mu))$ is positive at critical points if a conjectured quantum extension of the weak energy condition of general relativity is valid.

Let us now summarize this discussion of the positivity properties of trace anomaly coefficients. The free-field values $a_{\rm UV}$, $b_{\rm UV}$, $c_{\rm UV}$ are automatically positive. Positivity is rigorously required for $b_{\rm IR}$ and $c_{\rm IR}$, and it is a useful test of our understanding of the infrared dynamics to check this property in models. We will also explore the conjectured a-theorem and the related condition $a_{\rm IR}>0$. We will also show that the ''data'' for $N=1$ SUSY gauge theories in the non-Abelian Coulomb phase imply that there is no linear combination $u(a_{\rm UV}-a_{\rm IR})+v(c_{\rm UV}-c_{\rm IR})$ which is positive in all models (except for $v=0$, $u>0$).

B. Relation between Θ and $\partial_\mu R^\mu$ anomalies in SUSY/SG

In a supersymmetric theory in the external U(1) gauge and supergravity backgrounds discussed above, the divergence of the R^μ current and the trace of the stress tensor are components of a single superfield. Therefore the supersymmetry partner of the trace anomaly Θ of Eq. (2.1) is

$$\partial_\mu(\sqrt{g}R^\mu)=-\frac{1}{3g^3}\,\tilde\beta(g)(F\tilde F)-\frac{\tilde b(g)}{48\pi^2}\,(B\tilde B)+\frac{\tilde c(g)-a(g)}{24\pi^2}\,R\tilde R+\frac{5a(g)-3\tilde c(g)}{9\pi^2}\,(V\tilde V), \tag{2.14}$$

where R and $\tilde R$ on the right hand side are the curvature tensor and its dual. The ratio $-\frac{2}{3}$ between the first two terms of Eqs. (2.1) and (2.14) is well known in global supersymmetry, but the detailed relation of the anomaly coefficients of the gravitational section was first derived in Ref. [3] by evaluating the appropriate components of the curved superspace anomaly equation

$$\bar D^{\dot\alpha}J_{\alpha\dot\alpha}=\frac{1}{24\pi^2}\,(\tilde c W^2-a\Xi), \tag{2.15}$$

where $J_{\alpha\dot\alpha}$, W^2, and Ξ are the supercurrent, super-Weyl, and super-Euler superfields, respectively. This equation shows that all gravitational anomalies are described by the two functions $\tilde c(g)$ and $a(g)$, and this is also the reason why the coefficients of the third and fifth terms of Eq. (2.1) are related. An alternate derivation of Eq. (2.14) which does not require superspace technology was also given in Ref. [3].

The last three terms of Eq. (2.14) are essentially the same as the anomalies usually computed in studies of $N=1$ Seiberg duality. It is this fact that leads to immediate positivity constraints on supersymmetry anomalies which we can test easily in the various models in the literature which flow to infrared fixed points.

C. Computing infrared anomaly coefficients

In this section we discuss how the infrared central charges $b_{\rm IR}$, $c_{\rm IR}$, and $a_{\rm IR}$ are related to the conventional $U(1)_R F^2$, $U(1)_R$, and $U(1)_R^3$ anomalies. This is already quite clear, and some readers may wish to jump ahead to the final formulas at the end. However, we do think that it is useful to derive this relation using the formalism of the all-orders anomaly-free S^μ current introduced in Ref. [7]. The external anomalies of this current can be clearly seen to agree in the infrared limit with those of the R^μ current which is in the same multiplet as the stress tensor, and thus part of the $N=1$ superconformal algebra of the infrared fixed point theory. A very clear explanation of the S^μ current is given in Sec. III of [7] for the case of general gauge group G and arbitrary superpotential $W(\phi)$. We summarize and exemplify the argument for the slightly simpler case of cubic $W(\phi)$.

Gaugino fields are denoted by $\lambda^a(x)$, $a=1,\ldots,\dim G$, and scalar and fermionic components of $\Phi_i^\alpha(x)$ by $\phi_i^\alpha(x)$ and $\psi_i^\alpha(x)$, respectively. The canonical R^μ current (which is the partner of the stress tensor), and the matter Konishi currents K_i^μ for each representation are

$$R^\mu=\frac{1}{2}\,\bar\lambda^a\gamma^\mu\gamma^5\lambda^a-\frac{1}{6}\sum_i\,\bar\psi^i_\alpha\gamma^\mu\gamma^5\psi_i^\alpha+\frac{2}{3}\sum_i\,\bar\phi^i_\alpha\overleftrightarrow{D}_\mu\phi_i^\alpha,$$

$$K_i^\mu=\frac{1}{2}\sum_i\,\bar\psi^i_\alpha\gamma^\mu\gamma^5\psi_i^\alpha+\sum_i\,\bar\phi^i_\alpha\overleftrightarrow{D}_\mu\phi_i^\alpha. \tag{2.16}$$

Conservation of the Konishi current is spoiled by a classical violation for any nonvanishing W and a one-loop exact chiral anomaly. The internal anomaly of R^μ in Eq. (2.14) can also be generalized to include W. The divergences of these currents are then (external sources are dropped)

$$\partial_\mu K_i^\mu=\Phi_i^\alpha\,\frac{\partial W}{\partial\Phi_i^\alpha}\bigg|+\frac{T(R_i)}{16\pi^2}\,F\tilde F, \tag{2.17}$$

$$\partial_\mu R^\mu=\frac{1}{3}\sum_i\,\gamma_i\Phi_i^\alpha\,\frac{\partial W}{\partial\Phi_i^\alpha}\bigg|+\frac{1}{48\pi^2}\times\left[3T(G)-\sum_i\,T(R_i)(1-\gamma_i)\right]F\tilde F, \tag{2.18}$$

where the vertical bar indicates the θ^2 component of the superfield minus its adjoint. The anomaly-free R current usually stated in the literature for any given model is a specific linear combination (assumed unique here)

$$S_0^\mu = R^\mu + \frac{1}{3}\sum_i \gamma_i^* K_i^\mu, \tag{2.19}$$

which is conserved classically and nonanomalous to one-loop order. This means that all terms in its divergence

$$\partial_\mu S_0^\mu = \frac{1}{3}\sum_i (\gamma_i^* + \gamma_i)\Phi_i^\alpha \frac{\partial W}{\partial \Phi_i^\alpha}\Bigg| + \frac{1}{48\pi^2} \times \left[3T(G) - \sum_i T(R_i)[1-(\gamma_i^* + \gamma_i)]\right] F_{\mu\nu}^a \tilde{F}^{\mu\nu a} \tag{2.20}$$

cancel except those with coefficients γ_i. There is then a unique (flavor singlet) all-order conserved current

$$S^\mu = R^\mu + \frac{1}{3}\sum_i (\gamma_i^* - \gamma_i) K_i^\mu. \tag{2.21}$$

Its divergence vanishes,

$$\partial_\mu S^\mu = \frac{1}{3}\sum_i \gamma_i^* \phi_i^\alpha \frac{\partial W}{\partial \phi_i^\alpha}\Bigg| + \frac{1}{48\pi^2} \times \left[3T(G) - \sum_i T(R_i)(1-\gamma_i^*)\right] F\tilde{F} = 0, \tag{2.22}$$

and the vanishing of the coefficients of $F\tilde{F}$ and the independent cubic terms means that the γ_i^* are the unique set of numbers which make the gauge and various Yukawa β functions vanish. The γ_i^* then have the physical interpretation as IR anomalous dimensions of the superfields ϕ_i^α, assuming that there is an IR fixed point. In the infrared limit, $\gamma_i \to \gamma_i^*$ in Eq. (2.21), and $S^\mu \to R^\mu$. It is worth noting that the coefficient in front of the Konishi current in Eq. (2.21) is a manifestation of positive anomalous dimension of the anomalous Konishi current [16]. In physical correlators the infrared limit can be associated with large distance behavior. Therefore in the infrared (large distance) limit of correlators with an insertion of $R_\mu = S_\mu - \frac{1}{3}\Sigma_i(\gamma_i^* - \gamma_i)K_i^\mu$ the contribution of the Konishi current decreases faster than the contribution of the S_μ current which has no anomalous dimension. Thus the S^μ and R^μ operators and their correlators agree in the long distance limit, as is required at the superconformal IR fixed point. In the free UV limit $\gamma_i \to 0$, and $S^\mu \to S_0^\mu$. As we will see shortly this means that external anomalies of S^μ coincide with those computed in the literature.

We distinguish three classes of models in which one obtains unique S_0^μ and S^μ currents. The first is the set of models with chiral fields in N_f copies of a single (real) irreducible representation R (or N_f fields in $R \oplus \bar{R}$) and no superpotential. It is easy to see that the unique S^μ current in these two cases is

$$S^\mu = R^\mu + \frac{1}{3}\left(1 - \frac{3T(G)}{N_f T(R)} - \gamma(g(\mu))\right)\sum_i K_i^\mu,$$

$$S^\mu = R^\mu + \frac{1}{3}\left(1 - \frac{3T(G)}{2N_f T(R)} - \gamma(g(\mu))\right)\sum_i (K_i^\mu + \tilde{K}_i^\mu) \tag{2.23}$$

[where K_i^μ and $\tilde{K}_i^\mu$ are the Konishi currents of fields in the R and $\bar{R}$ representations, respectively, and we use $T(R) = T(\bar{R})$ and $\gamma = \tilde{\gamma}$]. Comparing with Eq. (2.2), one can see that the coefficient of the Konishi terms is proportional to $\tilde{\beta}(g(\mu))$ and thus vanishes in the infrared limit if there is a fixed point.

The second class of models are those of Kutasov [17] and generalizations [18,19] in which we add a superfield X in the adjoint representation to the previous matter content and take $W = f\mathrm{Tr}\, X^3$. We let K_X^μ and γ_X denote the Konishi current and anomalous dimension for the adjoint fields. The procedure outlined above leads to the unique currents

$$S^\mu = R^\mu + \frac{1}{3}\left(1 - \frac{2T(G)}{N_f T(R)} - \gamma(g,f)\right)\sum_i K_i^\mu - \frac{1}{3}\gamma_X K_X^\mu,$$

$$S^\mu = R^\mu + \frac{1}{3}\left(1 - \frac{T(G)}{N_f T(R)} - \gamma(g,f)\right) \times \sum_i (K_i^\mu + \tilde{K}_i^\mu) - \frac{1}{3}\gamma_X K_X^\mu, \tag{2.24}$$

for the cases of representations $R \oplus \mathrm{adj}$ and $R \oplus \bar{R} \oplus \mathrm{adj}$, respectively. If there is an IR fixed point, then both $\beta_f = 3f\gamma_X/2$ and $\tilde{\beta}(g)$ given in Eq. (2.2) must vanish, and it is easy to see that all coefficients of the Konishi terms in Eq. (2.24) vanish if this occurs. The procedure may be extended to more general models with $W = f\mathrm{Tr}\, X^{k+1}$, $k > 2$, using the modification of Eq. (2.20) (see Sec. III of [7]) for general superpotentials.

Another common class of models resembles the ''magnetic'' version of SU(N_c) SUSY QCD. There are N_f flavors of quark and antiquark fields q and $\tilde{q}$ in conjugate representations R' and $\bar{R}'$ of a dual gauge group G' plus a gauge singlet M in the $(N_f, \bar{N}_f)$ representation of the flavor group. The models have a cubic superpotential $W = f\tilde{q}Mq$. In this case the unique S^μ current is

$$S^\mu = R^\mu + \frac{1}{3}\left(1 - \frac{3T(G')}{2N_f T(R')} - \gamma_q\right)(K_i^\mu + \tilde{K}_i^\mu - 2K_M^\mu) - \frac{1}{3}(2\gamma_q + \gamma_M)K_M^\mu, \tag{2.25}$$

and one can check again that the coefficients of independent Konishi currents vanish exactly when $\beta_g = \beta_f = 0$.

Because the operator S^μ is exactly conserved without internal anomalies, 't Hooft anomaly matching [20] can be applied to calculate the anomalies of its matrix elements with other exactly conserved currents, such as $\partial_\mu \langle S^\mu T^{\rho\sigma} T^{\lambda\tau}\rangle$. One argument for this (Sec. III of Ref. [3]) is the following. The operator equation $\partial_\mu S^\mu = 0$ holds in the absence of sources, and it must remain local when sources are introduced. For an external metric source dimensional and symmetry considerations restrict the possible form of the matrix element to

$$\langle \partial_\mu S^\mu(x)\rangle = s_0 R\tilde{R}(x), \tag{2.26}$$

where the right hand side is local. *A priori* $s_0(g(\mu))$ could depend on the RG scale μ. However, S^μ in this case is an RG invariant operator, so matrix elements cannot depend on $g(\mu)$. Therefore s_0 must be a constant, hence one-loop exact. If we now use the fact that S and R coincide at long distances we have the chain of equalities

$$\partial\langle RTT\rangle_{\rm IR} = \partial\langle STT\rangle_{\rm IR} = \partial\langle STT\rangle_{\rm UV} = \partial\langle S_0 TT\rangle, \tag{2.27}$$

where the last term simply includes the one loop graphs of the current S_0 and gives the $\mathrm{U}(1)_R$ anomaly coefficient quoted in the literature. Similar arguments justify the conventional calculation of $\mathrm{U}(1)_R FF$ and $\mathrm{U}(1)_R^3$ anomalies.

Formulas for anomaly coefficients. The previous discussion enables us to write simple formulas for the infrared values of the anomaly coefficients in terms of the anomaly-free R charges quoted in the literature. For a chiral superfield Φ_i^α in the representation R_i of dimension $\dim R_i$ the R charge r_i is related to γ_i^* in the S_0^μ current (2.19) by $r_i = (2+\gamma_i^*)/3$.

The quantities $b_{\rm IR}$, $c_{\rm IR}$, and $a_{\rm IR}$ are the infrared values of the trace anomaly coefficients $\tilde{b}$, $\tilde{c}$, and a in Eq. (2.1). They are normalized by the free field values in Eqs. (2.4) and (2.5) and are related to R-current anomalies by Eq. (2.14). One then obtains

$$b_{\rm IR} = -3\mathrm{U}(1)_R F^2 = 3\sum_{ij} (\dim R_i)(1-r_i)T_i^j T_j^i,$$

$$c_{\rm IR} - a_{\rm IR} = -\frac{1}{16}\mathrm{U}(1)_R$$

$$= -\frac{1}{16}\left(\dim G + \sum_i (\dim R_i)(r_i-1)\right),$$

$$5a_{\rm IR} - 3c_{\rm IR} = \frac{9}{16}\mathrm{U}(1)_R^3$$

$$= \frac{9}{16}\left(\dim G + \sum_i (\dim R_i)(r_i-1)^3\right),$$

$$c_{\rm IR} = \frac{1}{32}[9\mathrm{U}(1)_R^3 - 5\mathrm{U}(1)_R]$$

$$= \frac{1}{32}\left(4\dim G + \sum_i (\dim R_i)(1-r_i)[5-9(1-r_i)^2]\right),$$

$$a_{\rm IR} = \frac{3}{32}[3\mathrm{U}(1)_R^3 - \mathrm{U}(1)_R]$$

$$= \frac{3}{32}\left(2\dim G + \sum_i (\dim R_i)(1-r_i)[1-3(1-r_i)^2]\right). \tag{2.28}$$

Note that the R charge of the fermionic component of Φ_i^α is r_i-1 and appears in these formulas, which are valid for theories in an interacting conformal phase with unique anomaly free R charges and no accidental symmetry. The treatment is extended to include accidental symmetry and theories with nonunique R charge in later sections.

The hypothesis that there is a nontrivial infrared fixed point in any given model is established by several consistency tests which in the past did not include the positivity conditions we have discussed. The set of infrared R charges assigned in the literature is not guaranteed to produce positive $b_{\rm IR}$, $c_{\rm IR}$, $a_{\rm IR}$ so the positivity constraints provide an additional check that the hypothesis of an interacting fixed point is correct.

The corresponding UV quantities are obtained from Eq. (2.28) by replacing $r_i \to \frac{2}{3}$, and one can check that Eqs. (2.4) and (2.5) are reproduced when this is done. Thus for flows without gauge symmetry breaking the total flow of the central charges from the UV to the IR is due to the difference between the canonical and nonanomalous R charges, and are given by the following formulas:

$$b_{\rm UV} - b_{\rm IR} = 3\sum_{ij} (\dim R_i)\left[\left(r_i - \frac{2}{3}\right)T_i^j T_j^i\right], \tag{2.29}$$

$$c_{\rm UV} - c_{\rm IR} = \frac{1}{384}\sum_i (\dim R_i)(2-3r_i)[(7-6r_i)^2 - 17], \tag{2.30}$$

$$a_{\rm UV} - a_{\rm IR} = \frac{1}{96}\sum_i (\dim R_i)(3r_i-2)^2(5-3r_i). \tag{2.31}$$

Higgs flows with spontaneous symmetry breaking of gauge symmetry are studied in Sec. III.

There is a rather interesting aspect of the formulas (2.29)–(2.31) for central charge flows. In perturbation theory about a UV free fixed point the quantity $(2-3r_i)$ is of order g^2. Thus our formulas are consistent with the two-loop calculations of Ref. [21] where it was found that radiative corrections to $c(g)$ begin at two-loop order [and quantitatively agree [3] with the perturbative limit of Eq. (2.30)], while corrections to $a(g)$ vanish at two-loop order. The ''input'' to Eq. (2.31) comes from one-loop chiral anomalies, so it is curious that the formula for $a_{\rm UV} - a_{\rm IR}$ ''knows'' about two-loop curved space computations.

The perturbative structure becomes more significant when we consider the physical requirement that a c function must be stationary at a fixed point, and that Zamolodchikov's c function actually satisfies $(\partial/\partial g)C(g)=0$ at a fixed point. A monotonic interpolating A function is not known in four dimensions but one can consider a candidate A function obtained from $a_{\rm IR}$ in Eq. (2.28) by replacing the infrared values of r_i by their values calculated along the flow, i.e., $r_i \to [2+\gamma_i(g(\mu))]/3$. This candidate A function naturally satisfies Zamolodchikov's stationarity condition at weak coupling. The analogous candidate C function obtained from $c_{\rm IR}$ of Eq. (2.28) does not.

III. MODELS WITH UNIQUE *R* CHARGE

In this section we discuss the positivity conditions $b_{\rm IR}>0$, $c_{\rm IR}>0$, $a_{\rm IR}>0$, and $a_{\rm UV}-a_{\rm IR}>0$ in a large set of models in the literature where the anomaly-free R charge is unique. While some of these models will be considered in more detail in the next two sections, here we are going to analyze some general aspects. It is worth emphasizing that even though the positivity of $b_{\rm IR}$ and $c_{\rm IR}$ follows generally from unitarity constraints, the fact that they turn out to be positive in our approach is additional evidence that our understanding of the infrared dynamics is correct.

The positivity constraint $a_{\rm UV}-a_{\rm IR}>0$ deserves some comments. As explained above, the gravitational effective action depends on the functions a and c. It is natural to assume that a candidate C function measuring the irreversibility of the RG flow may be a universal model independent linear combination $C=ua+vc$. We are going to show that the only combination $C=ua+vc$ which satisfies $\Delta C = u(a_{\rm UV}-a_{\rm IR})+v(c_{\rm UV}-c_{\rm IR})\geq 0$ for all models is just $C=a$. First note that since there are theories [e.g., SU(N_c) SUSY QCD with $N_f<3N_c$] with $c_{\rm UV}-c_{\rm IR}$ of either sign [3] and $a_{\rm UV}-a_{\rm IR}$ positive, one must take $u>0$. It is then sufficient to assume $u=1$. Consider the electric version of Seiberg's SU(N_c) QCD with N_f fundamental flavors in the conformal window $3N_c/2<N_f<3N_c$. In the weak coupling limit N_c, $N_f\to\infty$, and $N_c/N_f\to 3$, the work of Ref. [3] shows that $\Delta c<0$ and $0\leq\Delta a\leq|\Delta c|$. So we have $v\leq 0$. On the other hand, in the weak coupling limit $N_f\to\infty$ and $N_c/N_f\to 3/2$ of the magnetic theory one can see that $0\leq\Delta a\leq\Delta c$ so we have $v\geq 0$. Then $v=0$, and $a_{\rm UV}-a_{\rm IR}>0$ is the only universal a-theorem candidate.

Below we state simple sufficient conditions for the positivity constraints $b_{\rm IR}>0$, $c_{\rm IR}>0$, $a_{\rm IR}>0$, and also for $a_{\rm UV}-a_{\rm IR}>0$ in the case of RG flows from a free ultraviolet to an infrared fixed point. Remarkably enough, these sufficient conditions can be quickly seen to be satisfied in most of the conformal window of all renormalizable theories that we have analyzed. Closer examination is required for cases with accidental symmetry. There are also many examples of flows between interacting fixed points which are generated by various deformations. These situations are discussed in later sections.

A. Sufficient conditions

We first note that in part of the conformal window of some models, the unitarity bound $r\geq\frac{2}{3}$ fails for one or more composite operators of the chiral ring. Then the formulas (2.28) for infrared anomalies require correction for the ensuing accidental symmetry. Such cases are discussed separately in Sec. IV, and we consider here models without accidental symmetry, which necessarily have $r_i\geq\frac{1}{3}$ for all fields of the microscopic theory.

The simplest way in which the positivity conditions can be satisfied is if the contributions to $b_{\rm IR}$, $c_{\rm IR}$, and $a_{\rm IR}$ in Eq. (2.28), and to $a_{\rm UV}-a_{\rm IR}$ in Eq. (2.31), are separately positive for each contributing representation R_i. This leads to the following sufficient conditions: (i) $b_{\rm IR}>0$ if $r_i\leq 1$ for all chiral superfields Φ^i; (ii) $c_{\rm IR}>0$ if $1-\sqrt{5}/3=0.254\leq r_i\leq 1$ or $r_i\geq 1+\sqrt{5}/3=1.745$ for all Φ^i; (iii) $a_{\rm IR}>0$ if $1-1/\sqrt{3}=0.423\leq r_i\leq 1$ or $r_i\geq 1+1/\sqrt{3}=1.577$ for all Φ^i; (iv) $a_{\rm UV}-a_{\rm IR}\geq 0$ if $r_i\leq\frac{5}{3}$ for all Φ^i.

In all of the models examined we find, that in the part of the conformal window where no accidental symmetry is required, (a) remarkably, $r_i\leq\frac{5}{3}$ for all renormalizable models, so the a theorem is always satisfied, (b) $1-\sqrt{5}/3<r_i<1$ in all electric models without accidental symmetry (since electric and magnetic anomalies match in all models, we have $b_{\rm IR}>0$ and $c_{\rm IR}>0$ on both sides of the duality), and, (c) $1-1/\sqrt{3}<r_i<1$ is satisfied in part of the conformal window of all theories, but not always. But the sufficient condition is rather weak, and the positive contribution of the gauge multiplet $a_{\rm IR}$ always ensures $a_{\rm IR}>0$ in the nonaccidental region.

Thus, most of the positivity conditions, especially the a theorem, can be verified essentially by inspection of the tables of R charges presented in the literature on the various models. Actually, in many cases one can prove that $r_i<\frac{5}{3}$ as a consequence of asymptotic freedom in absence of accidental symmetry (i.e., when all $r_i\geq\frac{1}{3}$). Explicit check is then unnecessary. We illustrate this in three simple situations.

(i) For models with N_f copies of a single irreducible real representation R (or N_f copies of $R\oplus\bar{R}$), one can see from the S_μ current in Eq. (2.24) that $\gamma^*=1-3T(G)/N_fT(R)$ [or $\gamma^*=1-3T(G)/2N_fT(R)$] and asymptotic freedom gives $\gamma^*<0$ in both cases. Thus $r=(2+\gamma^*)/3<\frac{2}{3}$.

(ii) For renormalizable Kutasov-Schwimmer type models the current (2.25) immediately gives the same information, $r<\frac{2}{3}$ for the fields in R and $\bar{R}$ and $r_X=\frac{2}{3}$.

(iii) We also consider models which have the same structure as magnetic SU(N_c) SUSY QCD, namely, N_f fields q in a real representation R' of a dual gauge group G' (or N_f fields q, $\tilde{q}$ in $R'\oplus\bar{R}'$) plus a gauge singlet meson field in the $\mathbf{N}_f\otimes\mathbf{N}_f$ [or $(\mathbf{1},\mathbf{N}_f)\otimes(\mathbf{N}_f,\mathbf{1})$] representation of the flavor group SU(N_f) [or SU(N_f)$\times$SU(N_f)]. There is a superpotential $W=qMq$ (or $W=qM\tilde{q}$). Here again one can inspect the gauge β function [or the appropriate S_μ current (2.25)] and find $\gamma_q^*<0$ and $\frac{1}{3}\leq r_q<\frac{2}{3}$. The superpotential then tells us that $r_M=2-2r_q$ satisfies $\frac{2}{3}<r_M\leq\frac{4}{3}$ with the upper limit from unitarity without accidental symmetry. Thus again $r_i<\frac{5}{3}$ for all fields.

B. Flows between superconformal fixed points

A conformal fixed point is characterized by the values of b, c, and a. These values do not depend on the particular flow which leads to or from this conformal theory. Therefore one may be interested in a computation of the flow $a_{\rm UV}-a_{\rm IR}$ for a theory which interpolates between two interacting conformal fixed points. Such an interpolation may be obtained by deforming a superconformal theory with a relevant operator which generates an RG flow driving the theory to another superconformal fixed point. Since we know the conformal theories at both ultraviolet and infrared limits of this interpolating theory, the computation simply requires subtraction of the end-point central charges. In this case we do not need to construct any S-current interpolating between the ultraviolet and infrared conformal fixed points. However, it is interesting that in some cases one can construct such an S current and check directly the value of the flow $a_{\mathrm{UV}}-a_{\mathrm{IR}}$. We discuss below aspects of various types of deformations.

Mass deformations. The simplest case is a mass deformation. Consider a conformal theory (H) characterized by a^H, b^H, and c^H which contains a chiral superfield Φ in a real representation of the gauge group (or a pair of chiral superfields Φ and $\tilde{\Phi}$ in conjugate representations). Such a theory may be deformed by adding a gauge invariant mass term $W_m=(m/2)\Phi^2$ (or $W_m=m\Phi\tilde{\Phi}$). We assume that the heavy superfield Φ (or Φ and $\tilde{\Phi}$) decouples from the low-energy spectrum, and that the resulting theory flows to a new conformal fixed point with a smaller global symmetry group, and characterized by the values a^L, b^L, and c^L. Since the heavy fields of the original theory do not contribute to infrared anomalies, we have $a_{\mathrm{IR}}=a^L$, $b_{\mathrm{IR}}=b^L$, $c_{\mathrm{IR}}=c^L$. On the other hand, the heavy fields contribute to ultraviolet anomalies so that $a_{\mathrm{UV}}=a^H$, $b_{\mathrm{UV}}=b^H$, and $c_{\mathrm{UV}}=c^H$. Thus we have $a_{\mathrm{UV}}-a_{\mathrm{IR}}=a^H-a^L$. As a result we expect that $a_{\mathrm{UV}}>a_{\mathrm{IR}}$. This is indeed the case for all the models that we have analyzed.

One can obtain a simple analytic formula in the case of an electric type theory with N_f copies of $R\oplus \bar{R}$ representation and no superpotential. In this theory $r=1-T(G)/2N_fT(R)$ for the N_f quarks of the theory H. We consider a mass deformation of H which leaves N_f-n massless quarks in the theory L. These quarks have $r=1-T(G)/2(N_f-n)T(R)$. Substituting these charges in the formula (2.31) we subtract with the result

$$a_H-a_L=\frac{9\,\dim RT(G)^3}{128T(R)^2}\left(-\frac{1}{N_f^2}+\frac{1}{(N_f-n)^2}\right)>0.$$

In the special case of interpolation between an ultraviolet free theory and an infrared nontrivial conformal fixed point one can apply a more formal argument. In this case we consider the electric theory above with added mass term for the n massive quarks. The unique S_μ current of this new theory is

$$S_\mu=R_\mu+\frac{1}{3}\left(1-\frac{3T(G)}{2(N_f-n)T(R)}-\gamma_L\right)\times K_\mu^L+\frac{1}{3}(1-\gamma_H)K_\mu^H,$$

where the superscripts L and H indicate Konishi currents for the light and heavy quarks, respectively. Thus $\gamma_H^*=1$ and $r_H=1$ so that the heavy quarks do not contribute to $a_{\mathrm{IR}}=a_L$ in Eq. (2.31). For the light quarks $\gamma_L^*=1-3T(G)/2(N_f-n)T(R)$ and $r_L=1-T(G)/2(N_f-n)T(R)$ which is exactly the correct value in the low-energy theory of N_f-n flavors. Thus the S_μ current analysis leads to the same value of $a_{\mathrm{IR}}=a_L$ used above.

Higgs deformations. There are two qualitatively different types of Higgs deformations. The first is a deformation along flat directions of the potential for the scalar fields. Under such a deformation one generically breaks both the gauge and flavor symmetries. While the Goldstone bosons corresponding to the gauge symmetry breaking are absorbed by the Higgs mechanism, the Goldstone bosons of the flavor symmetry breaking remain in the massless spectrum of the theory. Therefore these Goldstone bosons (and their superpartners) have to be taken into account in the computation of the infrared values of a, b, and c of the resulting theory. It is implicitly assumed in the literature that these Goldstone superfields decouple from other light fields of the low-energy theory and are free in the infrared. We thus assign $r=\frac{2}{3}$ to these fields.

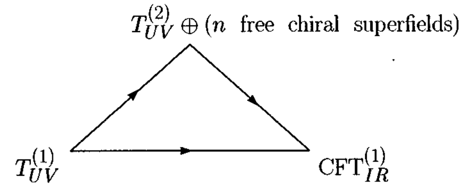

FIG. 1. The diagram of flows under Higgs deformations.

In general the positivity of the flow $a_{\mathrm{UV}}-a_{\mathrm{IR}}$ under the Higgs deformations is nontrivial evidence for the a theorem. In a simple situation of flow from the ultraviolet free theory deformed by the Higgs mechanism to an infrared conformal fixed point the positivity of $a_{\mathrm{UV}}-a_{\mathrm{IR}}$ follows from the following argument. Let us consider an asymptotically free theory T. Let us also consider an asymptotically free theory $T^{(1)}$ which is a Higgs deformed version of T along a flat direction and flows to a nontrivial conformal theory in the infrared $\mathrm{CFT}_{\mathrm{IR}}^{(1)}$. We are going to argue that the flow $a_{\mathrm{UV}}(T^{(1)})-a_{\mathrm{IR}}^{(1)}>0$. We assume that there are n Goldstone chiral superfields that decouple from the rest of the theory. It is convenient to define another asymptotically free theory $T^{(2)}$ which is just the theory $T^{(1)}$ with all massive fields dropped out plus n free chiral superfields. Let us assume that the interacting part of the theory $T^{(2)}$ is also in its conformal window and flows to a nontrivial conformal theory $\mathrm{CFT}_{\mathrm{IR}}^{(2)}$, and the a theorem is satisfied for this flow. We have $\mathrm{CFT}_{\mathrm{IR}}^{(1)}=\mathrm{CFT}_{\mathrm{IR}}^{(2)}\oplus(n$ free chiral superfields). Therefore instead of the flow $T^{(1)}\to\mathrm{CFT}_{\mathrm{IR}}^{(1)}$ one can consider the two step flow $T_{\mathrm{UV}}^{(1)}\to T_{\mathrm{UV}}^{(2)}\oplus(n$ free chiral superfields)$\to\mathrm{CFT}_{\mathrm{IR}}^{(1)}$ (see Fig. 1).

Since the a theorem is trivially satisfied for the flow $T_{\mathrm{UV}}^{(1)}\to T_{\mathrm{UV}}^{(2)}\oplus(n$ free chiral superfields) we arrive at the conclusion that $a_{\mathrm{UV}}(T^{(1)})-a_{\mathrm{IR}}^{(1)}>0$.

The second type of Higgs deformation is the magnetic counterpart of a mass term in the electric theory. To be concrete we consider $\mathrm{SU}(N_c)$ SUSY QCD with electric quarks Q_α^i and antiquarks $\tilde{Q}_i^\alpha$, where $\alpha=1,\ldots,N_c$, and $i=1,\ldots,N_f$ are color and flower indices, respectively. The magnetic theory has $G=\mathrm{SU}(N_f-N_c)$ with quarks, antiquarks, and meson q_i^α, $\tilde{q}_\alpha^i$, and M_j^i. The mass perturbation $W_m=mQ_\alpha^{N_f}\tilde{Q}_{N_f}^\alpha$ in the electric theory is mapped to $W_m=mM_{N_f}^{N_f}$ on the magnetic side [23] so that flavor symmetry is broken explicitly to $\mathrm{SU}(N_f-1)$. Analysis [23] of the magnetic equations of motion shows that there is a Higgs effect with $\langle q_{N_f}\tilde{q}^{N_f}\rangle\neq 0$, so the gauge group is broken to $\mathrm{SU}(N_f-N_c-1)$. The spectrum contains massive fields plus the light fields of the magnetic effective low-energy theory with $G = \mathrm{SU}(N_f-N_c-1)$ and N_f-1 flavors. If this theory is still in its conformal window, i.e., $N_f-1>\frac{3}{2}N_c$, then a_{IR} can be computed from Eq. (2.28) with the matter content and the gauge group of the low-energy theory.

As an example one may consider a special case of the flow from the Higgs deformed ultraviolet free theory to an infrared conformal fixed point. It should be no surprise that there is also a formal argument (based on a consideration of a conserved S_μ current) since the conserved R current on the electric side corresponds to a conserved current on the magnetic. One can verify that the magnetic theory, with $W_m = mM^{N_f}_{N_f}$, has a unique set of anomaly free R charges. There is an elaborate cancellation of the contributions of heavy fields to the $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_R^3$ anomalies, and only the expected contributions from fields of the low-energy effective theory remain.

Deformations of superpotential. One can also consider more general deformations of the superconformal theories by relevant operators. A particular type of deformation is obtained by adding a relevant chiral gauge invariant operator to the superpotential of a superconformal theory. As a result the deformed theory may flow to another fixed point along the RG flow generated by the deformation. In all renormalizable models that we studied the induced flow of a is positive but this is not true in nonrenormalizable models (see Sec. VI). Examples of interpolating flows are those between the k and $k-1$ Kutasov-Schwimmer models which are discussed in Sec. V.

IV. ACCIDENTAL SYMMETRIES

In this section we explain the computation of the infrared values of a, b, and c in the presence of accidental symmetry. The appearance of accidental symmetry is associated with an apparent violation of the unitarity bound $r\geq\frac{2}{3}$ for a primary gauge invariant chiral composite field M. The simplest hypothesis explored in the literature (for a review and discussion see Ref. [22]) is that this signals that the field M is actually decoupled from the interacting part of the theory, and becomes a free chiral superfield in the infrared [22].

On the other hand the R charge is equal to $\frac{2}{3}$ for a free chiral superfield, which contradicts the result of computation with the S_μ current. A plausible explanation is that there is an additional anomaly free global U(1) generated by the spin-1 component $J^{(M)}_\mu$ of the composite superfield $\bar{M}M$. The field M is charged with respect to the current $J^{(M)}_\mu$ but the other fields are not. In this case the perturbative anomaly free S_μ current can mix with the $J^{(M)}_\mu$ current under the RG flow because the scaling dimension of the latter tends to the canonical dimension 3 of a conserved current. Thus the infrared R current can be determined as an infrared limit of a linear combination

$$R^{\mathrm{IR}}_\mu = S_\mu + A_\mu, \tag{4.1}$$

where $A_\mu=\lambda J^{(M)}_\mu$. The coefficient λ is fixed by the condition that $R=\frac{2}{3}$ for the field M.

Assuming that this picture is correct one can easily compute the infrared values of the central functions a, b, and c. In the notation of Sec. II, one has to compute the three point correlators $\langle RRR\rangle_{\mathrm{IR}}$ and $\langle RTT\rangle_{\mathrm{IR}}$. Substituting the expression (4.1) for R_μ into these correlators one has (the subscript IR is omitted here)

$$\langle RRR\rangle = \langle SSS\rangle + 3\langle SSA\rangle + 3\langle SAA\rangle + \langle AAA\rangle,$$
$$\langle RTT\rangle = \langle STT\rangle + \langle ATT\rangle. \tag{4.2}$$

At this point we note that the correlators $\langle SSA\rangle$, $\langle SAA\rangle$, $\langle AAA\rangle$, and $\langle ATT\rangle$ are saturated by the free chiral field M and hence they can be easily computed, i.e., we have

$$\langle SSA\rangle = \langle SSA\rangle_{\mathrm{free}}, \quad \langle SAA\rangle = \langle SAA\rangle_{\mathrm{free}},$$
$$\langle AAA\rangle = \langle AAA\rangle_{\mathrm{free}}, \quad \langle ATT\rangle = \langle ATT\rangle_{\mathrm{free}}.$$

Thus the correlators $\langle RRR\rangle_{\mathrm{IR}}$ and $\langle RTT\rangle_{\mathrm{IR}}$ can be rewritten as follows:

$$\langle RRR\rangle_{\mathrm{IR}} = \langle SSS\rangle + \langle RRR\rangle_{\mathrm{free}} - \langle SSS\rangle_{\mathrm{free}},$$
$$\langle RTT\rangle_{\mathrm{IR}} = \langle STT\rangle + \langle RTT\rangle_{\mathrm{free}} - \langle STT\rangle_{\mathrm{free}}. \tag{4.3}$$

As we explained in Sec. II the central charges a_{IR} and c_{IR} are just given by linear combinations of the correlators $\langle RRR\rangle_{\mathrm{IR}}$ and $\langle RTT\rangle_{\mathrm{IR}}$. We consider the case where there is one accidental U(1) symmetry for the gauge invariant composite superfield M in an irreducible representation of the flavor group of dimension dim M (more general cases can easily be handled). The corrected infrared values of the central charges are

$$a_{\mathrm{IR}} = a^{(0)}_{\mathrm{IR}} + \frac{\dim M}{96}(2-3r_M)^2(5-3r_M),$$

$$c_{\mathrm{IR}} = c^{(0)}_{\mathrm{IR}} + \frac{\dim M}{384}(2-3r_M)[(7-6r_M)^2-17]. \tag{4.4}$$

Here we denoted by $a^{(0)}_{\mathrm{IR}}$ and $c^{(0)}_{\mathrm{IR}}$ the expressions for a and c given by Eqs. (2.28), and r_M stands for the S charge of the chiral field M, specifically the sum of the S charges of its elementary constituents. Since by assumption $r<\frac{2}{3}$ it is easy to see that the correction to a is always positive. The correction to c is positive at $r<(7-\sqrt{17})/6\approx 0.479$ and negative at $0.479\approx(7-\sqrt{17})/6<r<\frac{2}{3}$. In some models the accidental correction is required to make a_{IR} and c_{IR} positive, so the sign is important.

In general the formulas for the infrared values of flavor central functions should also be corrected due to the presence of accidental symmetries. The general formula for the corrected b can be easily obtained along the above lines and reads

$$b_{\mathrm{IR}} = b^{(0)}_{\mathrm{IR}} + 3T^i_jT^j_i\left(r_M-\frac{2}{3}\right).$$

Here we denoted by $b^{(0)}_{\mathrm{IR}}$ the expression for b_{IR} given in Eq. (2.28), T^i_j stands for the flavor generator associated with b. The correction dim $M(r_M-\frac{2}{3})$ is always negative.

Deformations of conformal fixed points with accidental symmetry. In the following we test various examples of su-

TABLE I. Flows from UV free theories to Seiberg's conformal QCD.

Gauge group	$a_{\rm UV}-a_{\rm IR}$ in electric theory	$a_{\rm UV}-a_{\rm IR}$ in magnetic theory
SU(N_c)	$\frac{N_f N_c}{48}\left(1-\frac{3N_c}{N_f}\right)^2\left(2+\frac{3N_c}{N_f}\right)$	$\frac{1}{12}\left(1-\frac{3N_c}{2N_f}\right)^2(3N_c^2+4N_cN_f+3N_f^2)$
SO(N_c)	$\frac{N_c(-6+2N_f+3N_c)(6+N_f-3N_c)^2}{96N_f^2}$	$\frac{N_c(-6-2N_f+3N_c)^2(3N_f^2-6N_c+4N_cN_f+3N_c^2)}{96N_f^2}$
Sp($2N_c$)	$\frac{(-3+N_f-3N_c)^2N_c(3+2N_f+3N_c)}{24N_f^2}$	$\frac{(3-2N_f+3N_c)^2(3N_f^2+3N_c+4N_cN_f+3N_c^2)}{24N_f^2}$

perconformal models and flows between them. In particular we will consider flows from superconformal models with accidental symmetries taken as an ultraviolet fixed point to different infrared fixed points. Such a flow may be generated by appropriate deformation of the ultraviolet theory with a relevant operator. It is important that the ultraviolet theory has to be taken together with the free chiral fields generating the accidental symmetry. In fact the deformation of the ultraviolet theory by a relevant operator generates a nontrivial coupling of the interacting part of the UV theory to the accidental chiral superfields. This turns out to be important for positivity of $a_{\rm UV}-a_{\rm IR}$.

V. EXAMPLES OF MODELS WITH UNIQUELY DEFINED *S* CURRENT AND THE FLOWS

In this section we give detailed results for the models that we have analyzed. We mainly focus on subtleties met in the computations of the infrared values of a and c.

A. Models with one type of irreducible representation

This class of models includes the SU(N_c) series, SO(N_c) series [23], Sp($2N_c$) series [24], Pouliot Spin (7) model [25], Distler-Karch models with exceptional groups [26].

Seiberg's QCD with $G=\mathrm{SU}(N_c)$, SO(N_c) with N_f, and Sp($2N_c$) with $2N_f$ fundamentals. Conformal windows are $3N_c/2<N_f(\mathrm{SU})<3N_c$, $3(N_c-2)/2<N_f(\mathrm{SO})<3(N_c-2)$, $3(N_c+1)/2<N_f(\mathrm{Sp})<3(N_c+1)$. There are no accidental symmetries. Since all R charges obey $r\leqslant\frac{5}{3}$ we always have $\Delta a=a_{\rm UV}-a_{\rm IR}>0$ for the flows from the free ultraviolet to conformal fixed points. The results of our computations are given Table I. It should be noted that all flows vanish quadratically in the respective weakly coupled limits of electric and magnetic theories. This agrees with the discussion of the perturbative limit at the end of Sec. II.

The models considered below have nonrenormalizable magnetic versions. Therefore we discuss only the electric versions that are renormalizable. The results of our computations are given in Table II. Aspects of the RG flows of nonrenormalizable theories are considered in the next section.

Spin (7) Pouliot model with N_f spinors **8**, Q_i. Conformal window: $7\leqslant N_f\leqslant 14$. We have in the infrared $r_{\mathbf 8}^{\rm IR}=1-5/N_f$. There is an accidental symmetry at $N_f=7$ due to decoupled QQ singlet. In Table II we separated the accidental corrections to $a_{\rm IR}$ and $c_{\rm IR}$ from the regular ones. Note that the correction to $c_{\rm IR}$ turns out to be negative.

G_2 with N_f **7**. Conformal window: $6\leqslant N_f\leqslant 11$. We have $R_{\mathbf 7}^{\rm IR}=1-4/N_f$. The accidental symmetry point appears at $N_f=6$ where QQ has $r=\frac{2}{3}$ and hence it is free. Therefore there are no accidental corrections to the central charges.

E_7 Distler-Karch model: four fundamentals **56**, Q_i; $r_Q=\frac{1}{4}$.

E_6 Distler-Karch model (I): six fundamentals **27**, Q_i; $r_Q=\frac{1}{3}$.

E_6 Distler-Karch model (II): $3\times(\mathbf{27}+\overline{\mathbf{27}})$ fundamentals Q_i; $r_Q=\frac{1}{3}$.

F_4 Distler-Karch model: five fundamentals **26**, Q_i; $r_Q=\frac{2}{5}$.

F_4 Distler-Karch model: four fundamentals **26**, Q_i; $r_Q=\frac{1}{4}$. There is an accidental symmetry associated with decoupling of meson fields $M_{ij}=Q_iQ_j$. In Table II we separated the accidental corrections to $a_{\rm IR}$ and $c_{\rm IR}$ from the regular ones. Again the correction to $c_{\rm IR}$ turns out to be negative.

Spin (8) Distler-Karch model: $4\times(\mathbf{8}_v+\mathbf{8}_c+\mathbf{8}_s)$ fundamentals Q; $r_Q=\frac{1}{2}$.

B. Deformations

Deformations of SU(N_c), SO(N_c), *and* Sp($2N_c$) *Seiberg* QCD *models*. Higgs deformation of the Seiberg superconformal models corresponds to N_c, $N_f\rightarrow N_c'=N_c-1$, $N_f'=N_f-1$. The infrared theory has $2(N_f-1)$ decoupled Goldstone gauge singlets for SU(N_c) and Sp($2N_c$) models and N_f-1 for SO(N_c).

(1) Consider first the SU(N_c) theory. In the region $3N_c/2<N_f\leqslant 3N_c-3$ both the ultraviolet and infrared theories are in their conformal windows and we have

$$\Delta a=\frac{1-N_f}{24}+\frac{3(2N_c-1)}{8}-\frac{9N_c^4}{16N_f^2}+\frac{9(N_c-1)^4}{16(N_f-1)^2}>0.$$

In the cases $N_f=3N_c-1$, $3N_c-2$ the infrared theory is free since $N_f'=3N_c'+1$ and $N_f'=3N_c'$, respectively. The infrared value $a_{\rm IR}$ is then computed using $r=\frac{2}{3}$ for all chiral superfields of the low-energy N_c', N_f' theory and the Goldstone fields. The results are

$$\Delta a=\frac{-9+76N_c-210N_c^2+180N_c^3}{48(-1+3N_c)^2}>0$$

TABLE II. The infrared a and c charges, and flows from the ultraviolet free theory to conformal fixed points.

Model	$a_{\rm IR}$	$c_{\rm IR}$	Electric theory, $a_{\rm UV}^{\rm free}-a_{\rm IR}$
Spin (7) with $N_f<7$ spinors **8** no accidental symmetry	$\frac{123}{16}-\frac{1125}{4N_f^2}>0$	$\frac{71}{8}-\frac{1125}{4N_f^2}>0$	$\frac{N_f}{12}\left(1-\frac{15}{N_f}\right)^2\left(2+\frac{15}{N_f}\right)$
Spin (7) with $N_f=7$ spinors **8** accidental symmetry	$\frac{1527}{784}+\frac{23}{168}=\frac{4903}{2352}$	$\frac{1229}{392}-\frac{13}{84}=\frac{3505}{1176}$	$\frac{3551}{1176}$
G_2 with $7\leqslant N_f\leqslant 11$ in **7** no accidental symmetry	$\frac{21}{4}-\frac{126}{N_f^2}>0$	$\frac{49}{8}-\frac{126}{N_f^2}>0$	$\frac{7N_f}{48}\left(1-\frac{12}{N_f}\right)^2\left(1+\frac{6}{N_f}\right)$
E_7 with 4 fundamentals **56**	$\frac{903}{64}$	$\frac{1043}{64}$	$\frac{2975}{192}$
E_6 with 6 fundamentals in **27**	$\frac{45}{4}$	$\frac{105}{8}$	$\frac{27}{4}$
E_6 with matter in $3\times(\mathbf{27}+\overline{\mathbf{27}})$	$\frac{45}{4}$	$\frac{105}{8}$	$\frac{27}{4}$
F_4 with $N_f=5$ in **26**	$\frac{1833}{200}$	$\frac{1079}{100}$	$\frac{247}{75}$
F_4 with $N_f=4$ in **26** accidental symmetry	$\frac{1209}{256}+\frac{7}{48}=\frac{3739}{768}$	$\frac{1625}{256}-\frac{1}{48}=\frac{4859}{768}$	$\frac{5413}{768}$
Spin (8) with matter in $4\times(\mathbf{8}_v+\mathbf{8}_c+\mathbf{8}_s)$	$\frac{51}{8}$	$\frac{61}{8}$	$\frac{7}{8}$

and

$$\Delta a=\frac{(-2+5N_c)(6-19N_c+12N_c^2)}{16(-2+3N_c)^2}>0.$$

(2) Consider the SO(N_c) theory. In the region $3(N_c-2)/2<N_f\leqslant 3N_c-8$ both the ultraviolet and infrared theories are at their conformal fixed points and we have

$$\Delta a=\frac{1-N_f}{48}+\frac{3}{32}\left[2N_f+8(N_f-N_c+2)-9\left(\frac{N_c}{N_f}-\frac{N_c-1}{N_f-1}\right)\times(N_f-N_c+2)^2+3\left(\frac{N_c}{N_f^2}-\frac{N_c-1}{(N_f-1)^2}\right)\times(N_f-N_c+2)^3\right]>0$$

In the cases of $N_f=3N_c-7$, $3N_c-8$ (in the latter case we limit ourselves to $N_c\geqslant 4$ for the ultraviolet theory to be in the conformal window) the infrared theory is free so that, respectively,

$$\Delta a=\frac{-882+1756N_c-1011N_c^2+180N_c^3}{96(-7+3N_c)^2}>0,$$

$$\Delta a=\frac{-192+372N_c-193N_c^2+30N_c^3}{16(-8+3N_c)^2}>0.$$

(3) Consider the Sp($2N_c$) theory. In the region $3(N_c+1)/2<N_f\leqslant 3N_c+1$ both the ultraviolet and infrared theories are at their conformal fixed points and we have

$$\Delta a=\frac{1-N_f}{24}+\frac{1}{32}\left[6(3-4N_f)+96(N_f-N_c-1)-108\left(\frac{N_c+1}{N_f}-\frac{N_c}{N_f-1}\right)(N_f-N_c-1)^2+36\left(\frac{N_c+1}{N_f^2}-\frac{N_c}{(N_f-1)^2}\right)(N_f-N_c-1)^3\right]>0.$$

In the cases of $N_f=3N_c+1$, $3N_c+2$ the infrared theory is free so that, respectively,

$$\Delta a=\frac{-3-16N_c+41N_c^2+138N_c^3}{16(1+3N_c)^2}>0$$

and

$$\Delta a=\frac{-28+86N_c+471N_c^2+414N_c^3}{48(2+3N_c)^2}>0.$$

The mass deformations obviously respect the a theorem because $\partial a/\partial N_f>0$ in all cases (see explicit computation in Sec. III).

Deformations of spin (7) *Pouliot model.* First consider the Higgs deformation of the spin (7) Pouliot model with $7\leqslant N_f\leqslant 14$ fundamentals to the G_2 model with N_f-1 fundamentals and N_f-1 Goldstone superfields.

TABLE III. Higgs deformations of Distler-Karch models.

Higgs deformation	$F_4 \to \mathrm{Spin}(8)$	$E_6 \to F_4$	$E_7 \to E_6$
$a_{\mathrm{UV}} - a_{\mathrm{IR}}$	$\frac{2623}{300}$	$\frac{2377}{1200}$	$\frac{175}{64}$

In the region $8 \leqslant N_f \leqslant 14$ there are no accidental symmetries either in the ultraviolet or in the infrared. Thus we have $r_{\mathbf{8}}^{\mathrm{UV}} = 1 - 5/N_f$, $r_{\mathbf{7}}^{\mathrm{IR}} = 1 - 4/(N_f - 1)$, and $r_{\mathbf{1}}^{\mathrm{IR}} = \frac{2}{3}$. The flow is

$$\Delta a = \frac{1}{144 N_f^2 (N_f - 1)^2} (13500 - 27000 N_f + 7523 N_f^2 - 141 N_f^3 + 69 N_f^4 + N_f^5) > 0.$$

Note that for $N_f = 13,14$ the infrared G_2 theory is free. In this case we have

$$\Delta a (N_f = 13) = \frac{3781}{2704}, \quad \Delta a (N_f = 14) = \frac{859}{588}.$$

For $N_f = 7$ the UV theory has an accidental symmetry. One has

$$\Delta a = \frac{1945}{2352}.$$

Mass deformations. By giving a mass to one of the flavors one can generate the flow $N_f \to N_f - 1$. Obviously, $a_{\mathrm{UV}} - a_{\mathrm{IR}} = a(N_f) - a(N_f - 1) > 0$.

The results of computations for the flows induced by Higgs deformation of Distler-Karch superconformal models are given in Table III.

Mass deformation of F_4 model [26]. By giving a mass to one of flavors the theory with $N_f = 5$ is driven to a new conformal fixed point with $N_f = 4$ flavors Q_i and $r_Q = \frac{1}{4}$. The theory has an accidental symmetry associated with decoupling of the 16 mesons $M_{ij} = Q_i Q_j$, $r_M = \frac{2}{3}$. For the flow from $N_f = 5$ to $N_f = 4$ we have

$$\Delta a = \frac{85693}{19200}.$$

C. Models with two types of irreducible representations with uniquely determined S current

This set of models includes those given in Refs. [17] for SU [18,19], SO, and Sp gauge groups. We discuss in detail only the SU Kutasov-Schwimmer models and the Pouliot spin (7) model with $N_c + 4$ flavors in **8** and singlets [25]. For these models we discuss also various flows between conformal fixed points.

Consider the Kutasov-Schwimmer model [17] with the $\mathrm{SU}(N_c)$ gauge group, N_f flavors of quarks, Q and $\tilde{Q}$ in the fundamental, and a chiral superfield X in the adjoint representation. The superpotential is $W = X^{k+1}$. The R charges are given in Table IV.

The theory has a dual with gauge group $\mathrm{SU}(kN_f - N_c)$, with N_f flavors of $(\square + \bar{\square})$, an adjoint and gauge singlets. The conformal window is presumed to be the region in N_f, N_c where both the electric and magnetic theories are asymptotically free,

TABLE IV. Matter content of Kutasov-Schwimmer models.

	$\mathrm{SU}(N_c)$	$\mathrm{SU}(N_f)_Q$	$\mathrm{SU}(N_f)_{\tilde{Q}}$	$\mathrm{U}(1)_R$
Q	$\square$	$\square$		$1 - \frac{2N_c}{(k+1)N_f}$
$\tilde{Q}$	$\bar{\square}$		$\square$	$1 - \frac{2N_c}{(k+1)N_f}$
X	adj			$\frac{2}{k+1}$

$$\frac{2N_c}{2k-1} < N_f < 2N_c.$$

There is an accidental symmetry in the range

$$\frac{2N_c}{2k-1} < N_f \leqslant \frac{3N_c}{k+1},$$

where it corresponds to $QX^j\tilde{Q}$ out of the unitary region for one or more values of j. This accidental symmetry may appear in the conformal window for any $k \geqslant 2$ (and sufficiently large N_c). In particular, for $k = 2$ it appears for $N_f \leqslant N_c$, and for $k = 3$ it appears for $N_f \leqslant 3N_c/4$.

The only explicitly renormalizable Kutasov-Schwimmer model corresponds to $k = 2$, and it is studied below. The $k = 3$ theory can be made renormalizable in part of its conformal window, and this is discussed in Sec. VI.

In the case of absence of the accidental symmetry we may use Eqs. (2.28). We have

$$a_{\mathrm{IR}} = \frac{9}{32} \left\{ \left[\left(\frac{2}{k+1} - 1 \right)^3 + 1 \right] (N_c^2 - 1) - \frac{16}{(k+1)^3} \frac{N_c^4}{N_f^2} + \frac{2}{3} \frac{N_c^2 + 1}{k+1} \right\},$$

$$c_{\mathrm{IR}} = \frac{9}{32} \left\{ \left[\left(\frac{2}{k+1} - 1 \right)^3 + 1 \right] (N_c^2 - 1) - \frac{16}{(k+1)^3} \frac{N_c^4}{N_f^2} + \frac{10}{9} \frac{N_c^2 + 1}{k+1} \right\}, \tag{5.1}$$

$$\Delta a = -\frac{9}{32} \left\{ \left[\left(\frac{2}{k+1} - 1 \right)^3 + \frac{7}{27} \right] (N_c^2 - 1) - \frac{16}{(k+1)^3} \frac{N_c^4}{N_f^2} + \frac{2}{3} \frac{N_c^2 + 1}{k+1} - \frac{4 N_f N_c}{27} \right\}.$$

It is obvious that $\Delta a > 0$ in the conformal window since for all chiral fields $r_{\mathrm{IR}} \leqslant \frac{5}{3}$.

At $k=2$ we have

$$\Delta a=\frac{N_c}{24}\left(1-\frac{2N_c}{N_f}\right)^2(N_c+N_f)\geqslant 0.$$

Note also that the first two equations in Eq. (5.1) agree with the results for Seiberg's QCD at $k=1$.

We now consider the contribution of the accidental symmetry. We concentrate on the renormalizable case $k=2$. In the region $2N_c/3<N_f\leqslant N_c$, the meson operator $M=Q\tilde{Q}$ has $r_M=2(1-2N_c/3N_f)<\frac{2}{3}$, so there is an accidental correction to $c_{\rm IR}$ and $a_{\rm IR}$ (5.1). First we note that for large N_c and $N_f\approx 2N_c/3$, the previous formulas (5.1) for the $k=2$ central charges without accidental contributions give

$$c_{\rm IR}^{(0)}=-\frac{1}{6},\quad a_{\rm IR}^{(0)}=-\frac{1}{24}N_c^2$$

and are negative. This is not surprising since the theory is effectively nonunitary if the decoupling of the meson field is not taken into account. Positivity is restored by the accidental contribution, and this is an interesting check on the entire hypothesis of accidental symmetry. The sum of Eq. (5.1) and the accidental correction (4.4) are

$$a_{\rm IR}=-\frac{3}{16}-\frac{N_f^2}{6}+N_fN_c-\frac{7N_c^2}{6}+\frac{2N_c^3}{3N_f}-\frac{N_c^4}{6N_f^2}>0,$$

$$c_{\rm IR}=-\frac{1}{8}-\frac{N_f^2}{12}+\frac{11N_fN_c}{12}-\frac{9N_c^2}{8}+\frac{2N_c^3}{3N_f}-\frac{N_c^4}{6N_f^2}>0. \tag{5.2}$$

We note that intrinsically positive accidental corrections to $a_{\rm IR}$ decrease $a_{\rm UV}-a_{\rm IR}$ and thus tend to threaten the a theorem. Nevertheless we find that with the accidental contribution included

$$\Delta a=\frac{11N_c^2}{8}+\frac{N_c^4}{6N_f^2}-\frac{2N_c^3}{3N_f}-\frac{23N_fN_c}{24}+\frac{N_f^2}{6}\geqslant 0. \tag{5.3}$$

The contribution of the accidental symmetry to b is always negative. However, we find that all positivity conditions, including $b>0$, are satisfied for N_f, N_c in the accidental window. For example, for the central charge of the $SU(N_f)_Q$ current we find for $k=2$

$$b_{\rm IR}=\frac{4}{3N_f}(2N_f^2-2N_fN_c+N_c^2)>0. \tag{5.4}$$

Deformations of Kutasov-Schwimmer superconformal models.

(i) *Consider now the $k\to k-1$ interpolation.* The simplest case is to consider $W={\rm Tr}\,X^{k+1}+{\rm Tr}\,X^k$ with $\langle X\rangle=0$ and unbroken gauge group [17].

As mentioned above our approach is not expected to work for $k>3$ where there is no renormalizable description of the theory. For $k=3$ and $N_c\geqslant N_f$ there is a renormalizable description that will be discussed in the next section. Here we just note that in this region in the absence of accidental symmetries the central charges are given by Eqs. (5.1) at $k=3$. In particular at $N_c=N_f\geqslant 3$ (the only point in the renormalizable conformal window with no accidental symmetry) and for the flow $k=3\to k=2$ we have

$$\Delta a=\frac{7}{768}+\frac{43N_c^2}{768}>0.$$

At $N_c=2$ the $k=2$ Kutasov-Schwimmer model is not defined since ${\rm Tr}\,X^3=0$. Instead one can consider the flow from the $k=3$ $N_f=2$ fixed point in the ultraviolet to $k=1$, i.e., to Seiberg's SU(2) SUSY QCD with $N_f=2$ flavors. This infrared theory is confining and the flat directions are lifted due to nonperturbative quantum corrections [27]. As a result the SU(4) global symmetry is broken to Sp(4). The infrared low-energy theory is described by five free chiral superfields with $r=\frac{2}{3}$. Thus we have

$$\Delta a=\frac{451}{768}.$$

Accidental symmetry. Consider first the $k=3\to k=2$ flow with an accidental symmetry $(Q\tilde{Q})$ in the IR and none in the UV. This corresponds to $3N_c/4<N_f<N_c$. We have

$$\Delta a=-\frac{3}{256}+\frac{N_f^2}{6}-N_fN_c+\frac{1121N_c^2}{768}-\frac{2N_c^3}{3N_f}+\frac{37N_c^4}{384N_f^2}>0.$$

In the region $2N_c/3<N_f<3N_c/4$ there is an accidental symmetry $(Q\tilde{Q})$ in both the IR and UV, and the above expression has to be corrected. Obviously, $\Delta a>0$ since the accidental contribution to the UV theory is positive.

For $N_f\leqslant 2N_c/3$ the infrared theory is the free magnetic $k=2$ theory [17] (again we must consider $N_c\geqslant 3$). The value of $a_{\rm IR}$ can be computed by assigning $r=\frac{2}{3}$ to all chiral superfields of the magnetic theory. In the region $6N_c/11\leqslant N_f<2N_c/3$ the ultraviolet theory has only one accidental symmetry $(Q\tilde{Q})$ and we have

$$\Delta a=-\frac{51}{256}+\frac{211N_f^2}{1152}+\frac{5N_f^4}{288}-\frac{3N_fN_c}{4}-\frac{N_f^3N_c}{64}+\frac{291N_c^2}{256}+\frac{5N_f^2N_c^2}{1152}-\frac{9N_c^2}{32N_f}-\frac{9N_c^4}{128N_f^2}>0.$$

For $N_f<6N_c/11$ there is an additional accidental symmetry $(QX\tilde{Q})$ so that $a_{\rm UV}$ increases and again $\Delta a>0$.

Consider the $k=2\to k=1$ flow. The infrared theory is just Seiberg's QCD in the conformal phase. There is no accidental symmetry in the physical window in the IR, for $N_f\geqslant N_c$. In the region $N_f\geqslant 3N_c/2$ the IR theory is at the conformal fixed point we have

$$\Delta a=-\frac{1}{48}-\frac{N_c^2}{24}+\frac{19N_c^4}{48N_f^2}>0.$$

For $N_f\leqslant 3N_c/2$ the IR theory is free. By using the magnetic description of Seiberg's QCD to compute $a_{\rm IR}$ we get

$$\Delta a=\frac{7}{48}-\frac{1}{48N_c^2}-\frac{N_c^2}{6N_f^2}+\frac{5N_f}{12N_c}-\frac{N_f^2}{4N_c^2}>0.$$

(ii) *Higgs deformation by* $\langle X\rangle \neq 0$. We now consider the nontrivial stationary point of the deformed superpotential [17] that corresponds to the breaking $SU(N_c)\to SU(N_c-1)\times U(1)$. Consider $N_c\to N_c-1$ and $k\to k-1$, $k-2$ and $k=2,3$.

The flow $k=2\to k=1$, $N_c\to N_c-1 (N_c\geq 3)$. The infrared theory is Seiberg's QCD (plus $2N_f$ free chiral superfields) so that we have to consider only $N_c\leq N_f<2N_c$. At $N_c\leq N_f\leq 3N_c/2$ the infrared theory is confining and can be described by the free magnetic theory with $r=\frac{2}{3}$ for all chiral superfields. In this case we have

$$\Delta a=-\frac{19}{48}-\frac{11N_f}{24}-\frac{N_f^2}{4}+\frac{3N_c}{8}+\frac{5N_fN_c}{12}+\frac{7N_c^2}{48}-\frac{N_c^4}{6N_f^2}>0.$$

At $3N_c/2<N_f<2N_c$ the infrared theory is in the non-Abelian Coulomb phase (plus $2N_f$ free chiral superfields) and we have

$$\Delta a=-\frac{7}{12}+\frac{9}{16N_f^2}-\frac{N_f}{24}+\frac{3N_c}{4}-\frac{9N_c}{4N_f^2}-\frac{N_c^2}{24}+\frac{27N_c^2}{8N_f^2}-\frac{9N_c^3}{4N_f^2}+\frac{19N_c^4}{48N_f^2}>0.$$

The flow $k=3\to k=2$, $N_c\to N_c-1 (N_c\geq 4)$. The infrared theory is in its non-Abelian Coulomb phase. If $N_c=N_f$ then there are no accidental symmetries either in the UV or IR. Thus we have

$$\Delta a=\frac{125}{256}+\frac{1}{6N_c^2}-\frac{2}{3N_c}-\frac{N_c}{24}+\frac{43N_c^2}{768}>0.$$

In the region $3N_c/4\leq N_f<N_c$ there is an accidental symmetry in the IR and none in the UV. We have

$$\Delta a=\frac{253}{256}+\frac{1}{6N_f^2}+\frac{2}{3N_f}+\frac{23N_f}{24}+\frac{N_j^2}{6}-\frac{7N_c}{3}-\frac{2N_c}{3N_f^2}-\frac{2N_c}{N_f}-N_fN_c+\frac{1121N_c^2}{768}+\frac{N_c^2}{N_f^2}+\frac{2N_c^2}{N_f}-\frac{2N_c^3}{3N_f^2}-\frac{2N_c^3}{3N_f}+\frac{37N_c^4}{384N_f^2}>0.$$

In the region $2N_c/5<N_f<3N_c/4$ both the UV and IR theories have accidental symmetries so that both $a_{\rm UV}$ and $a_{\rm IR}$ increase. This accidental contribution in the UV is crucial for $\Delta a>0$ in this region.

The flow $k=3\to k=1$, $N_c=3$. We have to consider $N_f=2,3$. In both cases the infrared theory is Seiberg's SU(2) QCD with N_f flavors in the confining phase. At $N_f=2$ the infrared theory contains just five free chiral superfields with $r=\frac{2}{3}$. The UV theory has an accidental symmetry $(Q\tilde{Q})$. Thus we have

$$\Delta a=\frac{1453}{1536}.$$

At $N_f=3$ the infrared theory is described by nine free mesons and two baryons $(r=2/3)$, and

$$\Delta a=\frac{605}{384}.$$

The flow $k=3\to k=1$, $N_c=2$. We have to consider $N_f=2$. The infrared theory is a U(1) gauge theory with two flavors, which is infrared free. We have

$$\Delta a=\frac{323}{768}.$$

(iii) *Higgs deformation along flat directions.* One can change $N_c\to N_c-1$ and $N_f\to N_f-1$ by turning on $\langle Q_{N_f}\rangle=\langle \tilde{Q}_{N_f}\rangle\neq 0$. One can show for sufficiently large k which correspond to nonrenormalizable models the a theorem is violated due to the negative contribution of Goldstone superfields. However, $\Delta a>0$ in the renormalizable cases $k\leq 3$. This is the first observed problem with the a theorem and we discuss it in Sec. VI after further study of nonrenormalizable cases.

(iv) *Massive deformations.* By adding a mass term to one of the flavors one can reduce $N_f\to N_f-1$. This obviously gives $\Delta a>0$ since $\partial a/\partial N_f<0$.

Spin (7) Pouliot model with N_c+4 spinors **8**, q_i, with $r_q=1-5/(N_c+4)$, singlets $M_{\{i,j\}}$ with $r_M=10/(N_c+4)$. There is a superpotential $Mq\tilde{q}$. We have

$$c=\frac{-398+87N_c+74N_c^2+38N_c^3-N_c^4}{16(4+N_c)^2},$$

$$a=\frac{3(-308+42N_c+44N_c^2+23N_c^3-N_c^4)}{32(4+N_c)^2}.$$

For the flow from the ultraviolet free theory to the conformal fixed point we have

$$\Delta a=\frac{(-11+N_c)^2(42+23N_c+5N_c^2)}{48(4+N_c)^2}>0.$$

Higgs deformation of the model one can check that $\Delta a>0$ under the flow $N_c\to N_c-1$ in the conformal window $(N_c\leq 10)$.

VI. NONRENORMALIZABLE KUTASOV-SCHWIMMER MODELS

In this section we shall study flows of central charges in models which are nonrenormalizable as fundamental theories with Kutasov-Schwimmer models for $k\geq 3$ as examples. It is open to question whether our method is correct for nonrenormalizable theories, but we analyze the data first and then discuss the situation. To simplify the presentation we shall restrict to large N_c and set $N_f=xN_c$, and we shall take $k\leq 5$ and $3N_c/(k+1)<N_f<2N_c$ to avoid complications of accidental symmetry. The upper limit is the naive asymptotic freedom condition. Many more cases were actually studied with results in the same pattern we report here.

In the large N_c, N_f region the value of $a_{\rm IR}$ in Eq. (5.1) for the case $W={\rm Tr}\, X^{k+1}$ is

$$a(k+1)=\frac{9N_c^2}{32}\left\{1+\frac{2}{3(k+1)}-\left(1-\frac{2}{k+1}\right)^3-\frac{16}{(k+1)^3x^2}\right\}, \tag{6.1}$$

which is positive in the region indicated above. The S-current method by which this value is computed implicitly assumes that there is a free ultraviolet fixed point and that the S_μ current is well defined along the RG flow. If we make this assumption then the a theorem is satisfied for the flow from this fixed point since $r<\frac{5}{3}$ for both adjoint and fundamentals.

We can also test the a theorem for flows which interpolate between nontrivial fixed points in the Kutasov-Schwimmer series. Indeed, evidence was given in Refs. [17,28] that in the perturbed theory with $W=\mathrm{Tr}\,X^{k+1}+\mathrm{Tr}\,X^k$, there are flows from the $(k+1)$-fixed point theory in the UV (where $\mathrm{Tr}\,X^k$ is an irrelevant operator) to the k-fixed point theory in the IR (where $\mathrm{Tr}\,X^{k+1}$ is irrelevant). Therefore the differences $a(k+1)-a(k)$ provide further tests of the theorem in the new situation of interacting critical theories at *both* ends of the flow. The differences and their signs are as follows:

$$a(3)-a(2)=\frac{9N_c^2}{32}\left[-0.148+\frac{1.407}{x^2}\right]>0,\quad \frac{3}{2}<x<2;$$

$$a(4)-a(3)=\frac{9N_c^2}{32}\left[-0.143+\frac{0.342}{x^2}\right]<0,$$

$$1.546<x<2>0,\quad 1<x<1.546;$$

$$a(5)-a(4)=\frac{9N_c^2}{32}\left[-0.125+\frac{0.122}{x^2}\right]<0,$$

$$0.988<x<2>0,\quad 0.75<x<0.988;$$

$$a(6)-a(5)=\frac{9N_c^2}{32}\left[-0.102+\frac{0.054}{x^2}\right]<0,$$

$$0.728<x<2>0,\quad 0.6<x<0.728. \tag{6.2}$$

We thus observe additional violations of the a theorem, which occur in the three nonrenormalizable cases above for x in the upper part of its allowed range. We will discuss this below, but let us digress briefly to discuss a special property of the $W=\mathrm{Tr}\,X^4$ theory, which will strengthen our inference that failure of the a theorem is due to nonrenormalizability.

We consider a theory whose field content is that of the Kutasov-Schwimmer model with an extra chiral superfield Y^α_β in the reducible adj$\oplus\mathbf{1}$ representation of the gauge group. The superpotential is $W=-\mathrm{Tr}\,Y^2+2\,\mathrm{Tr}(YX^2)$. The field Y is massive and may be integrated out to give $W_{\mathrm{eff}}=\mathrm{Tr}\,X^4$. Thus the new theory is equivalent to the $W=\mathrm{Tr}\,X^4$ Kutasov-Schwimmer theory in the infrared, and is renormalizable, asymptotically free, and without accidental symmetry in the reduced range $3N_c/4<N_f<N_c$. In the presence of the new chiral superfield Y the value of a_{UV} changes so that for the flow from the ultraviolet free fixed point to the infrared we have

$$\Delta a=\frac{7}{768}-\frac{N_fN_c}{24}+\frac{49N_c^2}{768}-\frac{9N_c^4}{128N_f^2}>0.$$

The computation above for $a(4)-a(3)$ was valid only for $x>1$ because we did not include accidental contributions. However, we can now add the previously computed contribution to $a(3)$, namely, $\Delta a(3)=N_c^2(1-x)^2(4-x)/6x$ [which should be multiplied by a step function $\theta(1-x)$]. The new result for the flow of a, namely, $a(4)-a(3)-\Delta a(3)$ is now valid for $0.75N_c<N_f<N_c$ and is positive in this range. So the observed violation above occurs only in the nonrenormalizable region.

We must consider the question whether one can expect the a theorem to hold for nonrenormalizable theories. In two dimensions, Zamolodchikov assumed Wilsonian renormalizability in his proof of the two-dimensional c theorem. The structure of the theory above some large cutoff Λ was not relevant to his demonstration that the c function $C(g(\mu))$ is monotonically decreasing toward the infrared below this scale. In the approach of Cappelli, Friedan, and Latorre [6] the ultraviolet central charge c_{UV} is expressed as an integral over a Lehmann weight function, and the integral diverges in a (power counting) nonrenormalizable two-dimensional theory. The well known Cardy sum rule $c_{\mathrm{UV}}-c_{\mathrm{IR}}\sim\int d^2x x^2\langle\Theta(x)\Theta(0)\rangle$ also diverges. It is entirely possible that in future work an A function can be identified and monotonicity proven without assumptions concerning the ultraviolet behavior. However, at present we have theoretical control of the Euler anomaly coefficient only at fixed points, and one must expect that this control is lost in the ultraviolet limit of a nonrenormalizable theory. One possible technical reason is a problem with the S-current method we have used. The S current can be viewed as the solution of the operator mixing problem for the current R^μ. In a renormalizable theory it can mix only with a flavor singlet combination of Konishi currents, but in a nonrenormalizable theory there are an infinite number of possibilities.

VII. THEORIES WITH ADDITIONAL GLOBAL U(1) SYMMETRIES

In theories with anomaly-free global $\mathrm{U}(1)_F$ symmetries the R symmetry is not unique and we *a priori* do not know which R-symmetry participates in the superconformal algebra of the infrared theory. As a result we cannot determine a_{IR}, b_{IR}, and c_{IR} by the procedure described above. For simplicity we assume that there is a single $\mathrm{U}(1)_F$ symmetry. In this situation the anomaly free R current is not unique, and there is a one parameter ambiguity in the choice of constants γ_i^* in the anomaly-free S_0^μ and S^μ currents of Eqs. (2.19) and (2.21). We choose any member of this one-parameter family as a particular R symmetry with current $\bar{S}^\mu$. This corresponds to a particular assignment of R charges $\bar{r}_i=(2+\gamma_i^*)/3$ for chiral superfields Φ_i^α, each of which has a unique flavor charge q_i. The most general R current is then $S^\mu=\bar{S}^\mu-vJ^\mu$, where v is a real parameter and J^μ is the flavor current, and the R charges for this current are $r_i(v)=\bar{r}_i-vq_i$. For one particular value of v this S current is in the same multiplet as the stress tensor at the IR fixed point, but it is usually possible to determine v only near the weakly coupled end of the conformal window, where the RG flow is perturbative.

We can compute the anomaly coefficients $a_{\rm IR}(v)$, $b_{\rm IR}(v)$, $c_{\rm IR}(v)$ as functions of v from Eq. (2.28) and use the various positivity conditions to constrain the value of v. A weak check of the a theorem and conformality is then provided by the constraint that there exist a region in v for which all of the positivity conditions are satisfied. Conversely, these positivity conditions constrain the scaling dimensions of operators at the fixed point. Furthermore, the physically allowed value of v is restricted by the assumption that all chiral composite fields have $r(v)>\frac{2}{3}$ so that unitarity is satisfied without accidental symmetry.

We now illustrate this procedure for the Sp$(2N_c)$ gauge theories with $2N_f$ fundamentals and one two-index symmetric tensor, previously studied in Ref. [29], where evidence for a non-Abelian Coulomb phase was given in the conformal window $0<N_f<2N_c+2$. The charges of the fields under the global symmetries are given below, with a simple choice for the anomaly-free S symmetry

	Sp$(2N_c)$	Su$(2N_f)$	U$(1)_F$	U$(1)_{\tilde{S}}$
S	$\square\square$	1	-1	0
Q	$\square$	$\square$	$\frac{N_c+1}{N_f}$	1

As discussed above the value of v is constrained by unitarity. For this model Q^2 and S^2 must have scaling dimension greater than one, or R charge greater than $\frac{2}{3}$. This requires v to lie in the range

$$\frac{1}{3}<v<\frac{2N_f}{3(N_c+1)}, \tag{7.1}$$

and also determines the lower limit on N_f in

$$\frac{N_c+1}{2}<N_f<2(N_c+1), \tag{7.2}$$

where the upper bound is from asymptotic freedom. Equations (7.1) and (7.2) determine the triangular ''physical region'' of the two parameters N_f and v. It is actually expected [29] that v exits from the triangular physical region below some value of N_f. In this case an accidental symmetry is required, and our analysis is valid only above this value of N_f. In the $v-N_f$ plane we plot the curves $c_{\rm IR}(v,N_f)=0$ and $a_{\rm IR}(v,N_f)=0$ for various values of N_c. The results, shown in Fig. 2, indicate that positivity $c_{\rm IR}>0$ and $a_{\rm IR}>0$ holds in the entire physical region. Further, the flow $a_{\rm UV}-a_{\rm IR}$ and the value of $b_{\rm IR}$ for both SU(N_f) and U$(1)_F$ central charges is positive in the entire region shown. Thus there is no constraint on the parameter v from any of the positivity conditions studied.

Near the edge of the conformal window, i.e., near the upper bound for N_f, we can determine the scaling dimensions of operators perturbatively, hence determining the correct R current order by order in the gauge coupling at the fixed point α_*. The anomalous dimensions for the operators Q^2 and S^2 are, near the point $N_f/(N_c+1)\sim 2$ [14,29],

$$\gamma_{S^2}=\frac{\alpha_*}{\pi}(N_c+1)+O(\alpha_*^2),$$

$$\gamma_{Q^2}=-\frac{\alpha_*}{2\pi}\left(N_c+\frac{1}{2}\right)+O(\alpha_*^2). \tag{7.3}$$

Defining $\varepsilon=2-N_f/(N_c+1)$, vanishing of the beta function $\beta\propto 4(N_c+1)-2N_f+2(N_c+1)\gamma_{S^2}+2N_f\gamma_{Q^2}$, to order ε determines the gauge coupling and anomalous dimensions,

$$-\gamma_{S^2}=\frac{\alpha_*}{\pi}(N_c+1)=\frac{\varepsilon}{2}+O(\varepsilon^2). \tag{7.4}$$

Since the scaling dimensions are proportional to the R charges, this fixes v to be

$$v=\frac{2}{3}\left(1+\frac{\gamma_{S^2}}{2}\right)=\frac{2}{3}-\frac{\varepsilon}{6}. \tag{7.5}$$

At the point $N_f=2(N_c+1)$, $a_{\rm UV}-a_{\rm IR}=0$. This point is a local minimum as a function of N_f and v, so the flow is necessarily positive as v moves away from the free field value. In fact, the perturbative analysis is certain to preserve positivity since $b_{\rm IR}$, $c_{\rm IR}$, and $a_{\rm IR}$ are large and positive near the free point.

VIII. REVIEW OF RESULTS

Let us summarize the conclusions of this paper. There are rigorous positivity constraints on the flavor current and Weyl^2 trace anomaly coefficients in any renormalizable four-dimensional theory which flows from a conformal theory in the UV to another in the IR. These constraints arise because the fixed-point values of the anomaly coefficients coincide with central charges of the conformal algebra at the fixed point, and the central charges must be positive by unitarity. This part of the argument was first presented in Ref. [2]. There are additional conjectured positivity conditions [15] on the Euler anomaly coefficient $a(g(\mu))$ and on its flow [5] from the UV to the IR. In particular the only viable candidate for a universal c theorem in four dimensions seems to be the inequality $a_{\rm UV}-a_{\rm IR}>0$. There is no proof of this result, so it is important to test it in models where both the UV and IR behavior are known. It is fortunate that many such models are now known from the study of $N=1$ Seiberg duality. Because of asymptotic freedom the UV values of the anomaly coefficients can be simply obtained from lowest order one-loop graphs, but the IR values are more difficult because the coupling is strong at long distance. It was first shown in Ref. [3] that the IR values can be easily computed from the U$(1)_R FF$, U$(1)_R$, and U$(1)_R^3$ anomalies which are usually calculated to establish the IR equivalence of the electric and magnetic duals. This is possible because of the close relation between the trace anomaly and the anomalous divergence of the U$(1)_R$ current in global and local supersymmetry. Results [3] of tests of the positivity conditions in the SU(N_c) series of SUSY gauge theories showed that all conditions were satisfied throughout the conformal window, and that other possible c-theorem candidates could be ruled out.

The major purpose of the present paper was to test the positivity constraints in many more models. For this purpose

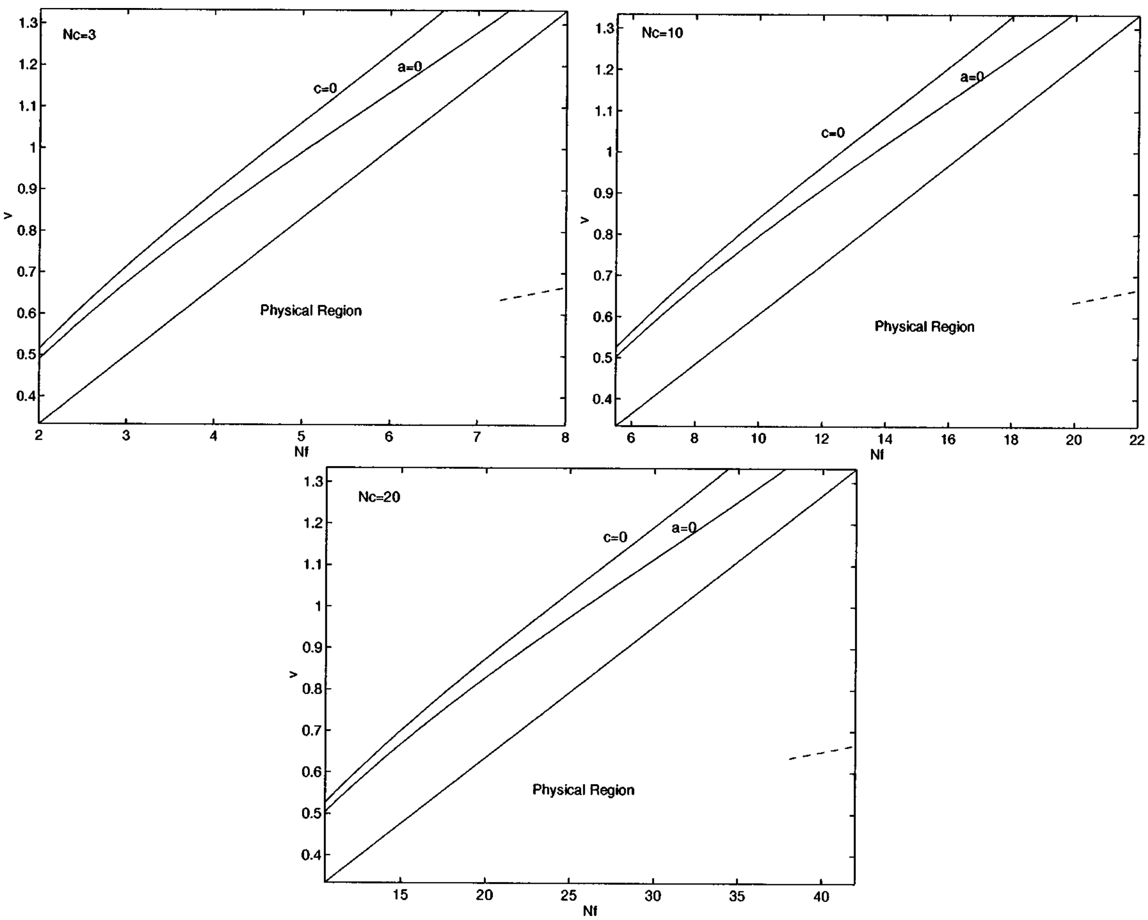

FIG. 2. Positivity conditions are satisfied below the $c=0$ and $a=0$ curves, which includes the entire physical region. The short dashed line is the weak coupling limit of v from Eq. (7.5). Results are shown for various N_c. The flow $a_{\rm UV}-a_{\rm IR}$ and $b_{\rm IR}$ are positive everywhere in the graphs.

we developed general formulas (2.28) for the infrared anomaly coefficients in terms of the anomaly free R charges. In models where the nonanomalous R charges are unique, a precise test of the positivity conditions can be carried out with little difficulty, and this has been done for the rigorous conditions $b_{\rm IR}>0$ and $c_{\rm IR}>0$ for flavor and Weyl^2 anomalies, as well as the a theorem itself and the associated condition $a_{\rm IR}>0$. In many cases positivity can be established from rather weak sufficient conditions, but a closer analysis is required for models with accidental symmetry and for flows between interacting fixed points generated by a relevant perturbation or Higgs deformations of the UV fixed point theory. All conditions are satisfied in the large number of renormalizable theories we have studied, but there are counterexamples for interpolating flows in nonrenormalizable theories where $a_{\rm UV}-a_{\rm IR}$ can have either sign. There is considerably less theoretical control in nonrenormalizable cases and, even in two dimensions, tests of the c theorem which involve the ultraviolet limit of a power-counting nonrenormalizable theory seem to be problematic. Provisionally, then, we believe that the cases of negative flows in nonrenormalizable should not be viewed as ruling out a universal a theorem.

The assignment of R charges in theories conjectured to be in the non-Abelian Coulomb phase is important for the understanding of infrared dynamics because the $N=1$ superconformal algebra necessarily includes the generator of $U(1)_R$ transformations. This assignment is not guaranteed to satisfy the rigorous positivity conditions, and the fact that these are satisfied is a broad consistency check of $N=1$ duality. The fact that $a_{\rm IR}>0$ and $a_{\rm UV}-a_{\rm IR}>0$ in all renormalizable models is very strong evidence that there is a universal a theorem, and that the RG flow is irreversible in four-dimensional supersymmetric theories, and perhaps more. We hope that this empirical result might stimulate a successful theoretical proof.

It is worth noting that the present approach is not immediately applicable to some superconformal models with $N=2$ [30–33] and $N=1$ [34]. It would be interesting to extend the present method to these cases. Note that an approach to the computation of the flavor $b_{\rm IR}$ in the $N=2$ theories has been recently suggested in Ref. [35].

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APPENDIX: TESTS OF A POSSIBLE *b* THEOREM

We present here tests of the inequality $b_{\mathrm{UV}}-b_{\mathrm{IR}}>0$ for the flow of flavor current central charges in the situations (i)–(iv) for which previous tests of the a theorem were discussed in Sec. II.

(i) Let us assume (as was done in [5]) that SU(N_c) QCD is realized in a confined phase with chiral symmetry breaking, so the massless spectrum consists of N_f^2-1 Goldstone bosons which decouple in the long distance limit. For the baryon number current one clearly has $b_{\mathrm{UV}}-b_{\mathrm{IR}}>0$ since there are no massless baryons. For a current of the vectorial SU(N_f) flavor group, on the other hand, we find $b_{\mathrm{UV}}\propto 4N_c$ and $b_{\mathrm{IR}}\propto N_f$ with a common constant of proportionality. Thus $b_{\mathrm{UV}}-b_{\mathrm{IR}}$ changes sign within the region of asymptotic freedom. Of course this could just mean that the conjectured Goldstone realization fails for $4N_c<N_f<11N_c/2$.

(ii) To investigate the b theorem for large N_c, N_f we can make use of the well known QED β function. Up to two-loop order it is given by $\beta_{\mathrm{QED}}(\alpha)=2\alpha^2/3\pi+\alpha^3/2\pi^2$. The graphs for the flavor current correlator in QCD are obtained from the identical QED graphs (see Fig. 3) by replacing the U(1) coupling by the SU(N_f) flavor matrix $T^A/2$ at each external vertex and by the gauge coupling matrix $gt^a/2$, where t^a is an SU(N_c) color matrix at each internal vertex. The point is that these replacements preserve the relative positive sign between the one and two-loop contributions. The current correlator then takes the form

$$\langle J_\mu^A(x)J_\nu^A(x)\rangle \sim (\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)\,\frac{\mathrm{Tr}(T^A)^2}{x^4}\,[N_c+\rho g^{*2}], \tag{A1}$$

where ρ is a positive constant and the fixed point value of the coupling is $g^{*2}/4\pi=(22N_c-4N_f)/75N_c^2$. The same is true for the correlator of baryon number currents. Thus $b_{\mathrm{UV}}-b_{\mathrm{IR}}\sim[N_c-(N_c+\rho g^{*2})]<0$.

(iii) One may also test a possible b theorem in the free magnetic phase of SU(N_c) SUSY QCD as follows. In the ultraviolet we compute b_{UV} from the free field $\langle R^\mu J^\nu J^\rho\rangle$ correlator in the electric theory. The infrared value b_{IR} is obtained from a similar free field computation in the magnetic theory. The difference is

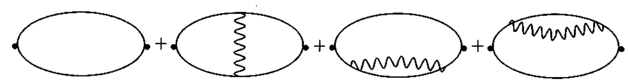

FIG. 3. The graphs for the flavor current correlator.

$$b_{\mathrm{UV}}-b_{\mathrm{IR}}=-\frac{1}{3}\left[\frac{2N_c^2N_f}{N_f-N_c}-2N_fN_c\right]=\frac{2N_fN_c}{N_f-N_c}\,[N_f-2N_c], \tag{A2}$$

which is negative in the entire free magnetic region. Hence the b theorem fails again.

(iv) In the entire conformal window $3N_c/2<N_f<3N_c$ of SU(N_c) supersymmetric QCD, it is known [3] that $b_{\mathrm{UV}}-b_{\mathrm{IR}}<0$ in both electric and magnetic theories for the baryon number central charge. We present here a more general computation for an electric type theory with N_f copies of $(R\oplus \underline{R})$, and we include a mass deformation, making n flavors massive. For a current of the low-energy SU(N_f-n) flavor group, we have, using $\mathrm{Tr}(T^A)^2=\frac{1}{2}$, the central charges $b_{\mathrm{UV}}=\dim R$ at the free UV point, and

$$b_n=3\dim R\,\frac{T(G)}{2(N_f-n)T(R)}$$

for the interacting fixed point theory with N_f-n massless flavors. One can then see that asymptotic freedom implies $b_{\mathrm{UV}}-b_n<0$ so the b theorem fails for a flow from the free UV fixed point to any of the IR fixed point theories. Furthermore, $b_{n_1}-b_{n_2}<0$ if $n_1<n_2$, so the flow between any pair of fixed point theories in which the number of massless quarks decreases also violates a b theorem. At this point one might think that an anti-b theorem holds in supersymmetric theories. However, this is not the case for Higgs deformations. To see this we consider the basic Higgs deformation of the SU(N_c), SU(N_f) theory, leading to the SU(N_c-1), SU(N_f-1) IR theory plus $2(N_f-1)$ decoupled Goldstone fields. For an SU(N_f-1) flavor current we have $b_{\mathrm{UV}}=N_c$ at the free UV point, while $b_{\mathrm{IR}}^*=3(N_c-1)^2/(N_f-1)+1$ in the Higgs deformed low-energy theory. The contribution $+1$ comes from Goldstone fields. One sees quite easily that $b_{\mathrm{UV}}-b_{\mathrm{IR}}^*$ can have either sign in the conformal window, and the same is true for the flow from the SU(N_c), SU(N_f) fixed point to that of the Higgs deformed theory.

The conclusion of this analysis is that the flow of flavor central charges does not have a recognizable universal property.

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